A parameterization method for the computation of invariant tori and their whiskers in quasi-periodic maps: Rigorous results

A. Haro\textsuperscript{a}, R. de la Llave\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a} Departament de Matemàtica Aplicada i Anàlisi, Facultat de Matemàtiques, Universitat de Barcelona, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain
\textsuperscript{b} Department of Mathematics, University of Texas at Austin, Austin, TX 78712, USA

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Abstract

In this paper we prove rigorous results on persistence of invariant tori and their whiskers. The proofs are based on the parameterization method of [X. Cabré, E. Fontich, R. de la Llave, The parameterization method for invariant manifolds. I. Manifolds associated to non-resonant subspaces, Indiana Univ. Math. J. 52 (2) (2003) 283–328; X. Cabré, E. Fontich, R. de la Llave, The parameterization method for invariant manifolds. II. Regularity with respect to parameters, Indiana Univ. Math. J. 52 (2) (2003) 329–360]. The invariant manifolds results proved here include as particular cases of the usual (strong) stable and (strong) unstable manifolds, but also include other non-resonant manifolds. The method lends itself to numerical implementations whose analysis and implementation is studied in [A. Haro, R. de la Llave, A parameterization method for the computation of invariant tori and their whiskers in quasi-periodic maps: Numerical algorithms, preprint, 2005; A. Haro, R. de la Llave, A parameterization method for the computation of invariant tori and their whiskers in quasi-periodic maps: Numerical implementation and examples, preprint, 2005]. The results are stated as \textit{a posteriori} results. Namely, that if one has an approximate solution which is not degenerate, then, one has a true solution not too far from the approximate one. This can be used to validate the results of numerical computations.

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* Corresponding author.
E-mail addresses: haro@cerber.mat.ub.es (A. Haro), llave@math.utexas.edu (R. de la Llave).
KAM theory without action-angle variables

R de la Llave\textsuperscript{1}, A González\textsuperscript{1}, À Jorba\textsuperscript{2} and J Villanueva\textsuperscript{3}

\textsuperscript{1} Department of Mathematics, University of Texas at Austin, University Station C1200, Austin, TX 78712-0257, USA
\textsuperscript{2} Dept. de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Barcelona, Spain
\textsuperscript{3} Dept. de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Barcelona, Spain

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Recommended by V Baladi

Abstract
We give a proof of a KAM theorem on existence of invariant tori with a Diophantine rotation vector for Hamiltonian systems. The method of proof is based on the use of the geometric properties of Hamiltonian systems which, in particular, do not require the Hamiltonian system either to be written in action-angle variables or to be a perturbation of an integrable one. The proposed method is also useful to compute numerically invariant tori for Hamiltonian systems. We also prove a translated torus theorem in any number of degrees of freedom.

Mathematics Subject Classification: 37J40

1. Introduction

The goal of this paper is to present a proof of a KAM theorem on existence of invariant tori with a Diophantine rotation vector for Hamiltonian systems (symplectic maps and Hamiltonian vector fields).

What we show is that, given an approximately invariant torus which is not too degenerate, there is a true invariant torus nearby. We will refer to such proofs as "polishing".

The proof is constructive and the KAM method presented here leads to an algorithm that can be implemented numerically and which uses very small requirements of memory and operation.

The methodology presented here does not require the system either to be a perturbation of an integrable system or to be written in action-angle variables. This is due to the fact that the method is based on the geometric properties of Hamiltonian systems which do not require such assumptions.

Of course, action-angle variables always exist in a neighbourhood of an invariant torus. However, a change of coordinates bringing the system to action-angle variables in general cannot be explicitly computed.
Homoclinic orbits to invariant tori near a homoclinic orbit to center–center–saddle equilibrium

Oksana Koltsova\textsuperscript{a}, Lev Lerman\textsuperscript{b, *}, Amadeu Delshams\textsuperscript{c}, Pere Gutiérrez\textsuperscript{c}

\textsuperscript{a} Department of Computational Mathematics and Cybernetics, University of Nizhny Novgorod, 23 Garinov Ave., 603600 Nizhny Novgorod, Russia
\textsuperscript{b} Department of Differential Equations, University of Nizhny Novgorod, and Research Institute for Applied Mathematics and Cybernetics, 10 Ul’yanov St., 603600 Nizhny Novgorod, Russia
\textsuperscript{c} Dept. de Matem. Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain

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Communicaed by C.K.R.T. Jones

Abstract

We consider a perturbation of an integrable Hamiltonian vector field with three degrees of freedom with a center–center–saddle equilibrium having a homoclinic orbit or loop. With the help of a Poincaré map (chosen based on the unperturbed homoclinic loop), we study the homoclinic intersections between the stable and unstable manifolds associated to persistent hyperbolic KAM tori, on the center manifold near the equilibrium. If the perturbation is such that the homoclinic loop is preserved (i.e. the perturbation also has a homoclinic loop inherited from the unperturbed one), we establish that, in general, the manifolds intersect along 8, 12 or 16 transverse homoclinic orbits. On the other hand, in a more generic situation (the loop is not preserved; a condition for this fact is obtained by means of a Melnikov-like method), the manifolds intersect along four transverse homoclinic orbits, though a small neighborhood of the loop has to be excluded. In a first approximation, those homoclinic orbits can be detected as nondegenerate critical points of a Melnikov potential defined on the 2-torus. The number of homoclinic orbits is given by Morse theory applied to the Melnikov potential.

