On the number of bases of bicircular matroids

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Abstract

Let $t(G)$ be the number of spanning trees of a connected graph G, and let $b(G)$ be the number of bases of the bicircular matroid $B(G)$. In this paper we obtain bounds relating $b(G)$ and $t(G)$, and study in detail the case where G is a complete graph K_n or a complete bipartite graph $K_{n,m}$.

1 Introduction and preliminaries

Given a connected graph G, let $M(G)$ and $B(G)$ be the cycle matroid and the bicircular matroid, respectively, defined on the edge set of G (precise definitions are given below). Let $t(G)$ be the number of spanning trees of G, which is the number of bases of $M(G)$, and let $b(G)$ be the number of bases of $B(G)$, which correspond to spanning subgraphs of G in which every connected component has a unique cycle.

The quantity $t(G)$ has been widely studied (see [3], among many references), but little is known about $b(G)$. The aim of this paper is to study the number $b(G)$ and to compare it to $t(G)$, both for arbitrary graphs and for some specific families of graphs. In Section 2 we obtain bounds relating $t(G)$ and $b(G)$. For any connected graph G with n vertices and m edges, we show that

$$t(G) \leq \frac{n}{m-n+1}\, b(G).$$

If in addition G is simple we show that

$$b(G) \leq 2.42^n\, t(G).$$

By means of examples we show that $b(G)$ can be exponentially larger than $t(G)$, so that the above bound is qualitatively of the right order. We also show that no bound like $b(G) \leq C^n\, t(G)$ with $C > 0$ can hold for the class of general non-simple graphs.

In Sections 3 and 4 we study the number $b(G)$ when G is a complete graph K_n or a complete bipartite graph $K_{n,m}$. Using generating functions and singularity analysis, we show that $b(K_n)$ is asymptotically $\gamma n^{n-1/4}$ for a constant γ whose approximate value is 0.274618. We also show that, for fixed m, $b(K_{n,m}) = p_m(n)m^n$, where $p_m(n)$ is a polynomial in n of degree m. This solves, in particular, a problem left open in [4].

Next we state several preliminaries needed in the paper. The cycle matroid $M(G)$ is well known; if G is connected its bases are the edge sets of spanning trees of G. The bicircular matroid $B(G)$ has as bases the edge sets of spanning subgraphs of G in which every connected

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component has a unique cycle. See [2] for additional properties of bicircular matroids, including the definition in terms of circuits, and [7] for general background on matroids.

If e is an element of a matroid M, then $M-e$ and M/e denote the matroids obtained from M by deleting and contracting e, respectively. If e is neither a loop nor a coloop, then the number of bases of M is equal to the number of bases of $M-e$ (those not containing e) plus the number of bases of M/e (in bijection with those containing e). It follows that if e is an edge of a connected graph G which is neither a loop nor a bridge (an edge whose deletion disconnects G), then

$$t(G) = t(G-e) + t(G/e).$$

If also follows that for any edge e we have

$$b(G) \leq b(G-e) + b(G/e).$$

Now we summarize the enumerative tools needed in Sections 3 and 4. We follow the powerful approach developed by Flajolet and Sedgewick in their forthcoming book "Analytic combinatorics" [1]. A class $\mathcal{A}$ of labelled combinatorial objects has an associated exponential generating function (EGF)

$$A(z) = \sum_{n\geq 0} A_n \frac{z^n}{n!},$$

where A_n is the number of objects in $\mathcal{A}$ of size n (in our case the objects are labelled graphs and the size is the number of vertices). Combinatorial operations on classes have a direct counterpart on EGFs, as summarized in the table below:

Construction		Operation on EGF
Disjoint union	$\mathcal{C} = \mathcal{A} \cup \mathcal{B}$	$C(z) = A(z) + B(z)$
Labelled product	$\mathcal{C} = \mathcal{A} * \mathcal{B}$	$C(z) = A(z)B(z)$
Set	$\mathcal{B} = \Pi\{\mathcal{A}\}$	$B(z) = \exp(A(z))$
Cycle	$\mathcal{B} = \mathcal{C}\{\mathcal{A}\}$	$B(z) = \log \dfrac{1}{1-A(z)}$

Here $\Pi\{\mathcal{A}\}$ is the class of sets of objects taken from $\mathcal{A}$, and $C\{\mathcal{A}\}$ is the class of objects taken from $\mathcal{A}$ arranged in cyclic order.

