Transversal matroids

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Let's Matroid CUSO doctoral school, Neuchâtel, March 25–27, 2015

What's ahead

- Definition, first properties, and examples
- Further properties of transversal matroids
- Characterizations of transversal matroids
- Other topics

Definition, first properties, and examples

Keywords: set systems, partial transversals, transversal matroids, matrix representations, lattice path matroids, bicircular matroids

Transversals of set systems

A set system is a collection $\mathcal{A}$ of subsets of a set S:

$$\mathcal{A} = (A_j : j \in J)$$

(both S and J finite)

Ex $S = [9]$, $\mathcal{A} = (A, B, C, D)$:

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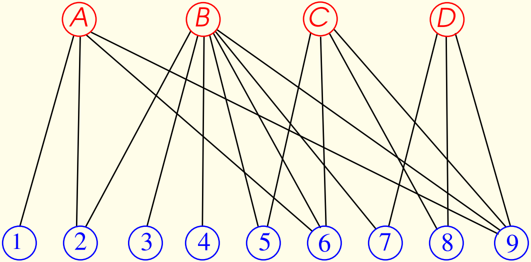
Set systems are in bijection with bipartite graphs with stable parts J and S



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Def A partial transversal of $\mathcal{A} = (A_j : j \in J)$ is a subset T of S such that there is an injective map $\varphi : T \to J$ such that

$$t \in A_{\varphi(t)} \text{ for all } t \in T$$

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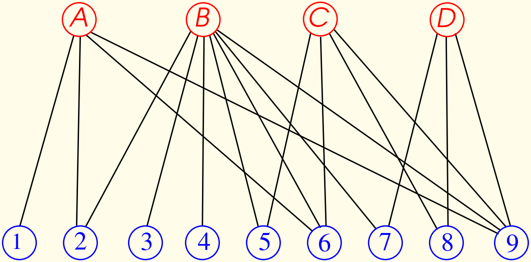

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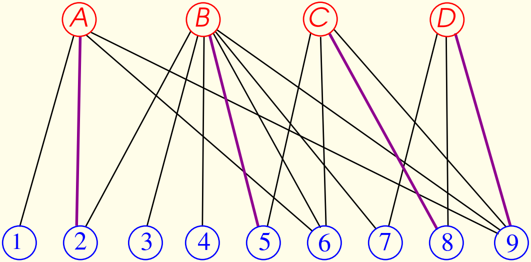

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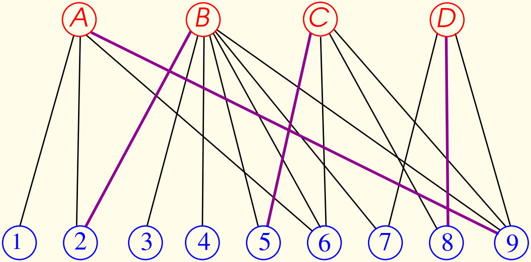

$\{2, 5, 8, 9\}$ is a partial transversal that corresponds to several matchings

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Proof: it will become transparent once we introduce another view on set systems and transversals. But first let's look at some examples

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  A rank-3 matroid with 2 lines with a common point, aka the cycle matroid of K_4 minus an edge
- Transversal matroids are closed under deletion

The matrix perspective

Given a set system $(A_j, j \in J)$ on S consider a $|J| \times |S|$ matrix with entries

$$m_{i,j} = \begin{cases} 0 & \text{if } j \notin A_i \\ x_{ij} & \text{if } j \in A_i \end{cases}$$

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Ex $A = \{1\,2\,6\,9\}, B = \{2\,3\,4\,5\,6\,7\,9\}, C = \{5\,6\,8\,9\}, D = \{7\,8\,9\}$

$$\begin{pmatrix} x_{11} & x_{12} & 0 & 0 & 0 & x_{16} & 0 & 0 & x_{19} \\ 0 & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & 0 & x_{29} \\ 0 & 0 & 0 & 0 & x_{35} & x_{36} & 0 & x_{38} & x_{39} \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{47} & x_{48} & x_{49} \end{pmatrix}$$

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- As all of the x_{ij} are algebraically independent,
  T is a partial transversal $\Leftrightarrow$ the columns corresponding to T are linearly independent

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A stronger result is

Thm (Piff and Welsh 70)
Every transversal matroid is representable over all sufficiently large fields

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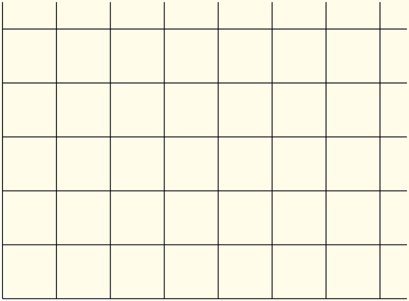
Morevoer, if there are no coloops, all presentations have exactly r sets

Example: lattice path matroids

Lattice paths start at the origin and have steps $E = (1, 0)$ and $N = (0, 1)$



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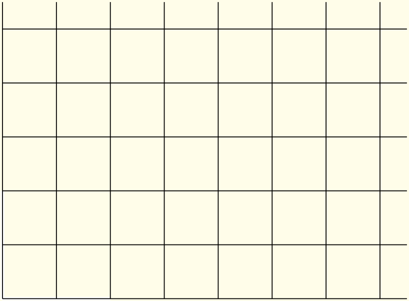