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Keywords: Homoclinic orbits; KAM tori; Center manifold; Morse theory; Melnikov potential

* Corresponding author. Tel.: +7 8312 389923; fax: +7 8312 390411.
E-mail addresses: koltsova@uic.nnov.ru (O. Koltsova), lermanl@mm.unn.ru (L. Lerman), amadeu.delshams@upc.edu (A. Delshams), pere.gutierrez@upc.edu (P. Gutiérrez).

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Integrable flows and Bäcklund transformations on extended Stiefel varieties with application to the Euler top on the Lie group $SO(3)$

Yuri N FEDOROV

Department of Mathematics and Mechanics
Moscow Lomonosov University, Moscow, 119 899, Russia
E-mail: fedorov@mech.math.msu.su

and

Department de Matemática I, Universitat Politecnica de Catalunya, Barcelona,
E-08028 Spain
E-mail: Yuri.Fedorov@upc.es

This article is a part of the special issue titled “Symmetries and Integrability of Difference Equations (SIDE VI)

Abstract

We show that the $m$-dimensional Euler–Manakov top on $so^*(m)$ can be represented as a Poisson reduction of an integrable Hamiltonian system on a symplectic extended Stiefel variety $\tilde{V}(k,m)$, and present its Lax representation with a rational parameter.

We also describe an integrable two-valued symplectic map $B$ on the 4-dimensional variety $V(2,3)$. The map admits two different reductions, namely, to the Lie group $SO(3)$ and to the coalgebra $so^*(3)$.

The first reduction provides a discretization of the motion of the classical Euler top in space and has a transparent geometric interpretation, which can be regarded as a discrete version of the celebrated Poinset model of motion and which inherits some properties of another discrete system, the elliptic billiard.

The reduction of $B$ to $so^*(3)$ gives a new explicit discretization of the Euler top in the angular momentum space, which preserves first integrals of the continuous system.

1 Introduction

In most publications the integrable $m$-dimensional Euler top is represented as a flow on the cotangent bundle $T^*SO(m)$ or on the coalgebra $so^*(m)$.

Recently, an alternative description of this problem as a system on a symplectic subvariety of the group product $SO(m) \times SO(m)$ was proposed in [4, 5].
Regularity of critical invariant circles of the standard nontwist map

A Apte, Rafael de la Llave and Nikola P Petrov

1 Department of Physics, University of Texas at Austin, 1 University Station C1500, Austin, TX 78712-0262, USA
2 Department of Mathematics, The University of Texas at Austin, 1 University Station C1200, Austin, TX 78712-0257, USA
3 Department of Mathematics and Michigan Center for Theoretical Physics, University of Michigan, Ann Arbor, MI 48109-1109, USA

E-mail: apte@physics.utexas.edu, llave@math.utexas.edu and npetrov@umich.edu

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Recommended by A Chenciner

Abstract
We study critical invariant circles of several noble rotation numbers at the edge of break-up for an area-preserving map of the cylinder, which violates the twist condition.

These circles admit essentially unique parametrizations by rotational coordinates. We present a high accuracy computation of about $10^7$ Fourier coefficients. This allows us to compute the regularity of the conjugating maps and to show that, to the extent of numerical precision, it only depends on the tail of the continued fraction expansion.

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1. Introduction

Area-preserving maps of a cylinder have been studied for several decades, both as low-dimensional models of physical systems and as interesting mathematical problems. Until recently, most of these studies have dealt with twist maps, which are the maps that satisfy a certain non-degeneracy condition. Lately, maps that violate this condition locally in phase space have been of great interest. These nontwist maps occur in various physical systems such as, e.g., in the study of magnetic field lines in toroidal plasma devices with reversed shear profile, channel flows, and other physical applications (see, e.g. [1–8]).
The obstruction criterion for non-existence of
invariant circles and renormalization

Rafael de la Llave\textsuperscript{1} and Arturo Olvera\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, The University of Texas at Austin, 1 University Station C1200, Austin, TX 78712-0257, USA
\textsuperscript{2} Departamento de Matemáticas y Mécanica, IIMAS–UNAM, 01000 México DF, Mexico

E-mail: llave@math.utexas.edu and aoc@mym.iimas.unam.mx

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Recommended by K M Khanin

Abstract

The goal of this paper is to show that the renormalization group and the
obstruction criterion can work together. We formulate a conjecture which
supplements the standard renormalization scenario for the breakdown of golden
circles in twist maps. We show rigorously that if the conjecture was true then
the following hold.

