Tutte uniqueness of line graphs

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Abstract

We prove that if a graph H has the same Tutte polynomial as the line graph of a d-regular, d-edge-connected graph, then H is the line graph of a d-regular graph. Using this result we prove that the line graph of a regular complete t-partite graph is uniquely determined by its Tutte polynomial. We prove the same result for the line graph of any complete bipartite graph.

1 Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E; we assume that G has no isolated vertices, but we allow loops and multiple edges. The rank of a subset A of E is defined as $r(A) := |V| - k(G|A)$, where $k(G|A)$ denotes the number of connected components of the spanning subgraph induced by A in G. The *Tutte polynomial* of G is given by

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}.$$

The Tutte polynomial contains a great deal of information about the graph G (see [2] for a useful survey). For instance, by evaluating the Tutte polynomial at given points of the plane (x, y) one can obtain the number of spanning trees, the number of forests and the number of acyclic orientations of G. Other specializations of the Tutte polynomial include the chromatic and the flow polynomials. In view of these results, a question that arises naturally is whether a graph is uniquely determined by the information contained in its Tutte polynomial. We say that two graphs G and H are *T-equivalent* if $T(G; x, y) = T(H; x, y)$. A graph G is *T-unique* if for any other graph H T-equivalent to G we have that $G \cong H$.

The T-uniqueness of several well-known families of graphs has been proved recently in [5, 6]. In this paper we focus on line graphs. In Section 2 we prove our main result, namely, that a graph T-equivalent to the line graph of

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a d-regular, d-edge-connected graph G is the line graph of a d-regular graph (if $d = 3$ we have to add the technical condition that G is triangle-free). In Section 3 we apply this theorem to prove the T-uniqueness of the line graphs of regular complete t-partite graphs. We also prove that the line graph of any complete bipartite graph is T-unique. We conclude the paper by constructing pairs of non-isomorphic T-equivalent line graphs.

The related problem of characterizing line graphs by means of the *characteristic* polynomial has received much attention in the literature (see, for example, Chapter 3 in [1]). In particular, it is known that $L(K_n)$ is determined by its characteristic polynomial except for $n = 8$; in this case there are three graphs cospectral but not isomorphic to $L(K_8)$. And $L(K_{n,n})$ is determined by its characteristic polynomial too except for $n = 4$; in this case there is one exceptional graph cospectral but not isomorphic to $L(K_{4,4})$.

Although the Tutte polynomial is defined for graphs that allow loops and parallel edges, in this paper we only consider simple graphs. In particular, a d-regular graph is a graph where each vertex has d distinct neighbours. Since the simplicity of a graph can be deduced from its Tutte polynomial (see Lemma 1.2 below), restricting to simple graphs does not affect T-uniqueness results.

We recall the definition of the line graph $L(G)$ of a graph G. It has as vertices the edges of G, and two vertices in $L(G)$ are adjacent if the corresponding edges in G are incident. The basic property we use of $L(G)$ is that it decomposes into edge-disjoint cliques, where a *clique* of order d, or a d-clique, is a subgraph isomorphic to K_d. Each of these cliques corresponds to the set of edges incident to a vertex. In fact, we have the following characterization of line graphs [4].

Theorem 1.1. *A graph G is a line graph if and only if the edges of G can be partitioned into cliques such that no vertex of G lies in more than two of the cliques. Furthermore, if $C_1, \dots, C_t$ are cliques of G such that every vertex belongs to exactly two of them (some of the cliques may be trivial), then G is the line graph of G_0, where G_0 has as vertices $\{c_1, \dots, c_t\}$ and an edge between c_i and c_j if C_i meets C_j.*

The following lemma summarizes the combinatorial information encoded by the Tutte polynomial that is relevant to this paper (see [6] for a proof of the less known statements). Recall that the *clique number* of a graph G is the maximum order of a clique of G.

Lemma 1.2. *Let $G = (V, E)$ be a 2-connected simple graph, and let H be a graph T-equivalent to G. Then H is a 2-connected simple graph with the same number of vertices and edges as G. Moreover, G and H share the following parameters.*

1. *The edge-connectivity λ.*

2. *The number of cliques of each size; in particular, the clique number.*

3. *The girth g and the number of cycles of shortest length.*

4. *The number of triangles and the number of cycles of length four with a chord.*

5. *For every i and j, the number of edge-sets A with $r(A) = i$ and $|A| = j$.*

2 Main result

We need the following result from [3] relating the connectivity of a graph and that of its line graph.

Lemma 2.1. *If a graph G is n-edge-connected, then its line-graph $L(G)$ is n-connected and $(2n-2)$-edge-connected.*

Next we prove the main result of the paper.