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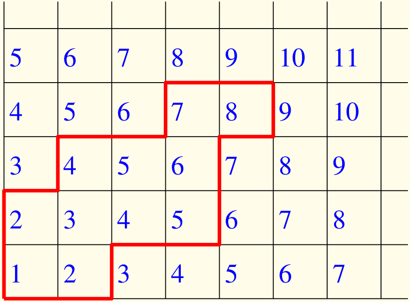

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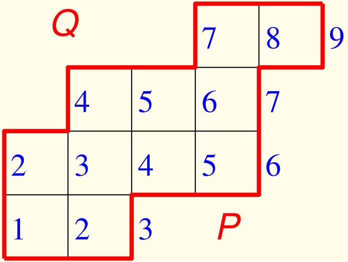

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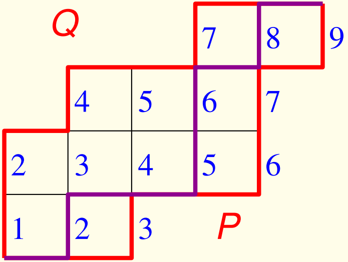

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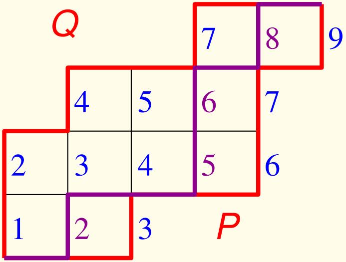

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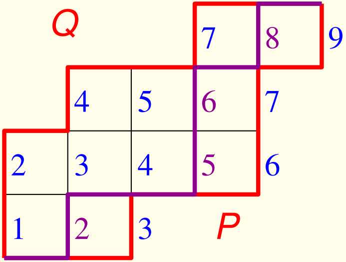

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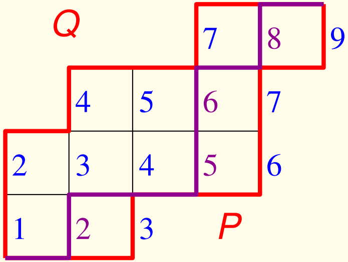

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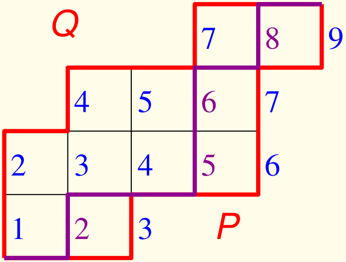

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Let N_i be the set of possible N steps at level i
Each path in $\mathcal{P}$ gives a transversal of $(N_1, \ldots, N_r)$

Example: lattice path matroids

Thm (Bonin, de Mier and Noy 2003)
Let P and Q be two non-crossing lattice paths ending at (m, r) and consider the transversal matroid $M[P, Q]$ with presentation $(N_1, \ldots, N_r)$
Then $M[P, Q]$ has rank r, $m + r$ elements and its bases are in bijection with lattice paths in $\mathcal{P}$

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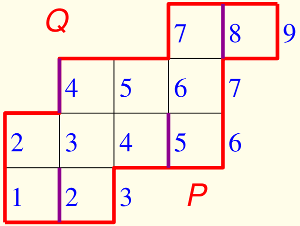

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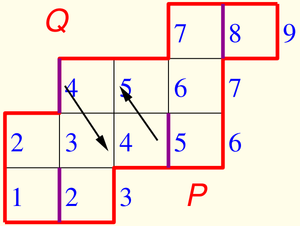

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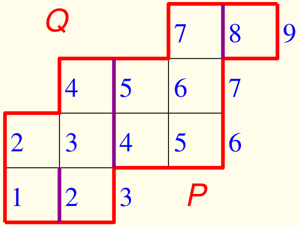

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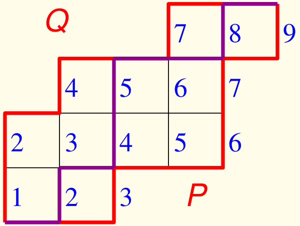

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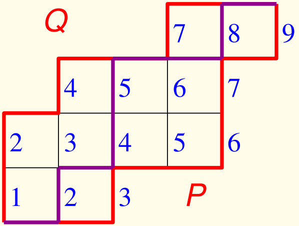

A lattice path matroid is a matroid isomorphic to $M[P, Q]$ for some P, Q

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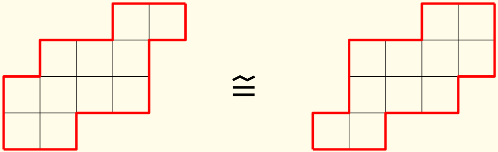

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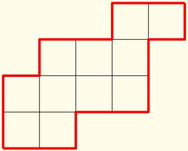

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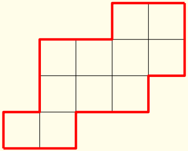

Thm (Bonin and de Mier 06)
The case described above is the only situation where different paths yield isomorphic lattice path matroids

Example: bicircular matroids

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For $v \in V$, let $A_v = \{e \in E : e \text{ is incident with } v\}$