(1) The stable manifold of the non-trivial fixed point would be part of the
boundary between the existence of smooth invariant tori and hyperbolic
orbits with golden mean rotation number. In particular, the boundary of
the set of twist maps with a torus with a golden mean rotation number
would include a smooth submanifold in the space of analytic mappings.
Moreover, if the conjecture was true, in the domain of universality (i.e.
a small neighbourhood of the non-trivial fixed point), we would have the
following (2), (3), (4).

(2) The obstruction criterion for non-existence of golden mean invariant circles
(Olvera and Simó 1987 \textit{Physica D} 26 181–92) is sharp. That is, for maps in
the universality class there is either a golden invariant circle or the condition
in Olvera and Simó for non-existence of golden circles applies.

of invariant circles if and only if there the residues of approximating orbits
are finite would be valid. That is, for maps in the universality class there
would be a smooth invariant circle if and only if the residue of periodic
orbits approximating the circle goes to zero.

(4) If there is no invariant circle, there are uniformly hyperbolic sets with
golden mean rotation number.
THE PARAMETERIZATION METHOD FOR ONE-DIMENSIONAL IN Variant MANIFOLDS OF HIGHER DIMENSIONAL PARABOLIC fixed POINTS

IMMACULADA BALDOMÁ
Departament d'Enginyeria Informática i Matemàtiques
Universitat Rovira i Virgili
Campus San Pau, Avinguda de les Corts Catalanes 250
43003, Tarragona, Spain

ERNEST FONTICH
Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
Gran Via 585. 08007 Barcelona, Spain

RAFAEL DE LA LLAVE
Department of Mathematics, 1 University Station C1200
The University of Texas at Austin
Austin, TX 78712-1082, U.S.A.

PAU MARTÍN
Departament de Matemàtica Aplicada IV
Universitat Politècnica de Catalunya
Ed-05, Jordi Girona, 1-3, 08034 Barcelona, Spain
(Communicated by Amadeu Delshams)

Abstract. We use the parameterization method to prove the existence and properties of one-dimensional submanifolds of the center manifold associated to the fixed point of $C^r$ maps with linear part equal to the identity. We also provide some numerical experiments to test the method in these cases.

1. Introduction. We consider $C^r$ maps of $\mathbb{R}^{1+n}$ having a parabolic fixed point and study the existence of one-dimensional invariant manifolds passing through this fixed point.

We assume that the fixed point is the origin and that the linear part of the map at the fixed point is the identity. Then a whole neighborhood of the origin is a center manifold. However there may exist invariant submanifolds of points which go to the origin by the iteration of the map. In this setting we refer to such submanifolds as stable manifolds. In the same way we can speak of unstable manifolds.

These problems appear naturally in Celestial Mechanics. In these applications, often the fixed point is the image of infinity under a suitable transformation and the invariant manifolds are the separation from bounded and unbounded motions. See for example, [23, 18, 19, 6, 15] for studies of parabolic invariant manifolds in $\mathbb{R}^2$ and applications to Celestial Mechanics.

2000 Mathematics Subject Classification. Primary: 37D10; Secondary: 37H05.
Key words and phrases. Parabolic point, invariant manifold, parameterization method.
Discussion

Comment on “Reconnection scenarios…”

Amit Apte a,*, Rafael de la Llave b, Emilia Petrisor c

a Department of Physics, The University of Texas at Austin, 1 University Station C1600/C1 500, Austin, TX 78712-0264, USA
b Department of Mathematics, The University of Texas at Austin, 1 University Station C1200, Austin, TX 78712-0257, USA
c Department of Mathematics, “Politehnica” University of Timisoara, P-ta Regina Maria No 1, 300040 Timisoara, Romania

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Abstract

We point out that Proposition 3.1 in [E. Petrisor. Reconnection scenarios and the threshold of reconnection in the
we suggest that for near integrable mappings, the results of [E. Petrisor. Reconnection scenarios and the threshold of
and quantitatively very approximate.

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1. Introduction

The paper [3] contains a detailed study of the reconnection phenomena in now-twist maps. The study in [3] is based
on the primitive function of these maps and contains a wealth of numerical and rigorous studies.

We refer to [3] for more details on notation. See also [2]. We recall that given an exact diffeomorphism f, its primitive
function $S_f$ (unique up to an additive constant) is defined by $f^*x = x + dS_f$ ($x$ is the 1-form such that $dx = \omega$).

Proposition 3.1 in [3] claims that if $p_1$, $p_2$ are fixed points of $f$ with a heteroclinic connection between them
$S_f(p_1) = S_f(p_2)$. Unfortunately, this result is, strictly speaking, false.