Theorem 2.2. *Let G be a d-regular d-edge-connected graph on n vertices, and assume that $d \geq 3$ and, if $d = 3$, then G is triangle-free. If a graph H is T-equivalent to $L(G)$ then $H = L(G_0)$, where G_0 is a d-regular connected graph on n vertices.*

Proof. Using Lemmas 1.2 and 2.1, we deduce that H is 2-connected, has $nd/2$ vertices and $nd(d-1)/2$ edges, and is $(2d-2)$-edge-connected. Then the minimum degree of H is at least $2d-2$ and, since the sum of the degrees of all vertices must be $nd(d-1)$, it follows that H is $(2d-2)$-regular.

Under the hypotheses, $L(G)$ has clique number d and has exactly n cliques of order d. We have excluded triangles in G when $d = 3$ since they also give rise to triangles in $L(G)$. Since H is T-equivalent to $L(G)$, we deduce that H has exactly n cliques of order d; let us denote them by $C_1, C_2, \ldots, C_n$. The key ingredient is the following claim.

Claim. For $i \neq j$, the cliques C_i and C_j meet in at most one vertex of H.

Proof. In $L(G)$ every edge belongs to either $d-1$ or $d-2$ triangles; we prove first that this also holds for H. For an edge $e \in E(L(G))$, let $t(e)$ be the number of triangles of $L(G)$ that contain e; similarly, for an edge $f \in E(H)$, denote by $t'(f)$ the number of triangles of H that contain f. Note that all cycles of length four with a chord consist of two triangles with a common edge. Lemma 1.2 implies the following two equalities.

$$\sum_{e \in E(L(G))} t(e) = \sum_{f \in E(H)} t'(f), \qquad \sum_{e \in E(L(G))} \binom{t(e)}{2} = \sum_{f \in E(H)} \binom{t'(f)}{2}.$$

If there exist two edges $f_1, f_2 \in E(H)$ such that $|t'(f_1) - t'(f_2)| > 1$ then, by elementary properties of binomial numbers,

$$\sum_{e \in E(L(G))} \binom{t(e)}{2} < \sum_{f \in E(H)} \binom{t'(f)}{2}.$$

Therefore $t'(f)$ is either $d-1$ or $d-2$ for all $f \in E(H)$.

Suppose now that two cliques C_i and C_j meet in a complete subgraph $M \cong K_p$ for some p with $2 \le p \le d-1$. If f is an edge of M, then $t'(f) \ge 2d-p-2$; since $t'(f) \le d-1$, we deduce that $p = d-1$. Hence $C_i \cup C_j$ induces a subgraph in H isomorphic to K_{d+1}^-, the complete graph K_{d+1} minus an edge. Observe that $L(G)$ contains no such subgraph. Since K_{d+1}^- is the only simple graph with rank d and size $\binom{d+1}{2}-1$, we can deduce from the knowledge of the Tutte polynomial that H contains no subgraph isomorphic to K_{d+1}^-. Therefore $C_i \cap C_j$ is either a vertex or the empty set. □

Since H has $n\binom{d}{2}$ edges, the previous claim implies that each edge of H belongs to exactly one of the d-cliques; since H is $(2d-2)$-regular, every vertex of H belongs to at most two of the cliques. Actually, since there are n cliques of order d and H has $nd/2$ vertices, each clique must intersect exactly d other cliques, and hence every vertex is in exactly two of the cliques. By Theorem 1.1, H is the line graph of a graph G_0 on n vertices, which is the intersection graph of the cliques $\{C_1, \ldots, C_n\}$. The graph G_0 is clearly d-regular, and it is connected since the line graph of a disconnected graph is not 2-connected.

□

We can in fact obtain more information about the graph G_0 in the above theorem.

Theorem 2.3. *With the hypothesis and notation as in Theorem 2.2, G_0 has the same number of triangles as G.*

Proof. We show how to deduce the number of triangles in a d-regular graph G' with n vertices from the knowledge of the Tutte polynomial of $L(G')$. Let $t_1(G')$ be the number of triangles in G'. Since triangles in $L(G')$ arise either from triangles in G' or from three adjacent edges in G', we have

$$t_1(L(G')) = t_1(G') + n\binom{d}{3}.$$

As we have seen in the previous proof, n and d can be deduced from $T(L(G'); x, y)$. By Lemma 1.2 we can also deduce the value of $t_1(L(G'))$, and so we know $t_1(G')$. □

3 Complete multipartite graphs

Using the previous results we prove that line graphs of regular complete multipartite graphs are T-unique. We begin with a simple application of Theorem 2.2 that covers the case of a complete graph.