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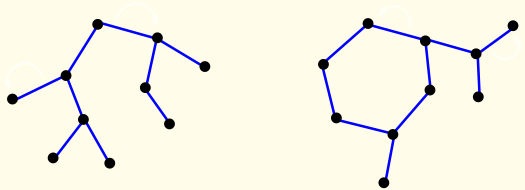

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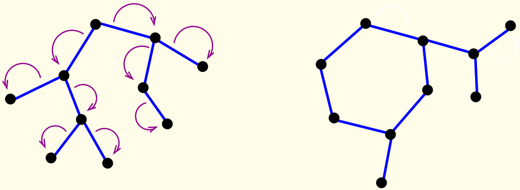

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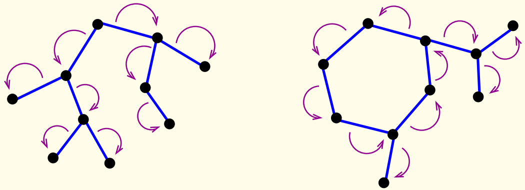

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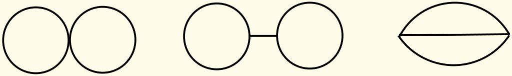

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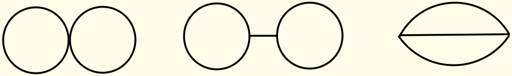

Note that bicircular matroids are loopless

Further properties of transversal matroids

Keywords: geometric representation, cyclic flats, contraction, duality, maximal and minimal presentations

Geometric representation of transversal matroids

Consider a matrix representation with non-negative entries and where all column sums are 1:

Ex $A = \{1\,2\,6\,9\}, B = \{2\,3\,4\,5\,6\,7\,9\}, C = \{5\,6\,8\,9\}, D = \{7\,8\,9\}$

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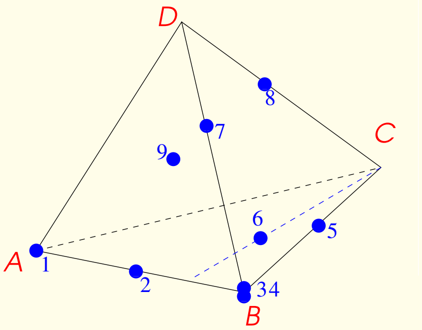

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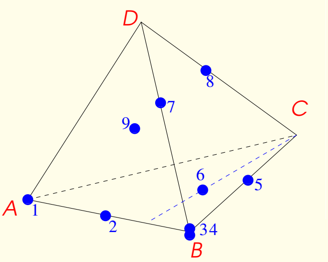

Independent sets in the matroid correspond to affinely independent sets of points

All affine relationships are dictated by the faces of the simplex

Points lie on the faces "as freely as possible"

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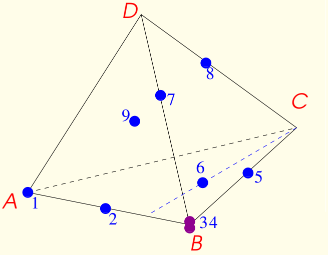

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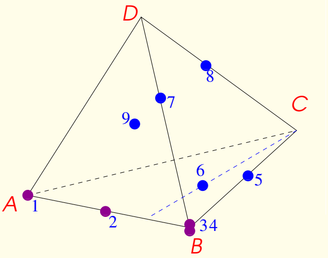

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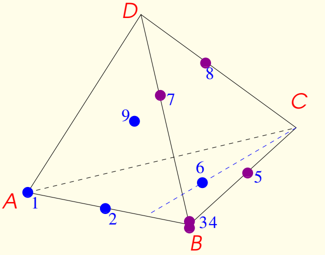

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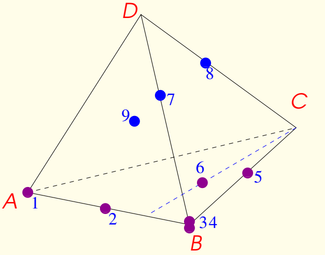

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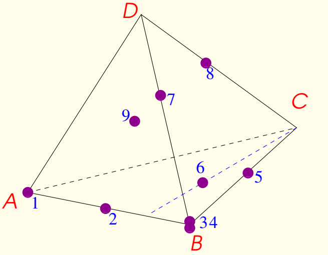

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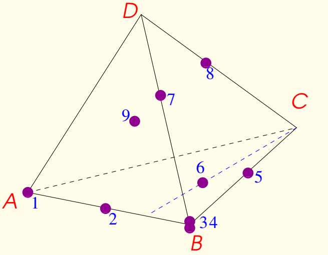

All rank-k cyclic flats lie on $(k-1)$-dimensional faces of the simplex

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It is straightforward to go from the set system presentation to the affine representation on the simplex:

- label vertices of the simplex by sets in the presentation
- place each element x in the span of the vertices corresponding to the sets that contain x

So, if we are given the simplex representation, to recover the sets we just need to look at the complements of the facets