Another ingredient we need is Lagrange's inversion theorem that we now state.

Lagrange inversion. *Let $\phi(u)$ be a formal power series with $\phi(0) \neq 0$, and let $Y(z)$ be the unique formal power series solution of the equation $Y = z\phi(Y)$. Then the coefficient of z in $\psi(Y)$, for an arbitrary series ψ, is given by*

$$[z^n]\psi(Y(z)) = \frac{1}{n}[u^{n-1}]\phi(u)^n \psi'(u).$$

Finally, we also need to apply the method of *singularity analysis* to obtain asymptotic estimates of the coefficients of a generating function. In particular, the analysis of singularities of a composition of series as described in Chapter 5 of [1]. Let $f(z)$ and $a(z)$ be two series with non-negative coefficients, and such that 1 is a dominant singularity of $f(z)$, that is, a singularity of minimum modulus. In order to estimate the coefficients of $f(a(z))$ we compute $a(R) = \lim_{x\to R^-} a(x)$, where R is the radius of convergence of $a(z)$. If $a(R) = 1$ we are in the *critical case*. This is often the case when $a(z)$ is defined by means of a Lagrange equation $a(z) = z\phi(a(z))$. Our generating functions in Section 3 follows this pattern, hence we can apply the approach described in [1, Section 5.7].

2 Bounds

In this section $G = (V, E)$ is a connected graph with n vertices and m edges. Recall that $b(G)$ is the number of bases of the bicircular matroid $B(G)$ and $t(G)$ is the number of spanning trees of G. We begin with a simple bound relating $b(G)$ and $t(G)$.

Theorem 2.1. *For a connected graph G, we have*

$$b(G) \geq \frac{m-n+1}{n}\, t(G).$$

Proof. Let $\mathcal{U}$ be the set of pairs (U, e), where U is a connected spanning unicyclic subgraph of G and $e \in E$ is an edge belonging to the unique cycle of U, and let $\mathcal{T}$ be the set of pairs (T, f), where T is a spanning tree of G and $f \in E$ is an edge not in T. Since $U \setminus \{e\}$ is a spanning tree and $T \cup \{f\}$ is a spanning unicyclic subgraph whose unique cycle contains f, it follows that $\mathcal{U}$ and $\mathcal{T}$ have the same cardinality.

But clearly $|\mathcal{T}| = (m-n+1)t(G)$; and $|\mathcal{U}| \leq b(G)n$, since any U is, in particular, a basis of $B(G)$, and the unique cycle of U has at most n edges. The bound then follows directly. □

We remark that Oxley [6] proved the inequality $t(G) \geq \frac{6n}{19} c(G)$, where $c(G)$ is the number of cycles of G (we are grateful to an anonymous referee for pointing out this fact).

Our next result gives an upper bound for $b(G)$ in terms of $t(G)$ when G is simple.

Theorem 2.2. *If G is a connected simple graph, then $b(G) \leq 2.42^n\, t(G)$.*

Proof. The proof is by induction on the number of edges, using the contraction and deletion rules stated in the first section. However, since contraction of an edge can produce loops and multiple edges giving rise to a non-simple graph, we need to be careful with the induction.

We say that an edge of a connected graph is *ordinary* if it is neither a loop nor a bridge. We say that G is an *ordinary minor* of G' if G is obtained from G' by a sequence of deletions and contractions of ordinary edges, and then we write $G \prec G'$. Now define the class $\mathcal{H}_n$ as

$$\mathcal{H}_n = \{G \colon G \prec K_n\}.$$

Notice that every graph in $\mathcal{H}_n$ is connected. Since $\mathcal{H}_n$ contains in particular all simple connected graphs on n vertices, it is enough to prove the following claim.

Claim. For every $G \in \mathcal{H}_n$, we have $b(G) \leq 2.42^n\, t(G)$.