The heuristic reason for the falsehood of Proposition 3.1 is that if $p_1$ and $p_2$ have a transverse heteroclinic intersection
for $f$, their connection persists for all maps $g$ sufficiently $C^1$ close to them. (This is a result of the implicit function
theorem and the smooth dependence on parameters of the stable manifolds. See, e.g., [1].) Hence we can find open sets
of maps with heteroclinic intersections. On the other hand, equality of the primitive functions at two fixed points is a
codimension 1 phenomenon since it involves the agreement of the function at two points.
INVARIANT PRE-FOLIATIONS FOR NON-RESONANT NON-UNIFORMLY HYPERBOLIC SYSTEMS

ERNEST FONTICH, RAFAEL DE LA LLAVE, AND PAU MARTÍN

Abstract. Given an orbit whose linearization has invariant subspaces satisfying some non-resonance conditions in the exponential rates of growth, we prove existence of invariant manifolds tangent to these subspaces. The exponential rates of growth can be understood either in the sense of Lyapunov exponents or in the sense of exponential dichotomies. These manifolds can correspond to “slow manifolds”, which characterize the asymptotic convergence.

Let \( \{x_i\}_{i \in \mathbb{N}} \) be a regular orbit of a \( C^2 \) dynamical system \( f \). Let \( S \) be a subset of its Lyapunov exponents. Assume that all the Lyapunov exponents in \( S \) are negative and that the sums of Lyapunov exponents in \( S \) do not agree with any Lyapunov exponent in the complement of \( S \). Denote by \( E^S_x \) the linear spaces spanned by the spaces associated to the Lyapunov exponents in \( S \). We show that there are smooth manifolds \( W^S_x \) such that \( f(W^S_x) \subset W^S_{x_{i+1}} \) and \( T_x W^S_x = E^S_x \). We establish the same results for orbits satisfying dichotomies and whose rates of growth satisfy similar non-resonance conditions. These systems of invariant manifolds are not, in general, a foliation.

1. Introduction and statement of results

When studying the behavior of an orbit of a dynamical system \( f \), it is natural to study the behavior of its linearization and wonder whether there are non-linear analogues for the features found in the study of the linearization.

Very often we can classify the tangent vectors along an orbit into subspaces with different rates of exponential growth either in the future or in the past. In the literature, there are several precise definitions of rates of growth. We will discuss them later in Section 1.1.

Since the subspaces corresponding to a rate of growth and combinations of them are invariant, the question of existence of invariant objects for the full system related to these linear spaces naturally arises. In particular, we may be interested in the spaces that converge the slowest, since these slowest convergences will dominate the long-term behavior.

The goal of this paper is to show that, under appropriate non-resonance conditions for the rates of growth, indeed one can find smooth manifolds tangent to the spaces invariant under the linearization. We also give examples that show that the non-resonance conditions are necessary for the existence of such invariant smooth manifolds.

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2000 Mathematics Subject Classification. Primary 37D10, 37D25, 34D09, 70K45.

Key words and phrases. Lyapunov exponents, invariant manifolds, resonances, normal forms.

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Orbits of unbounded energy in quasi-periodic perturbations of geodesic flows

Amadeu Delshams\textsuperscript{a}, Rafael de la Llave\textsuperscript{b,\,*}, Tere M. Seara\textsuperscript{a}

\textsuperscript{a}Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain
\textsuperscript{b}Department of Mathematics and ICES, University of Texas, Austin, TX 78712-0257, USA

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Abstract

We show that certain mechanical systems, including a geodesic flow in any dimension plus a quasi-periodic perturbation by a potential, have orbits of unbounded energy.

The assumptions we make in the case of geodesic flows are:

(a) The metric and the external perturbation are smooth enough.
(b) The geodesic flow has a hyperbolic periodic orbit such that its stable and unstable manifolds have a transverse homoclinic intersection.
(c) The frequency of the external perturbation is Diophantine.
(d) The external potential satisfies a generic condition depending on the periodic orbit considered in (b).

The assumptions on the metric are $C^2$ open and are known to be dense on many manifolds. The assumptions on the potential fail only in infinite codimension spaces of potentials.

The proof is based on geometric considerations of invariant manifolds and their intersections. The main tools include the scattering map of normally hyperbolic invariant manifolds, as well as standard perturbation theories (averaging, KAM and Melnikov techniques).