Contractions and duality

- Transversal matroids are not closed under contractions

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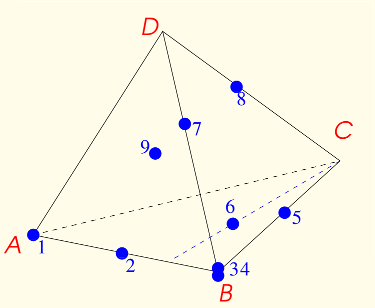
- Duals of transversal matroids are called cotransversal or strict gammoids. The smallest minor closed class containing transversal matroids is the class of gammoids

Presentations

Transversal matroids typically have several presentations



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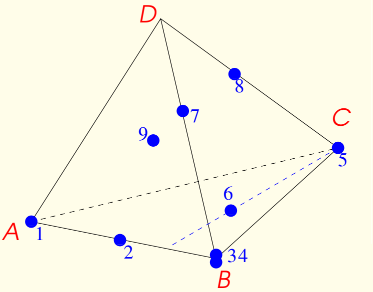

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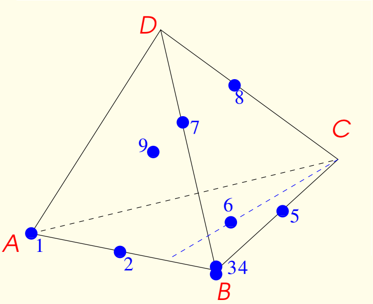

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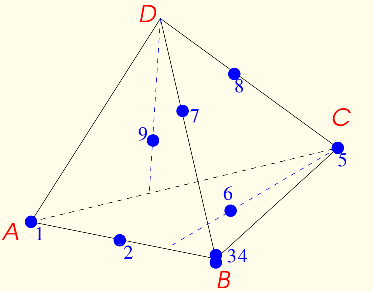

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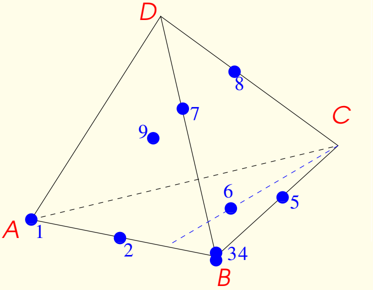

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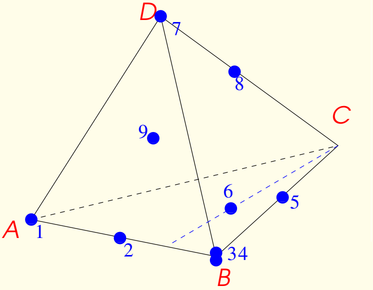

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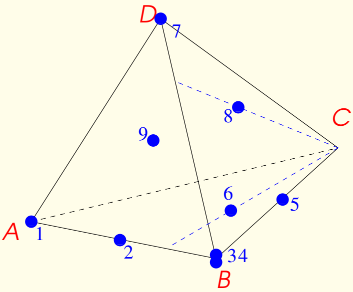

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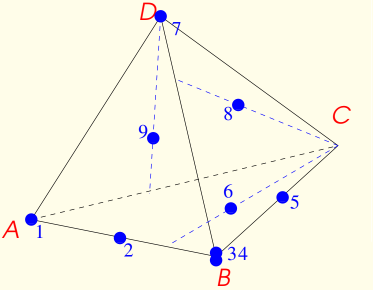

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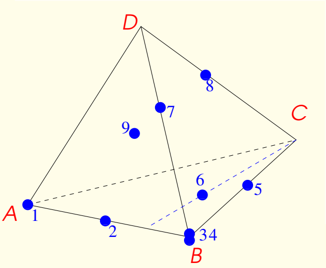

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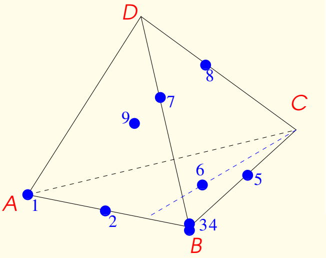

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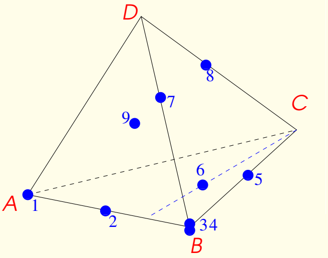

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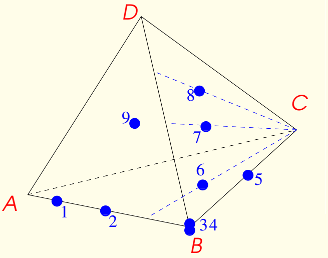

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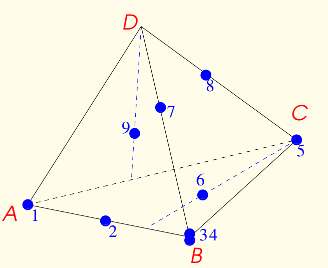

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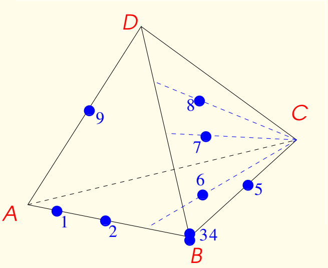

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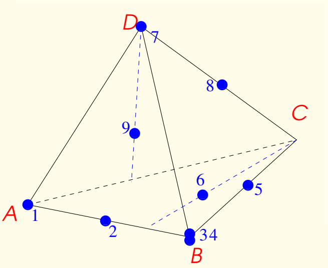