Assume first G has an ordinary edge e. Then $G - e$ and G/e are both in $\mathcal{H}_n$ and, by induction on the number of edges,

$$b(G) \leq b(G-e) + b(G/e) \leq 2.42^n\, t(G-e) + 2.42^n\, t(G/e) = 2.42^n\, t(G).$$

If G has no ordinary edges, then the only possible cycles of G are loops; this is because every edge in a cycle of length at least two is ordinary. Hence G is a tree without multiple edges and with a number of loops attached to the vertices. Let k be the number of vertices of G (observe that $k \leq n$), let $V(G) = \{v_1, \ldots, v_k\}$, and let $l_i \geq 0$ be the number of loops at vertex v_i.

¿From the definition of $\mathcal{H}_n$ it follows that G is constructed from K_n by partitioning $V(K_n) = V_1 \cup \cdots \cup V_k$, identifying all the vertices of V_i to a single vertex v_i, and removing edges until obtaining a tree with loops attached. If $d_i = |V_i|$, then after contraction there at most $\binom{d_i-1}{2}$ loops at v_i, those coming from edges joining vertices in V_i different from v_i, so that

$$2l_i \leq (d_i - 1)(d_i - 2).$$

On the other hand, it is easy to check that $d_i \geq 1$ implies

$$(d_i - 1)(d_i - 2) + 2 \leq (d_i + \sqrt{2} - 1)^2.$$

It is clear that $t(G) = 1$. A bicircular basis of G contains at most one loop at each vertex (two loops at the same vertex form a circuit of $B(G)$) plus a subset of the $k-1$ bridges of G. Hence

$$b(G) \leq (l_1 + 1) \cdots (l_k + 1) 2^k = \prod (2l_i + 2) \leq \left(\prod (d_i + \sqrt{2} - 1) \right)^2.$$

Since $\sum (d_i + \sqrt{2} - 1) = n + (\sqrt{2} - 1)k$, the last product is maximized when all the terms in the sum are equal to $(n + (\sqrt{2}-1)k)/k = \sqrt{2} - 1 + n/k$. If we set $\alpha = n/k$ (notice that $\alpha \geq 1$), then

$$b(G) \leq \left(\left(\sqrt{2} - 1 + \frac{n}{k} \right)^k \right)^2 = \left((\sqrt{2} - 1 + \alpha)^{2/\alpha} \right)^n.$$

A simple computation shows that the unique maximum of $(\sqrt{2} - 1 + \alpha)^{2/\alpha}$ is at $\alpha = 1.849537\ldots$ and substituting we get $b(G) \leq 2.419326^n \, t(G) \leq 2.42^n \, t(G)$ as claimed. □

The following example shows that $b(G)$ can be exponentially larger than $t(G)$. For $n = 3k$, let G_n the graph constructed as follows: take a path P_k with k vertices, and attach to each vertex of P_k a triangle. Then G_n has $n = 3k$ vertices. A spanning tree of G_n has to use necessarily all the edges of P_k, and for every triangle we have three choices for a spanning tree; hence $t(G_n) = 3^k = 3^{n/3}$. A lower bound on $b(G_n)$ is obtained as follows. Let i be the number of triangles fully contained in a basis of $B(G_n)$. For the remaining $k-i$ triangles choose a spanning tree in 3^{k-i} ways, then complete to a bicircular basis in any way. This construction gives

$$b(G_n) \geq \sum_{i=0}^{k} \binom{k}{i} 3^{k-i} = 4^k = (4/3)^{n/3} \, t(G_n).$$

The constant $(4/3)^{1/3} = 1.100642\ldots$ can be improved by attaching to every vertex of P_k a complete graph K_6 instead of a triangle K_3. If H_n denotes the resulting graph with $n = 6k$ vertices, computing the *exact* number of bicircular bases using generating functions we obtain $b(H_n) \sim \alpha^n \, t(H_n)$, where α is a computable algebraic number whose approximate value is 1.308716. We believe that both this constant and the constant 2.419326 in the upper bound can be improved, but since it seems unlikely to obtain matching bounds we leave the issue at this point.

The next example shows that no bound like $b(G) \leq C^n \, t(G)$ can hold if we remove the requirement of G being simple. For $n = 2k$ let H_n be the graph consisting of a path P_{2k} with $2k$ vertices in which the 1st, 3rd, 5th and every other edge is replaced by s edges in parallel. Notice that H_n has $n = 2k$ vertices, and clearly $t(H_n) = s^k = s^{n/2}$. On the other hand, taking two edges in each parallel class (and no other edges) gives a basis of $B(H_n)$. Hence

$$b(H_n) \geq \binom{s}{2}^k = \left(\frac{s-1}{2} \right)^{n/2} t(H_n),$$

and the basis of the exponential $\left(\frac{s-1}{2}\right)^{1/2}$ is unbounded as s gets large.