Theorem 3.1. *The graph $L(K_p)$ is T-unique for $p \ge 3$.*

Proof. Let H be a graph T-equivalent to $L(K_p)$. If $p \geq 5$, the hypotheses of Theorem 2.2 hold, and therefore we know that H is the line graph of a $(p-1)$-regular graph with p vertices. Since the only such graph is K_p, the result follows.

The two cases remaining have to be treated separately. For $p = 3$, we have that $L(K_3) = K_3$. By Lemma 1.2, all cycles are T-unique. If $p = 4$, the line graph of K_4 is $K_{2,2,2}$; its T-uniqueness follows from Theorem 4.1 in [6], where it is proved that all complete multipartite graphs are T-unique, with the only exception of $K_{1,p}$. $\square$

Now take $G = K(p,t)$, the complete t-partite graph with parts of order $p \geq 2$, and suppose H is a graph T-equivalent to $L(G)$. The graph $K(p,t)$ is $(tp-p)$-regular and $(tp-p)$-edge-connected. If $p = t = 2$, then $L(K(2,2)) = C_4$, a cycle of length four; as we have seen above, this graph is T-unique. For all the remaining cases the hypotheses of Theorem 2.2 hold. Then we know that $H = L(G_0)$, where G_0 is a $(tp-p)$-regular graph with tp vertices. By Theorem 2.3, we also know that G_0 and $G = K(p,t)$ have the same number of triangles, namely $p^3\binom{t}{3}$. Then it is enough to prove the following extremal result.

Lemma 3.2. *If G is a $(tp-p)$-regular graph with tp vertices, then G has at least $p^3\binom{t}{3}$ triangles. Moreover, equality holds if and only if $G = K(p,t)$.*

Proof. Let $N(x)$ denote the set of vertices adjacent to a vertex x. For every edge $e \in E(G)$, label its ends arbitrarily as x and y, and define the following quantities:

$$\begin{aligned}\alpha(e) &= |N(x) \cap (V \backslash N(y))|,\\ \beta(e) &= |N(x) \cap N(y)|,\\ \gamma(e) &= |(V \backslash N(x)) \cap (V \backslash N(y))|.\end{aligned}$$

Then, from the equalities

$$\alpha(e) + \beta(e) = d_G(x) = tp - p \quad \text{and} \quad \alpha(e) + \gamma(e) = tp - d_G(y) = p,$$

it follows that $\beta(e) = tp - 2p + \gamma(e)$. Since $\beta(e)$ equals the number of triangles that contain e, the total number of triangles in G is

$$\frac{1}{3}\sum_{e \in E(G)} (tp - 2p + \gamma(e)) \;\geq\; \frac{1}{3}|E(G)|\,(tp-2p) = p^3\binom{t}{3}.$$

This proves the first part of the claim.

If we have an equality, then $\gamma(e) = 0$ for all $e \in E(G)$. This means that given any edge xy and a third vertex z, either x or y is adjacent to z.

Let s be the chromatic number of G, and let $V = V_1 \cup \cdots \cup V_s$ be a partition of the vertex set of G into s stable subsets. From the regularity of G it follows that there are at least $tp - p$ vertices outside each part; therefore, $|V_i| \leq p$ for all i, and $s \geq t$.

Let us prove that G is complete s-partite. By definition of the chromatic number, for any two different parts V_i and V_j there exists an edge xy with $x \in V_i$

and $y \in V_j$. Let z be any vertex in $V_j \backslash y$; since $\gamma(xy) = 0$ and y and z are in the same part, it follows that xz is an edge. Repeated application of this argument shows that any pair of vertices in different parts are adjacent.

Since G is complete s-partite, $(tp - p)$-regular, and has tp vertices, it follows that $s = t$ and $G = K(p, t)$. $\square$

Corollary 3.3. *The graph $L(K(p, t))$ is T-unique for $t \geq 2$.*

Our next goal is to prove the T-uniqueness of $L(K_{p,q})$, a case not covered by Theorem 2.2 since $K_{p,q}$ is not regular. For this we need a simple combinatorial lemma.

Lemma 3.4. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of a positive integer n with $2 \leq k < n$ and $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$. Let μ be the conjugate partition of λ, that is, $\mu_j = |\{i \colon \lambda_i \geq j\}|$. Then*

$$\sum_i \binom{\lambda_i}{2} + \sum_j \binom{\mu_j}{2} \leq \binom{n-1}{2} + 1.$$

Proof. By induction on n, starting with the trivial case $n = 2$. Assume the claim holds for n, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of $n + 1$ into at least two parts, and let $\mu = (\mu_1, \ldots, \mu_l)$ be the conjugate partition of λ.