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- Contractions of bicircular matroids are bicircular, if loopless
- Bicircular matroids are not closed under duality
  For instance, $U_{2,n}$ is bicircular but $U_{n-2,n}$ is not if $n \geq 7$

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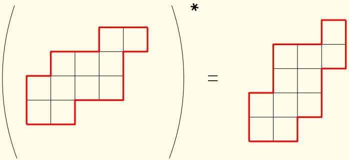

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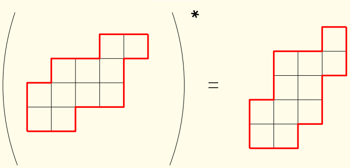

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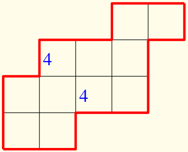

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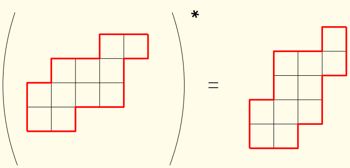

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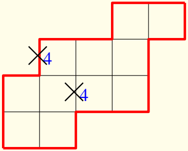

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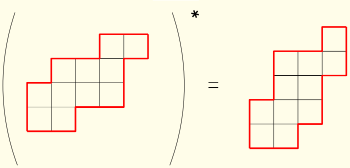

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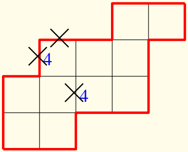

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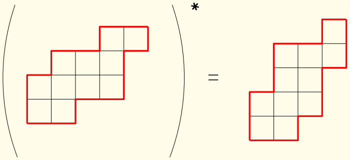

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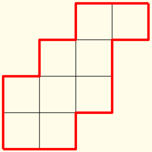

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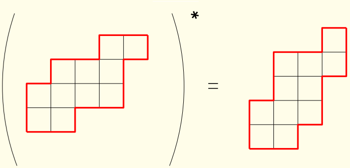
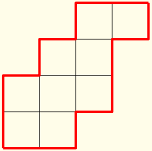


- Lattice path matroids are closed under deletion and contraction



- The excluded minors for the minor-closed class of lattice path matroids are known (Bonin 10)

Characterizations of transversal matroids

Keywords: fundamental transvesal matroids, Mason-Ingleton inequalities, the β function

The Mason-Ingleton characterization

How can one tell if a given matroid is transversal?

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Thm (Mason 71, Ingleton 77)
The following are equivalent

- M is transversal
- for all non-empty collections $\mathcal{F}$ of cyclic flats

$$r\left(\bigcap_{F\in\mathcal{F}} F\right) \leq \sum_{\mathcal{F}'\subseteq\mathcal{F}} (-1)^{|\mathcal{F}'|+1} r\left(\bigcup_{F'\in\mathcal{F}'} F'\right)$$

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A transversal matroid M is called fundamental transversal if there is a basis $B = \{b_1, \ldots, b_r\}$ of M such that in some simplex representation of M the elements $b_1, \ldots, b_r$ are placed on vertices

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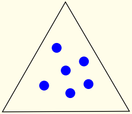

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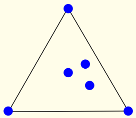

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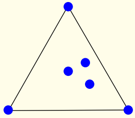

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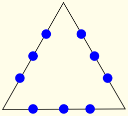

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Recall that a rank-k cyclic flat always lies on a $(k-1)$-dimensional face of the simplex

Thus, if F is a cyclic flat of a fundamental transversal matroid, then

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And similarly, for a collection $\mathcal{F}$ of cyclic flats,

$$r\left(\bigcup_{F \in \mathcal{F}} F\right) = \left|B \cap (\bigcup_{F \in \mathcal{F}} F)\right| = \left|\bigcup_{F \in \mathcal{F}} B \cap F\right|$$

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Thus, for a fundamental transversal matroid, the Principle of Inclusion and Exclusion gives

$$r\Big(\bigcap_{F\in\mathcal{F}} F\Big) = \sum_{\mathcal{F}'\subseteq\mathcal{F}} (-1)^{|\mathcal{F}'|+1} r\Big(\bigcup_{F'\in\mathcal{F}'} F'\Big)$$

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And for a collection $\mathcal{F}$ of cyclic flats of M,

$$r\left(\bigcup_{F\in\mathcal{F}} F\right) = r'\left(\bigcup_{F_1\in\mathcal{F}_1} F_1\right)$$
$$r\left(\bigcap_{F\in\mathcal{F}} F\right) \leq r'\left(\bigcap_{F_1\in\mathcal{F}_1} F_1\right)$$

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We have proved the $\Downarrow$ implication of

Thm (Mason 71, Ingleton 77)

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Let $\mathcal{Z}(M)$ be the set of all cyclic flats of M

Suppose we had a map $\beta : \mathcal{Z}(M) \to \{0, 1, 2, \ldots\}$ such that $\beta(F)$ is the number of times F^c appears in the maximal presentation of M