We conclude this section with a general upper bound on $b(G)$ not related to $t(G)$.

Theorem 2.3. *If $(d_1, \ldots, d_n)$ is the degree sequence of a graph G, then*

$$b(G) \leq d_1 \cdots d_n.$$

Proof. A mapping $f\colon [n] \to [n]$ can be represented as a direct graph $D(f)$ on the set of vertices $[n] = \{1, 2, \dots, n\}$ with an edge directed from x to $f(x)$ for every x. The graph obtained has the property that the outdegree of each vertex is equal to 1. It follows that $D(f)$ consists of a collection of rooted trees with edges directed towards the root and whose roots determine a collection of directed cycles. Hence the underlying indirected graph is a collection of unicyclic graphs, whose cycles can be of any length including one and two.

Take $\{1, 2, \dots, n\}$ as the vertex set of G, and consider mappings $f\colon [n] \to [n]$ with the condition that $f(x)$ must be adjacent to x. The number of such mappings is $d_1 \cdots d_n$, and by the previous argument every basis of $B(G)$ appears as the underlying graph of some such mapping. Hence the result follows. ☐

The above bound can be very crude (for instance if G is a cycle) but, as shown in Theorem 3.1, for complete graphs K_n is only $O(n^{1/4})$ off from the true value.

3 Complete graphs

In this section we study the numbers $b(K_n)$ of bases of complete graphs. Let

$$B(z) = \sum_{n \geq 0} b(K_n) \frac{z^n}{n!}$$

be the associated exponential generating function (EGF) and let $T(z)$ be the EGF of rooted labelled trees. As is well-known, $T(z)$ satisfies the Lagrange equation

$$T(z) = ze^{T(z)}.$$

Our next result is an expression for $B(z)$ in terms of $T(z)$ and a precise asymptotic estimate of $b(K_n)$ for large n.

Theorem 3.1. *If $B(z)$ is the EGF of the numbers $b(K_n)$, then*

$$B = \frac{1}{\sqrt{1-T}} \exp\left(-\frac{T}{2} - \frac{T^2}{4}\right).$$

Also, as $n \to \infty$, the following asymptotic estimate holds

$$b(K_n) \sim \gamma n^{n-1/4},$$

where

$$\gamma = \frac{(2e^{-3})^{1/4}\sqrt{\pi}}{\Gamma(1/4)} = 0.274618\dots$$

Proof. A *unicyclic* (labelled) graph is a connected graph with a unique cycle. We can think of a uniyclic graph as a cycle, of length at least three, with a rooted tree attached to each vertex of the cycle. According to the table given in the preliminaries, if $U(z)$ denotes the EGF of unicyclic graphs then

$$U(z) = \frac{1}{2}\left(\log\frac{1}{1-T} - T - \frac{T^2}{2}\right),$$

where the terms subtracted from the logarithm correspond to the fact that cycles of length one and two are not allowed, and the factor $1/2$ makes the unique cycle non-oriented. Now a basis of

$B(K_n)$ is an unordered set of unicyclic graphs; since the set construction corresponds to taking exponentials, $B(z) = \exp(U(z))$ and the first result follows.

For the second part we use singularity analysis. The series

$$\frac{1}{\sqrt{1-z}} \exp\left(-\frac{z}{2} - \frac{z^2}{4}\right)$$

has a unique dominant singularity at $z = 1$, and the singular expansion is

$$e^{-3/4}(1-z)^{-1/2} + O((1-z)^{1/2}).$$

On the other hand, the unique dominant singularity of $T(z)$ is at e^{-1} and $\lim_{z \to (e^{-1})^-} T(z) = 1$; hence we are in the critical case. The singular expansion of $T(z)$ at $z = e^{-1}$ is given by

$$T(z) = 1 - \sqrt{2}(1-ez)^{1/2} + O(1-ez).$$

Now take the first order approximation $1 - T(z) \sim \sqrt{2}(1-ez)^{1/2}$ and plug it into $B(z)$ obtaining the expansion

$$B(z) = e^{-3/4}2^{-1/4}(1-ez)^{-1/4} + O((1-ez)^{1/4}).$$

Using the transfer theorems of singularity analysis (see again [1, Chapter 5]) it is routine to deduce the estimate

$$b(K_n)/n! \sim (2e^3)^{-1/4} e^n \frac{n^{-3/4}}{\Gamma(1/4)}.$$

Finally, Stirling's approximation $n! \sim n^n e^{-n}\sqrt{2\pi n}$ gives the estimate for $b(K_n)$. □