Define an auxiliary partition of n as $\lambda' = (\lambda_1, \lambda_2, \ldots, \lambda_k - 1)$, where the last term is omitted if $\lambda_k = 1$ (if $\lambda = (n, 1)$ then λ' would have only one part, but the claim in this case can be checked directly). Letting $h = \lambda_k$, the conjugate partition of λ' is $\mu' = (\mu_1, \ldots, \mu_{h-1}, \mu_h - 1, \mu_{h+1}, \ldots, \mu_l)$. Then, by induction hypothesis,

$$\begin{aligned} \sum_i \binom{\lambda_i}{2} + \sum_j \binom{\mu_j}{2} &= \sum_i \binom{\lambda'_i}{2} + \sum_j \binom{\mu'_j}{2} + (\lambda_k - 1) + (\mu_h - 1) \leq \\ &\binom{n-1}{2} + \lambda_k + \mu_h - 1 \leq \binom{n}{2} + 1, \end{aligned}$$

the last inequality because $\lambda_k + \mu_h \leq \lambda_k + k \leq n + 1$. $\square$

Theorem 3.5. *The graph $L(K_{p,q})$ is T-unique for all $p \leq q$.*

Proof. The case $p = q$ is covered by Corollary 3.3, so let us assume $p < q$. Suppose H is T-equivalent to $L(K_{p,q})$. Then, by Lemma 1.2, H is 2-connected with pq vertices and $p\binom{q}{2} + q\binom{p}{2}$ edges. Since $L(K_{p,q})$ is $(p + q - 2)$-edge-connected, so is H. Then the minimum degree of H is at least $p + q - 2$ and, as a consequence, H is $(p + q - 2)$-regular.

Since $L(K_{p,q})$ has clique number q and has exactly p cliques of order q, so does H. Let $C_1, \ldots, C_p$ be the cliques of order q in H.

Claim. The cliques C_i are vertex disjoint.

Proof. It cannot be that $V(C_i) \cap V(C_j) = \{x\}$, since then the degree of x in H would be at least $2q-2 > p+q-2$. Suppose $|V(C_i) \cap V(C_j)| > 1$ and let $x \in V(C_j) \backslash V(C_i)$. Then the vertex set $C_i \cup \{x\}$ induces a subgraph in H of rank q and size at least $\binom{q}{2}+2$. We now prove that such a subgraph does not exist in $L(K_{p,q})$. This will give a contradiction because H and $L(K_{p,q})$ are T-equivalent.

Let $A \subseteq E(L(K_{p,q}))$ be of rank q and let us bound the size of A. The graph $L(K_{p,q})$ can be thought of as a $p \times q$ grid in which two vertices in the same row or column are adjacent. Let $R_1, \ldots, R_p$ be the vertex sets corresponding to the rows, and let S_i be the vertices in R_i incident to an edge of A; we allow some of the S_i to be empty. In order to maximize $|A|$ we may assume that the vertices of S_i are left-justified, that is, they form a consecutive set of vertices in R_i starting in the first column; this means that A is connected and thus $|S_1| + \cdots + |S_p| = q+1$.

Assume without loss of generality that $|S_1| \geq |S_2| \geq \cdots \geq |S_p|$ (note that $L(K_{p,q})$ is not affected by permuting the rows). Since $|R_i| = q$ and $|A| = q+1$, the set S_2 is nonempty. If we set $\lambda_i = |S_i|$, then $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of $q+1$, where k is the largest index for which S_i is not empty; clearly, $k \leq p < q$, and hence the hypotheses of Lemma 3.4 hold. If μ is the conjugate partition of λ, then the size of A is equal to

$$\sum_i \binom{\lambda_i}{2} + \sum_j \binom{\mu_j}{2}.$$

By the previous lemma this quantity is at most $\binom{q}{2}+1$, thus proving that there are no subgraphs in $L(K_{p,q})$ with rank q and size $\binom{q}{2}+2$. □

Since the cliques C_i are disjoint, $V(C_1) \cup \cdots \cup V(C_p)$ accounts for all the pq vertices in H. Each vertex is adjacent to the $q-1$ vertices in its own clique; since the degree is $p+q-2$, it must be adjacent to other $p-1$ vertices. If a vertex $x \in V(C_i)$ were adjacent to $y, z \in V(C_j)$ with $j \neq i$, then $C_j \cup \{x\}$ would induce a subgraph of rank q and size $\binom{q}{2}+2$, which we just proved is not possible. Summarizing, each vertex is adjacent to exactly one vertex in each clique different from its own.