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Thus,

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Moreover, as each F must intersect $r(F)$ sets of the presentation,

$$\sum_{Y \in \mathcal{Z}, F \cap Y^c \neq \emptyset} \beta(Y) = r(F)$$

The β function

Substracting the previous two equations, the map β should satisfy

$$\sum_{Y \in \mathcal{Z}, F \subseteq Y} \beta(Y) = r(M) - r(F)$$

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For any matroid M, define the map $\beta : 2^S \to \mathbb{Z}$ on all subsets of elements as

$$\beta(X) = r(M) - r(X) - \sum_{Y \in \mathcal{Z}, X \subset Y} \beta(Y)$$

(Ingleton and Piff 72, Mason 71)

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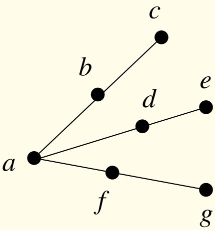
Lem If M satisfies the Mason-Ingleton inequalities, then $\beta(X) \geq 0$ for all X

The β function

$$\beta(S) = 0, \ \beta(X) = r(M) - r(X) - \sum_{Y \in \mathcal{Z}(M) : X \subset Y} \beta(Y)$$



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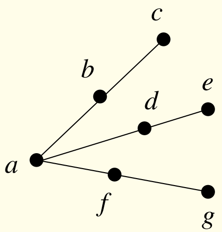

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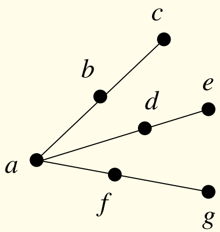

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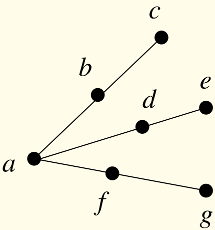

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$$\beta(\{a\}) = 3 - 1 - (1 + 1 + 1 + 0) = -1$$

The β function

It is not very difficult to check that if M is a matroid with $\beta(X) \geq 0$ for all $X \subseteq S$, then the set system

$$(F^c : F \in \mathcal{Z}, \beta(F) > 0)$$

is a presentation of the matroid M

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So all together:

Thm The following are equivalent

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- $\beta(X) \geq 0$ for all subsets X of the ground set

Other topics

Keywords: Tutte polynomials of lattice path matroids, transversal extensions, cyclic ordering conjecture

Tutte polynomials of lattice path matroids

Thm (Colbourn, Provan and Vertigan 95)
Computing $t(M; x, y)$ is $\#P$-complete for transversal matroids

Can we do better for lattice path matroids?

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The Tutte polynomial of a lattice path matroid can be computed in polynomial time

Tutte polynomials of lattice path matroids

Recall:

$$t(M;x,y) = \sum_{B \in \mathcal{B}(M)} x^{i(B)} y^{e(B)}$$

where $i(B), e(B)$ are the numbers of internally/externally active elements with respect to basis B and some linear order $<$ on the ground set

$x \notin B$ is externally active if
whenever $B - b \cup x$ is a basis we have $x < b$

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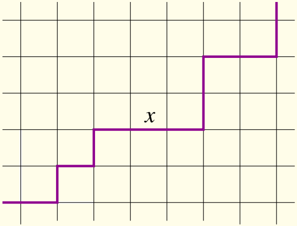
If B is a basis of a lattice path matroid, can one easily tell which elements are internally/externally active?

Tutte polynomials of lattice path matroids

Let M be a LPM of rank r and ground set $[m+r]$; take the linear order $1 < 2 < \cdots$

Let $B = \{b_1 < \cdots < b_n\}$ be a basis of a lattice path matroid and $x \notin B$; then $b_i < x < b_{i+1}$ for some i



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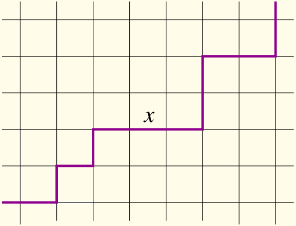

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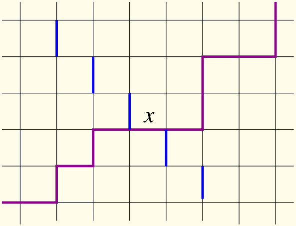

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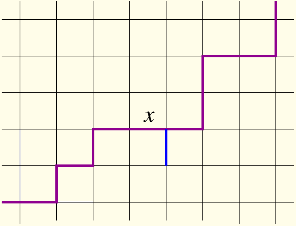

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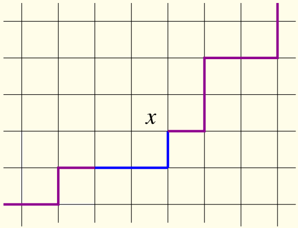

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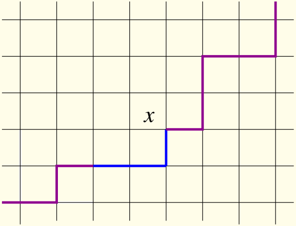

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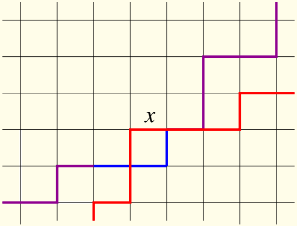

Can we modify the path to include x by removing one step from $b_1, \ldots, b_i$? If we cannot, x is externally active

We can use x in place of b_i, provided we are not in the lower path!