We do not need to assume that the metric is Riemannian and we obtain results for Finsler or Lorentz metrics. Indeed, there is a formulation for Hamiltonian systems satisfying scaling...
Quasi-Chaplygin Systems and Nonholonomic Rigid Body Dynamics

YURI N. FEDOROV$^{1,2}$ and BOŽIDAR JOVANOVIĆ$^{3}$

$^1$Department of Mathematics and Mechanics, Moscow Lomonosov University, Moscow 119 899, Russia. e-mail: fedorov@mech.math.msu.su

$^2$Departament de Matemàtica I, Universitat Politecnica de Catalunya, Barcelona E-08028, Spain. e-mail: Yuri.Fedorov@upc.es

$^3$Mathematical Institute SANU, Kneza Mihaila 35, 11000 Belgrade, Serbia. e-mail: bozaj@mi.sanu.ac.yu

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Abstract. We show that the Suslov nonholonomic rigid body problem studied in Fedorov and Kozlov (Am. Math. Soc. Transl. Ser. 2 168:141–171, 1995), Jovanović (Reg. Chaot. Dyn. 8(1):125–132, 2005), and Zenkov and Bloch (J. Geom. Phys. 34(2):121–136, 2000) can be regarded almost everywhere as a generalized Chaplygin system. Furthermore, this provides a new example of a multidimensional nonholonomic system which can be reduced to a Hamiltonian form by means of Chaplygin reducing multiplier. Since we deal with Chaplygin systems in the local sense, the invariant manifolds of the integrable examples are not necessary tori.


Keywords. Chaplygin reducing multiplier, Suslov problem, integrable nonholonomic systems, topology of invariant manifolds.

1. Introduction

We start the letter with the definition of nonholonomic Chaplygin systems, their reductions and Hamiltonization.

Chaplygin systems. Suppose we are given a natural nonholonomic system on the $n$-dimensional Riemannian manifold $(N, \kappa)$ with local coordinates $x_i$, Lagrangian $L(x, \dot{x}) = \frac{1}{2} \sum x_i \dot{x}_i - v(x)$ and $k$-dimensional distribution $D \subset TN$ describing kinematic constraints: a curve $x(t)$ is said to satisfy the constraints if $\dot{x}(t) \in D_{x(t)}$ for all $t$. The trajectory of the system $x(t)$ that satisfies the constraints is a solution to the Lagrange–d’Alembert equation

$$\sum_{i=1}^{n} \left( \frac{\partial L}{\partial \dot{x}_i} \frac{d}{dt} \frac{\partial L}{\partial x_i} \right) \eta_i = 0, \quad \text{for all } \eta \in D_x. \quad (1)$$
Abstract: The SRB measures of a hyperbolic system are widely accepted as the measures that are physically relevant. It has been shown by Ruelle that they depend smoothly on the system. Furthermore, Ruelle showed by a separate argument that the first derivative, i.e., the linear response function, admits a geometric interpretation.

In this paper, we consider thermodynamic limits of SRB measures in lattices of coupled hyperbolic attractors. In a previous paper, using Markov partitions and thermodynamic formalism, we had established the smooth dependence of thermodynamic limits of SRB measures. Here, we establish that the linear response function admits a geometric interpretation. The formula is analogous to the one found by Ruelle for finite dimensional systems if one term is reinterpreted appropriately. We show that the limiting derivative is the thermodynamic limit of the derivatives in finite volume. We also obtain similar results for the derivatives of the entropy.

1. Introduction: Derivative Formulas of the SRB Measure and its Entropy

For uniformly hyperbolic systems, Ruelle proved that the (generalized) SRB measure depends differentiably on the system and also gave a formula for the derivative in terms of geometric features of the map and the deformation [16, 18].

The derivative of the SRB state with respect to the underlying system is also called the linear response function and is used to develop the theory of non-equilibrium statistical mechanics [17] and, hence, it is natural to try to express it in terms of geometric objects that are as simple as possible and directly observable. We note that the problem of proving differentiability of the SRB measures is somewhat different to the problem of giving a geometric formula for the derivative. Indeed, many proofs of the existence of derivatives use a Markov partition to transform the problem into a problem in equilibrium states with a changing potential. The use of a Markov partition makes it very hard to

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On the numerical computation of Diophantine rotation numbers of analytic circle maps

Tere M. Seara*, Jordi Villanueva

Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain

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Abstract

In this paper we present a numerical method to compute Diophantine rotation numbers of circle maps with high accuracy. We mainly focus on analytic circle diffeomorphisms, but the method also works in the case of (enough) finite differentiability. The keystone of the method is that, under these conditions, the map is conjugate to a rigid rotation of the circle. Moreover, although it is not fully justified by our construction, the method turns out to be quite efficient for computing rational rotation numbers. We discuss the method through several numerical examples.

Keywords: Circle maps; Rotation number; Numerical approximation

1. Introduction

The main purpose of this work is to introduce a new numerical method to compute the rotation number of a circle map. This problem has been formerly considered by many other authors, and several algorithms have been developed. See, for instance, [32,5,21,25,24,12,13,8]. On the one hand, the level of complexity of these algorithms ranges from the definition itself to sophisticated methods of frequency analysis. On the other hand, some of them are efficient for the computation of rational rotation numbers and some others work better for irrational ones.