Better approximations of $b(K_n)$ can be obtained at the expense of taking more terms in the singular expansions. Also, notice that since the number of spanning trees of K_n is well-known to be n^{n-2}, the ratio $b(K_n)/t(K_n)$ is of order $n^{7/4}$.

The above result answers a question posed in [4], where a rather involved formula for $b(K_n)$ is derived. Let us mention, however, that the entries in Table 2 from [4] are not correct after the 15th term. For instance, by expanding the series $B(z)$ above with a computer algebra system one obtains $b(K_{16}) = 1722832666898627865$, which differs from the value given in the reference.

Another problem considered in [4] is to count the number $u(K_n)$ of *connected* bases of $B(K_n)$, which correspond to spanning unicyclic subgraphs of K_n. The EGF of the numbers $u(K_n)$ is precisely the series $U(z)$ above, and a similar analysis shows that

$$u(K_n) \sim (\pi/8)^{1/2} n^{n-1/2}.$$

In fact, this result was known long ago (see Corollary 7.1.2 in [3] and the references therein). It follows from the fact that the number of unicyclic subgraphs of K_n whose unique cycle is of length k is equal to $\frac{1}{2}n^{n-k-1}n(n-1)\cdots(n-k+1)$. We remark that the values of $u(K_n)$ given in Table 1 of [4] are not correct from the 15th term on. At the end of the paper we tabulate the first 20 values of $b(K_n)$ and $u(K_n)$.

4 Complete bipartite graphs

Next we investigate the numbers $b(K_{n,m})$ of bases of complete bipartite graphs. Since we have two indices we need a bivariate EGF:

$$B(x,y) = \sum_{m,n \geq 0} b(K_{n,m}) \frac{x^n}{n!} \frac{y^m}{m!}.$$

We also need to introduce the bivariate EGF $t(x,y)$ defined by means of

$$t(x,y) = x\exp(ye^{t(x,y)}).$$

If we call the vertices in the two parts of the bipartition of $K_{m,n}$ black and white vertices, respectively, then $t(x,y)$ is the EGF for spanning trees of $K_{m,n}$ rooted at a black vertex. Similarly, $ye^{t(x,y)}$ is the EGF for spanning trees rooted at a white vertex, and

$$T(x,y) = t(x,y)ye^{t(x,y)}$$

is the EGF for ordered pairs (τ, τ') of spanning trees, τ being rooted at a black vertex and τ' at a white vertex.

Theorem 4.1. *If $B(x,y)$ is the EGF of the numbers $b(K_{n,m})$, then*

$$B = \frac{1}{\sqrt{1-T}} \exp(-T/2).$$

Also, for fixed m, $b(K_{n,m}) = p_m(n)m^n$, where $p_m(n)$ is a polynomial of degree m in n.

Proof. In a spanning unicyclic subgraph of $K_{n,m}$ the unique cycle is of even length, at least four, and black and white vertices alternate in the cycle. Hence we can think of a spanning unyciclic subgraph as a cycle (of length at least two) of ordered pairs of trees, rooted at a black and a white vertex respectively. From general principles it follows that

$$B(x,y) = \exp\left(\frac{1}{2}\left(\log\frac{1}{1-T} - T\right)\right),$$

where, as in the previous section, the term subtracted from the logarithm throws away cycles of length one and the 1/2 factor makes the cycle non-oriented. Hence the first claim is proved.