Finally, let us consider the number of p-cliques in H. It must be the same as in $L(K_{p,q})$, that is, $p\binom{q}{p}+q$. The number of p-cliques contained in the q-cliques C_i is $p\binom{q}{p}$; hence there must exist q additional ones. The fact that there are no edge-sets with rank q and size $\binom{q}{2}+2$ implies that each p-clique intersects each q-clique in one vertex. This combined with the conclusion of the last paragraph shows that H has the structure of a $p \times q$ grid as defined above, and this finishes the proof. □

4 Concluding Remarks

The results above might lead us to conjecture that if two line graphs $L(G_1)$ and $L(G_2)$ have the same Tutte polynomial, then $L(G_1) \cong L(G_2)$ (this is equivalent to $G_1 \cong G_2$, except if G_1 is a triangle and G_2 a star $K_{1,3}$ [4, Theorem 8.3]). However, using a construction of Tutte [7] we can provide examples of pairs of non-isomorphic T-equivalent line graphs, which can even be chosen to arise from d-regular graphs.

A graph R is called a *rotor of order* n if there is a subset of the vertices $\{x_1, \ldots, x_n\} \subseteq V(R)$ and an automorphism φ of R such that $\varphi(x_i) = x_{i+1}$ for all i, where the indices are modulo n. The set $\{x_1, \ldots, x_n\}$ is called the *border* of R. We have the following theorem from [7].

Theorem 4.1. *Let R be a rotor of order n with $3 \leq n \leq 5$, and let S be a graph with n selected vertices $\{y_1, \ldots, y_n\} \subseteq V(S)$. Let G be the graph formed from $R \cup S$ by identifying x_i with y_i for all $1 \leq i \leq n$. Similarly, let H be the graph formed from $R \cup S$ by identifying x_{n-i+1} with y_i for all $1 \leq i \leq n$. Then G and H have the same Tutte polynomial.*

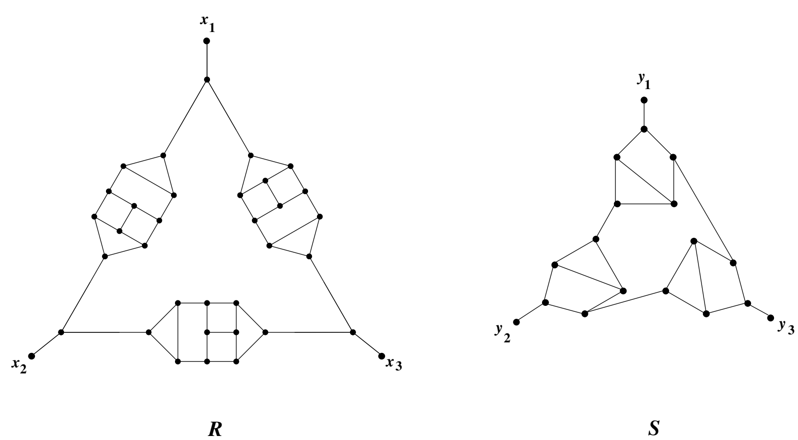

Figure 1: The line graphs of R and S are used by the rotor construction to produce T-equivalent line graphs.

We show next how to use this construction to produce pairs of non-isomorphic T-equivalent line graphs. Take as R any rotor of order n with $3 \leq n \leq 5$ such that the vertices $\{x_1, \ldots, x_n\}$ have degree one; denote by e_i the only edge incident with x_i. Let R' be the line graph $L(R)$; R' is a rotor of the same order as R whose border are the vertices corresponding to the edges $\{e_1, \ldots, e_n\}$; moreover, each of these vertices belongs to only one non-trivial clique of $L(R)$. Similarly, take as S any graph with a selected set of vertices $\{y_1, y_2, \ldots, y_n\}$ such that y_i is incident with only one edge f_i. Denote by S' the line graph of S. The vertices of S' that correspond to the edges $\{f_1, \ldots, f_n\}$ also belong to only one clique in S'. By Theorem 1.1, the graphs G and H that are obtained from Theorem 4.1 with the rotors R' and S' are line graphs, since their edges can be partitioned into cliques and every vertex belongs to at most two of the cliques. By choosing conveniently the rotor R and the graph S so that G and H are not isomorphic, we obtain a pair of non-isomorphic T-equivalent line graphs.

Furthermore, if we choose R and S such that all vertices except $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ have degree d, then G and H are the line graphs of a pair of d-regular graphs (see Figure 1 for an example of such R and S). We have thus proved the following corollary.

Corollary 4.2. *There exist pairs of non-isomorphic T-equivalent line graphs $L(G)$ and $L(H)$ such that G and H are d-regular.*

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