In this paper we are mainly concerned with analytic circle diffeomorphisms having Diophantine rotation number. So, we take strong advantage of the fact that the map is analytically conjugate to a rotation. The method we present is based on the computation of suitable averages of the iterates of the map, followed by Richardson’s extrapolation. The keystone of this procedure is that we know a priori which is the asymptotic behavior of these averages when the number of iterates goes to infinity. This algorithm provides numerical approximations to the rotation number, with very high accuracy in general.

To develop this method, we use the hypotheses on the map to be analytically conjugate to a rigid rotation and to have a (good) Diophantine rotation number. Although we focus on the analytic case, the same procedure can be used for smooth circle diffeomorphisms, but we only expect the method to be efficient if the conjugation is regular enough.

Of course, the set up of this method is restrictive and excludes a lot of cases. For instance, if we consider a (generic) one-parameter family of circle homeomorphisms, the set of parameters for which the rotation number is rational, and hence the map is not conjugate to a rotation (in general), is a dense set with (non-empty) interior. However, if these maps are smooth perturbations of a rotation, then, under general hypotheses, the set of parameters for which the rotation number is Diophantine has big relative measure. On the other hand, if the rotation number is eventually rational, the method provides quite good results. We do not have a complete justification of this fact, but we refer to Remark 9 for a tentative explanation and to Section 4 for examples with rational rotation numbers.

From the practical point of view, the numerical method presented here is suitable if we are able to compute the iterates of the map with high precision, for instance if we can work with a computer arithmetic having a large number of decimal digits. In this case, we can try to use the method with high-order extrapolation and, then, we can hope to obtain a good
Breakdown of Heteroclinic Orbits for Some Analytic Unfoldings of the Hopf-Zero Singularity

I. Baldomá and T. M. Seara

1 Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 587, 08007 Barcelona, Spain
   e-mail: barraca@mat.ub.es
2 Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain
   e-mail: terc.m-seara@upc.edu

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Summary. In this paper we study the exponentially small splitting of a heteroclinic connection in a one-parameter family of analytic vector fields in $\mathbb{R}^3$. This family arises from the conservative analytic unfoldings of the so-called Hopf zero singularity (central singularity). The family under consideration can be seen as a small perturbation of an integrable vector field having a heteroclinic orbit between two critical points along the $z$ axis. We prove that, generically, when the whole family is considered, this heteroclinic connection is destroyed. Moreover, we give an asymptotic formula of the distance between the stable and unstable manifolds when they meet the plane $z = 0$. This distance is exponentially small with respect to the unfolding parameter, and the main term is a suitable version of the Melnikov integral given in terms of the Borel transform of some function depending on the higher-order terms of the family. The results are obtained in a perturbative setting that does not cover the generic unfoldings of the Hopf singularity, which can be obtained as a singular limit of the considered family. To deal with this singular case, other techniques are needed. The reason to study the breakdown of the heteroclinic orbit is that it can lead to the birth of some homoclinic connection to one of the critical points in the unfoldings of the Hopf-zero singularity, producing what is known as a Shilnikov bifurcation.

Key words. Exponentially small splitting, Hopf-zero bifurcation, Melnikov function, Borel transform

1. Introduction

One of the most frequently studied problems in the last century was the existence of transversal intersections between stable and unstable manifolds of one or more critical
ON SQUIRT SINGULARITIES IN HYDRODYNAMICS

DIEGO CÓRDOBA†, CHARLES FEFFERMAN‡, AND RAFAEL DE LA LLAVE§

Abstract. We consider certain singularities of hydrodynamic equations that have been proposed in the literature. We present a kinematic argument that shows that if a volume preserving field presents these singularities, certain integrals related to the vector field have to diverge. We also show that if the vector fields satisfy certain partial differential equations (Navier–Stokes, Boussinesq), then the integrals have to be finite. As a consequence, these singularities are absent in the solutions of the above equations.

Key words. singularities, Boussinesq equations, Navier–Stokes equations

AMS subject classifications. 76D03, 76D05, 35Q35

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1. Introduction. One way to make progress towards settling the question of existence of singularities in incompressible fluid motion is to conjecture plausible scenarios for the formation of singularities supported by numerical evidence. Then, it becomes a natural object to develop mathematically rigorous arguments that derive quantitative consequences of the different scenarios and possibly show that these singularities cannot occur in solutions of hydrodynamic equations.

In this note we introduce some classes of singularities which we call “squirter” singularities in which some portion of material is ejected from a set of positive measure. These squirt singularities include as particular cases several other singularities that had been considered in the literature (for example, the “potato chip” singularities, the “saddle collapse,” and the “tube collapse”; see section 2 for precise definitions).