For the second part, notice that the equation defining $t(x,y)$ is in Lagrange form with respect to variable x. Let

$$\phi(t) = e^{ye^t}, \qquad \psi(t) = (1-tye^t)^{-1/2}e^{-tye^t/2}.$$

Then, by Lagrange inversion, we have

$$[x^n]B(x,y) = \frac{1}{n}[t^{n-1}]\phi(t)^n\psi'(t) = [t^{n-1}]\frac{y^2(t+t^2)e^{2t+nye^t-tye^t/2}}{2n(1-tye^t)^{3/2}}.$$

In order to obtain $b(K_{n,m})$ we have to extract the coefficient of y^mt^{n-1} from the expression above. A straightforward but tedious computation gives that the coefficient of y^m is

$$\frac{1}{2n}(t+t^2)e^{mt} \sum_{\substack{0\le m_i\le m-2\\ m_1+m_2+m_3=m-2}} \frac{(-1)^{m_3}t^{m_1+m_3}n^{m_2}(-1/2)^{m_1}\binom{-3/2}{m_3}}{m_1!m_2!}.$$

Next we need to extract the coefficient of t^{n-1} and multiply by $n!m!$. The contributions come from the terms $(t+t^2)e^{mt}$ and $t^{m_1+m_3}$ inside the summation and, after some manipulation, we obtain

$$b(K_{n,m}) = \sum_{\substack{0\le m_i\le m-2\\ m_1+m_2+m_3=m-2}} m^n n^{m_2-1}\left(K(m_1,m_2,m_3)\binom{n}{m-m_2} + L(m_1,m_2,m_3)\binom{n}{m-m_2+1}\right),$$

where $K(m_1,m_2,m_3)$ and $L(m_1,m_2,m_3)$ are expressions that depend only on m_1, m_2 and m_3. Taking into account that, for fixed k, $\binom{n}{k}$ is a polynomial in n of degree k, and grouping the terms in the summation that do not depend on n into a constant C, we obtain that $b(K_{n,m}) = Cm^nP(n)$, where P is a polynomial in n of degree $(m_2-1)+(m-m_2+1) = m$. □

For instance, we have that

$$b(K_{n,2}) = \frac{n(n-1)}{8}\, 2^n, \qquad b(K_{n,3}) = \frac{n(n-1)(4n+1)}{27}\, 3^n.$$

Since the number of spanning trees of $K_{n,m}$ is equal to $m^{n-1}n^{m-1}$, for fixed m the quotient $b(K_{n,m})/t(K_{n,m})$ is of order n. A table of values for $b(K_{n,m})$ is given below.

Rather more difficult seems to determine the asymptotic behavior of $b(K_{n,n})$ for large n. The EGF we should look at is $\sum_{n\geq 0} b(K_{n,n})z^n/n!^2$, which is the *diagonal* of $B(x,y)$. Although there exist methods for dealing with diagonals of bivariate GFs (see Section 6.3 in [8]), they do not apply in our case due to the complicated nature of the series $B(x,y)$. However, using Theorems 2.1 and 2.3 and the fact $t(K_{n,n}) = 2^{2n-2}$, at least we can say that

$$n^{2n-2}(n-1) \leq b(K_{n,n}) \leq n^{2n}.$$

Finally, we remark that [5] gives a complicated expression for $b(K_{n,m})$, and that the numerical values obtained there agree in this case with those in our table.

n	$b(K_n)$	$u(K_n)$
3	1	1
4	15	15
5	222	222
6	3670	3660
7	68820	68295
8	1456875	1436568
9	34506640	33779340
10	906073524	880107840
11	26154657270	25201854045
12	823808845585	787368574080
13	28129686128940	26667815195274
14	1035350305641990	973672928417280
15	40871383866109888	38132879409281475
16	1722832666898627865	1594927540549217280
17	77242791668604946560	70964911709203684440
18	3670690919234354407000	3347306760024413356032
19	184312149879830557190940	166855112441313024389625
20	9751080154504005703189791	8765006377126199463936000

Table 1: Numbers of bicircular bases and connected bicircular bases of K_n.

$n\backslash m$	2	3	4	5	6	7	8
2	1						
3	6	78					
4	24	612	8442				
5	80	3780	86280	1371040			
6	240	20250	735840	17476200	319899150		
7	672	98658	5556096	191908500	4907358540	102285464274	
8	1792	449064	38427648	1892030000	66236032800	1838097932496	43089882142552

Table 2: Numbers of bicircular bases of $K_{n,m}$.

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