Tutte polynomials of lattice path matroids

Lem An element $x \notin B$ is externally active if the E step of B corresponding to x belongs to the lower bounding path
Dually, an element $b \in B$ is internally active if the N step of B corresponding to b belongs to the upper bounding path

Tutte polynomials of lattice path matroids

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In other words,

$$t(M[P, Q]; x, y) = \sum_{\pi \in \mathcal{P}} x^{Q_N(\pi)} y^{P_E(\pi)}$$

where $P_E(\pi)/Q_N(\pi)$ are the number of E/N steps of π in common with P/Q

Tutte polynomials of lattice path matroids

A byproduct:

As turning the paths 180 degrees around is a matroid isomorphism that switches the paths P and Q, and we should get the same Tutte polynomial no matter how we compute it, one gets that the pairs

$$(Q_N(\pi), P_E(\pi)) \text{ and } (P_N(\pi), Q_E(\pi))$$

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(For more on similar results, see Elizalde and Rubey 13+)

Tutte polynomials of lattice path matroids

Thm (Bonin, de Mier and Noy 03)
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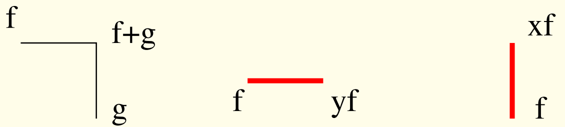
Tutte polynomials of lattice path matroids

Thm (Bonin, de Mier and Noy 03)
The Tutte polynomial of a lattice path matroid can be computed in polynomial time

- Label point $(0, 0)$ with 1
- Recursively, label each point in the region determined by the bounding paths according to



- The label of the point (m, r) is $t(M[P, Q]; x, y)$

Transversal extensions

A (single-element) extension of a matroid M on E is a matroid N on $E \cup x$ such that $M = N \backslash x$ (and, for us, $r(N) = r(M)$)

- The theory of extensions is well-understood: extensions of M are in bijection with some families of subsets called "modular cuts of flats" (Crapo 65)

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- If $(B_1, \ldots, B_r)$ is a presentation of N, then $(B_1 \backslash x, \ldots, B_r \backslash x)$ is a presentation of $M = N \backslash x$

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- The theory of extensions is well-understood: extensions of M are in bijection with some families of subsets called "modular cuts of flats" (Crapo 65)

A transversal extension of a transversal matroid M is an extension of M that is also transversal

- If $(B_1, \ldots, B_r)$ is a presentation of N, then $(B_1 \backslash x, \ldots, B_r \backslash x)$ is a presentation of $M = N \backslash x$
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- But could it be that we get repetitions? How can we ensure we have all extensions?

Transversal extensions

Let $\mathcal{A} = (A_1, \ldots, A_r)$ be a presentation of M
For $I \subseteq [r]$, let

$$\mathcal{A}^I = \begin{cases} A_i \cup x, & \text{if } i \in I, \\ A_i, & \text{otherwise.} \end{cases}$$

The matroid $M[\mathcal{A}^I]$ is a transversal extension of M

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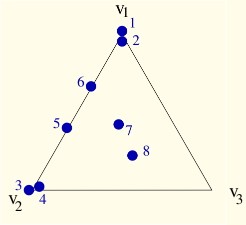

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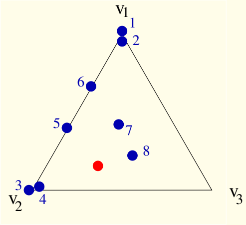

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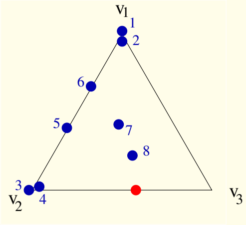

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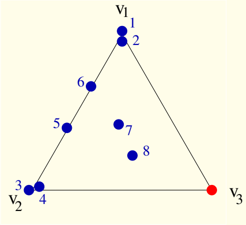

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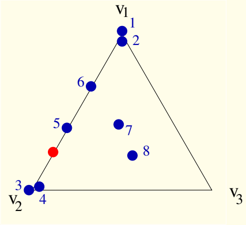

Transversal extensions

Some results from Bonin and de Mier 15

Thm The following are equivalent:

(i) if $I \neq J$ then $M[\mathcal{A}^I] \neq M[\mathcal{A}^J]$

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Thm If N is a transversal extension of M, there exist a minimal presentation $\mathcal{A}$ of M and a set $I \subseteq [r]$ such that $N = M[\mathcal{A}^I]$

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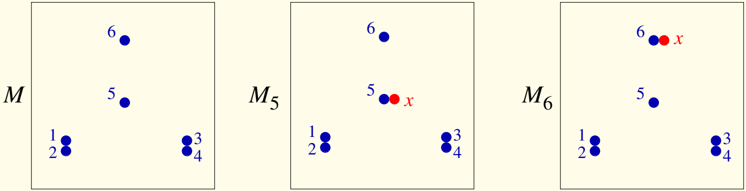
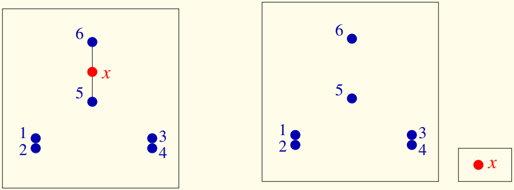
Fact: the set of all extensions of a matroid M is a lattice under the weak order

Question: is the set of all transversal extensions of a transversal matroid also a lattice under the weak order?