In section 3 we present a very simple argument that shows that if a volume preserving vector field $u$ presents a squirt singularity at time $T$, then

$$\int_0^T \|u\|_{L^\infty} \, dt = \infty.$$  

In section 4 we show that if the vector field satisfies certain partial differential equations (e.g., Navier–Stokes in two and three dimensions, Boussinesq equations in two and three dimensions with positive viscosity), then

$$\int_0^T \|u\|_{L^\infty} \, dt < \infty.$$  

As a consequence of the results in sections 3 and 4, we conclude that volume preserving vector fields satisfying the partial differential equations considered in section 4 do not experience any of the squirt singularities.
TOPOLOGICAL METHODS IN THE INSTABILITY PROBLEM OF HAMILTONIAN SYSTEMS

MARIAN GIDEA
Department of Mathematics
Northeastern Illinois University
Chicago, IL 60625, USA

RAFAEL DE LA LLAVE
Department of Mathematics
University of Texas at Austin
Austin, TX 78712, USA

ABSTRACT. We use topological methods to investigate some recently proposed mechanisms of instability (Arnold diffusion) in Hamiltonian systems.

In these mechanisms, chains of heteroclinic connections between whiskered tori are constructed, based on the existence of a normally hyperbolic manifold \( \Lambda \), so that: (a) the manifold \( \Lambda \) is covered rather densely by transitive tori (possibly of different topology), (b) the manifolds \( W^s_\Lambda \), \( W^u_\Lambda \) intersect transversally, (c) the systems satisfy some explicit non-degeneracy assumptions, which hold generically.

In this paper we use the method of correctly aligned windows to show that, under the assumptions (a), (b), (c), there are orbits that move a significant amount.

As a matter of fact, the method presented here does not require that the tori are exactly invariant, only that they are approximately invariant. Hence, compared with the previous papers, we do not need to use KAM theory. This lowers the assumptions on differentiability.

Also, the method presented here allows us to produce concrete estimates on the time to move, which were not considered in the previous papers.

1. Introduction. The paper [1] described a mechanism of global instability in Hamiltonian systems which are arbitrarily close to integrable. The paper showed that in the remarkable two parameter family

\[
H_{\varepsilon, \mu}(A_1, A_2, \varphi_1, \varphi_2, t) = H_0 + \varepsilon H_\varepsilon + \mu H_\mu
\]

\[
= \frac{1}{2} A_1^2 + \frac{1}{2} A_2^2 + \varepsilon (\cos \varphi_1 - 1) + \mu (\cos \varphi_1 - 1) (\sin \varphi_2 + \cos t)
\]

(1)

for all \( 0 < |\mu| \ll |\varepsilon| \ll 1 \), the system (1) admits orbits for which the action \( A_2 \) changes by 1 over time.

The mechanism of [1] is based on the existence of whiskered tori. The perturbation is chosen in such a way that it does not affect the tori but causes the stable
We study numerically the disappearance of normally hyperbolic invariant tori in quasiperiodic systems and identify a scenario for their breakdown. In this scenario, the breakdown happens because two invariant directions of the transversal dynamics come close to each other, losing their regularity. On the other hand, the Lyapunov multipliers associated with the invariant directions remain more or less constant. We identify notable quantitative regularities in this scenario, namely that the minimum angle between the two invariant directions and the Lyapunov multipliers have power law dependence with the parameters. The exponents of the power laws seem to be universal. © 2006 American Institute of Physics. [DOI: 10.1063/1.2150947]

Quasiperiodically forced systems occur in many situations in physics, mathematics, engineering, etc. In many cases, the external quasiperiodic perturbations induce quasiperiodic motions, which correspond to invariant tori. It is important to understand when these invariant tori persist under perturbations, and to identify the mechanisms of their breakdown. It has been known for a long time that the persistence of a torus is related to the exponential growth of the linearization along certain directions (normal hyperbolicity), and that normal hyperbolicity may be lost because of bifurcations such as saddle-node and period doubling, among others. The common feature of these standard bifurcations is that some Lyapunov multipliers approach 1, while the invariant directions remain smooth. In this paper we propose a new mechanism, in which two invariant directions of the linearized dynamics come close to each other, losing their regularity, and the corresponding Lyapunov multipliers remain more or less constant, away from each other and away from 1. Hence, the phenomenon is described by two observables: the minimum angle between the invariant directions and the Lyapunov multipliers. We also identify notable universal power laws of these observables.

I. INTRODUCTION

The long-term behavior of a dynamical system is organized by its invariant objects. Hence, it is important to understand which invariant objects persist under perturbations of the system. It has been known for a long time that the persistence of an invariant object is related to the exponential rate of growth of the perturbations of orbits starting on it.

For example Refs. 1–3 show that a manifold persists under all C1 small changes in the map if it is normally hyperbolic. The fact that this condition is also necessary for C1 persistence was proven in Ref. 4.

A problem that has received a great deal of attention5–7 is the study of the breakdown of normally hyperbolic invariant manifolds.

We recall that a manifold is normally hyperbolic if all the perturbations transversal to the manifold grow at an exponential rate (either in the future or in the past), and that this exponential rate is bigger than the exponential rate of growth for perturbations tangential to the manifold. We will not give a precise definition of normal hyperbolicity in general, but we refer to the literature quoted above. In Eq. (4) we will give the definition of a more general concept (exponential dichotomy) tailored for the systems that we will consider in this paper, which are quasiperiodically forced maps [see Eq. (1)].

We note that the definition of hyperbolicity has two measures of quality. One is the asymptotic rate of growth [called λ± in Eq. (4)] and another is the prefactor in the exponential [called C in Eq. (4)], which measures how long it is necessary to wait to observe the asymptotic rate of growth.

Even if most rigorous studies of loss of hyperbolicity are concerned with situations in which the rates of growth λ± degenerate, we emphasize (see also Ref. 8) that hyperbolicity may well be lost because the prefactors C become unbounded.

Indeed, in this paper we report two situations in which the hyperbolicity (or more generally the exponential dichotomy) is lost because the prefactors grow unbounded even though the asymptotic exponential rates remain more or less constant. In other words, even though the asymptotic rates of growth remain more or less constant we have to observe them for increasingly long time intervals till they manifest themselves.

Interestingly, we find in the situations above that there are quantitative regularities and scaling behaviors for several observables. The scaling exponents seem to be universal in a wide class of systems.

In this paper, we find invariant tori and study in detail the dynamics of the linearized equations around them [see Eq. (3)]. In particular, we compute quite accurately the invariant directions and their corresponding Lyapunov multi-

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WHISKERED AND LOW DIMENSIONAL TORI IN NEARLY INTEGRABLE HAMILTONIAN SYSTEMS

R. DE LA LLAVE AND C. E. WAYNE

Abstract. We show that a nearly integrable hamiltonian system has invariant tori of all dimensions smaller than the number of degrees of freedom provided that certain nondegeneracy conditions are met. The tori we construct are generated by the resonances of the system and are topologically different from the orbits that are present in the integrable system. We also show that the tori we construct have stable and unstable manifolds and point out how to construct other types of interesting orbits.

The method of proof is a combination of different perturbation methods.
A PARAMETERIZATION METHOD FOR THE COMPUTATION OF INVARIANT TORI AND THEIR WHISKERS IN QUASI-PERIODIC MAPS: NUMERICAL ALGORITHMS

À. HARO
Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
Gran Via de les Corts Catalanes 585
08007 Barcelona (Spain)

R. DE LA LLAVE
Department of Mathematics
University of Texas at Austin
Austin TX 78712 (USA)

(Communicated by Angel Jorba)

Abstract. In this paper we develop several numerical algorithms for the computation of invariant manifolds in quasi-periodically forced systems. The invariant manifolds we consider are invariant tori and the asymptotic invariant manifolds (whiskers) to these tori.

The algorithms are based on the parameterization method described in [36], where some rigorous results are proved. In this paper, we concentrate on numerical issues of algorithms. Examples of implementations appear in the companion paper [34].

The algorithms for invariant tori are based essentially on Newton method, but taking advantage of dynamical properties of the torus, such as hyperbolicity or reducibility as well as geometric properties.

The algorithms for whiskers are based on power-matching expansions of the parameterizations. Whiskers include as particular cases the usual (strong) stable and (strong) unstable manifolds, and also, in some cases, the slow manifolds which dominate the asymptotic behavior of solutions converging to the torus.

1. Introduction. This is the second in a series of three papers concerned with the computation of invariant tori and their whiskers in quasiperiodically forced systems. In the first paper [36], we proved rigorous theorems which establish existence, regularity and a posteriori estimates. In this paper, we aim to present algorithms and discuss running times and storage requirements. In the third paper [34], we consider actual implementations of the algorithms in concrete examples, and the improved accuracy allows us to formulate very precise conjectures (some of them were announced in [35]). Even if the subject matter of the papers is the same – invariant tori and their whiskers in quasi-periodic systems – the point of view and the problems considered are very different, hence, the overlap is rather minimal and we have strived to make the papers readable independently of each other.

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Integrable nonholonomic geodesic flows on compact Lie groups

Yuri N. Fedorov
Department of Mathematics and Mechanics
Moscow Lomonosov University, Moscow, 119 899, Russia
e-mail: fedorov@mech.math.msu.su

and

Departament de Matemàtica I,
Universitat Politecnica de Catalunya, Barcelona, E-08028 Spain
e-mail: Yuri.Fedorov@upc.es

and

Božidar Jovanović
Mathematical Institute, SANU
Kneza Mihaila 35, 11000, Belgrade, Serbia
e-mail: bozaj@mi.sanu.ac.yu

August 24, 2004

Abstract
This paper is a review of recent results on integrable nonholonomic geodesic flows of left-invariant metrics and left- and right-invariant constraint distributions on compact Lie groups.

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