Transversal extensions



Join $M_5 \vee M_6$ and meet $M_5 \wedge M_6$:



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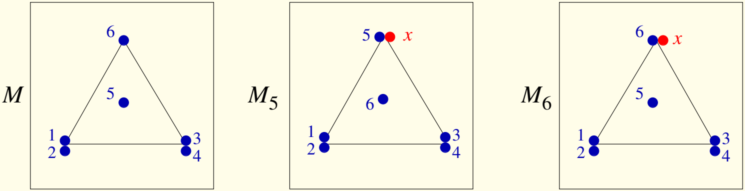

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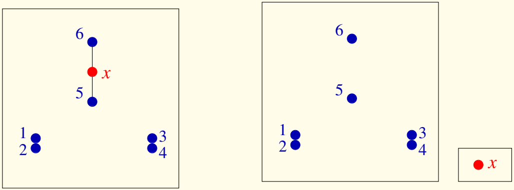

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But the ordinary join $M_5 \vee M_6$ is not transversal

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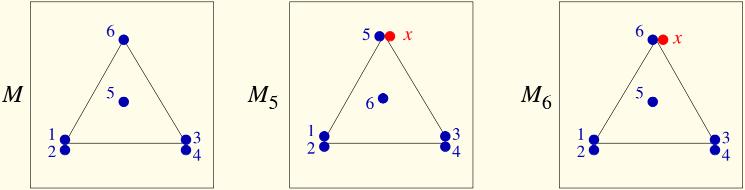

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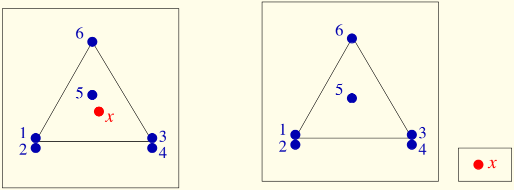

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The cyclic ordering problem

Question (Gabow 76)

Let B_1, B_2 be bases of a matroid M

Is there an ordering of the elements

$$b_1, b_2, \ldots, b_r, b_{r+1}, \ldots, b_{2r}$$

such that

- $B_1 = \{b_1, \ldots, b_r\}$,
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(In general, it is true for the larger class of strongly-base oderable matroids
For more basis-exchange problems, see Bonin 13)

References from the text

Edmonds and Fulkerson 65: Transversals and matroid partition, *J. Res. Nat. Bur. Standards Sect. B* 69B

Piff and Welsh 70: On the vector representations of matroids, *J. London Math. Soc* 2

Bonin, de Mier and Noy 03: Lattice path matroids: enumerative aspects and Tutte polynomials, *J. Combin. Theory Ser. A* 104

Bonin and de Mier 06: Lattice path matroids: structural properties, *Eur. J. Combin.* 27

Simoes-Pereira 72: On subgraphs as matroid cells, *Math. Z.* 127

Matthews 77: Bicircular matroids, *Quart. J. Math.* 28

Brylawski 75: An affine representation for transversal matroids, *Studies in Appl. Math.* 54

Bondy and Welsh 71: Some results on transversal matroids and constructions for indentically self-dual matroids, *Quart. J. Math.* 22

Bondy 72: Presentations of transversal matroids, *J. London Math. Soc* 5

Mason 69: *Representations of Independence Spaces* (PhD Dissertation)

Bonin 10: Lattice path matroids: the excluded minors *J. Combin. Theory Ser. B* 100

Mason 71: A characterization of transversal independence spaces, in: *Théorie des Matroïdes* (Lecture Notes in Math., 211)

References from the text

Ingleton 77: Transversal matroids and related structures, in: *Higher Combinatorics* (Proc. NATO Adv. Study Inst.)

Bonin, Kung and de Mier 11: Characterizations of transversal and fundamental transversal matrids, *Elec. J. Combin.*

Ingleton and Piff 72: Gammoids and Transversal Matroids, *J. Combin. Theory Ser. B* 15

Elizalde and Rubey 13: Bijections for pairs of non-crossing lattice paths and walks in the plane, *preprint*

Crapo 65: Single-element extensions of matroids, *J. Res. Nat. Bur. Standards Sect. B* 69

Bonin and de Mier 15: Extensions and presentations of transversal matroids, *Eur. J. Combin.*

Gabow 76: Decomposing symmetric exchanges in matroid bases, *Math. Programming* 10

Kotlar and Ziv 13: On serial symmetric exchanges of matroid bases, *J. Graph Theory* 73

Cordovil and Moreira 93: Bases-cobases grahs and polytopes of matroids, *Combinatorica* 13

Bonin 13: Basis-exchange properties of sparse paving matorids, *Adv. Appl. Math.* 50

Some general references about transversal matroids

J. Bonin, *An introduction to transversal matroids* (lecture notes online), 2010

R. Brualdi, Transversal matroids, in *Combinatorial geometries*, Cambridge Univ. Press, 1987

J. Oxley, *Matroid Theory*, 2n ed, Oxford Univ. Press, 2011 [Sections 1.6, 2.4, 11.2]