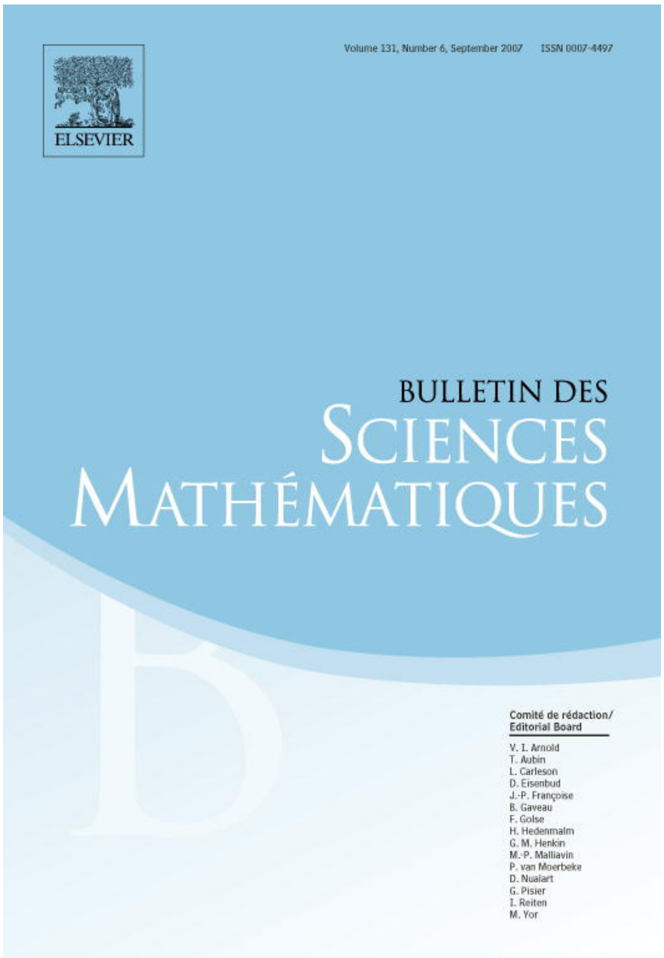
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Limit cycles and Lie symmetries

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Abstract

Given a planar vector field U which generates the Lie symmetry of some other vector field X, we prove a new criterion to control the stability of the periodic orbits of U. The problem is linked to a classical problem proposed by A.T. Winfree in the seventies about the existence of isochrons of limit cycles (the question suggested by the study of biological clocks), already answered by Guckenheimer using a different terminology. We apply our criterion to give upper bounds of the number of limit cycles for some families of vector fields as well as to provide a class of vector fields with a prescribed number of hyperbolic limit cycles. Finally we show how this procedure solves the problem of the hyperbolicity of periodic orbits in problems where other criteria, like the classical one of the divergence, fail.

Résumé

Etant donné un champ de vecteurs U du plan qui produit une symétrie de Lie d'un autre champ de vecteurs X, nous présentons un nouveau critère pour contrôler la stabilité des orbites périodiques du champ U. Ce problème est lié à un problème classique proposé par A.T. Winfree dans les années '70 au sujet de

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l'existence des isochrones de cycles limite (une question apparue dans l'étude des horloges biologiques), déjà répondu par Guckenheimer en utilisant une terminologie différente. Nous appliquons notre critère pour donner un borne supérieur du nombre de cycles limite pour quelques familles de champs de vecteurs aussi bien que pour fournir à une classe de champs de vecteurs un nombre prescrit de cycles limite hyperboliques. Finalement, on montre comment ce procédé résout le problème de l'hyperbolicité de cycles limite dans des problemes où d'autres critères, comme le critère classique de la divergence, échouent.

MSC: primary 34C07; secondary 34A26, 34C14, 37C27

Keywords: Limit cycles; Isochrons; Lie symmetries; Hyperbolicity

1. Introduction and main results

Starting from the interpretation of a Lie symmetry, this paper strongly intersects with three issues in dynamical systems and applications: (1) stability and (strong) hyperbolicity of limit cycles; (2) isochrons of limit cycles; (3) non-uniqueness of limit cycles.

In this introduction, we mention our new results along with others already known in order to describe the framework that Lie symmetries provide to these three issues. We differ the more technical results and details to the remaining sections.

1.1. Lie symmetries: switching from period functions to return maps

The use of Lie brackets for proving questions related to the time of the orbits of a planar vector field X is not new. One of the pioneering works wondering about isochronicity using this approach was the paper of Pleshkan [14]. A revival has come since the papers of Sabatini and Villarini [15,20], in the early 90s in which they linked the isochronicity of centres to the existence of commutators. Recall that, given an open set $\mathcal{V} \subset \mathbb{R}^2$, it is said that U is a (*transversal*) *generator of a Lie symmetry* or a *transversal normalizer* for X in $\mathcal{V}$ if, on this subset, X and U are transversal and $[X, U] = \mu X$, being [,] the standard Lie bracket. In fact the geometric interpretation of the existence of a Lie symmetry is that the flow of U sends orbits of X to orbits of X (orbital symmetry) and the function μ controls the relation between the parameterization of the orbits. In [6,7], we give a quantitative relation in case that X has a centre. It reads as follows.

Theorem 1. (Freire et al. [7]) *Consider a $\mathcal{C}^1$ vector field X having a centre at a point p with period annulus $\mathcal{P}$. Let U be a vector field, $U \in \mathcal{C}^1(\mathcal{P})$, transversal to X in $\mathcal{P} \setminus \{p\}$, and such that $[X, U] = \mu X$ on $\mathcal{P}$, for some $\mathcal{C}^1$ function $\mu : \mathcal{P} \subset \mathbb{R}^2 \to \mathbb{R}$. Denote by $\psi = \psi(s)$ a trajectory of U such that $\psi(s_0) \in \mathcal{P}$. Then,*

$$T'(s_0) = \int_0^{T(s_0)} \mu\big(x(t), y(t)\big)\,\mathrm{d}t, \tag{1}$$

where $(x(t), y(t))$ is the orbit of X such that $(x(0), y(0)) = \psi(s_0)$ and $T(s)$ the period of the orbit of X passing through $\psi(s)$.

Notice that the above result can be useful to prove isochronicity of the centre ($T'(s) \equiv 0$), to prove monotonicity of the period function ($T'(s) \neq 0$) or to study the number of critical periods of T (the solutions of $T'(s)=0$). With Theorem 1 in mind we thought about the possibility of getting a similar result but concerning to the Poincaré return map associated to a limit cycle of X. We quickly realized that if X possesses a transversal normalizer in a neighbourhood of a periodic orbit γ, it can never be a limit cycle. In fact the symmetry and the transversality force γ to live in a continuum of periodic orbits. After checking several papers about the subject we confirmed that the only way for X to have a limit cycle and, at the same time, having a Lie symmetry, is breaking the transversality. In other words, essentially, the following idea is used to study the limit cycles of planar vector fields X possessing a normalizer U: the limit cycles should live in the set where X and U are parallel, see [17–19]. Hence, from this point of view, when the above approach is useful and a symmetry can be computed, the limit cycles can be explicitly computed and they are included in the set $X \cdot U^{\perp} = 0$. This can be useful, for instance, when the system possesses algebraic limit cycles, but in general using this point of view is equivalent to localize the limit cycles, a very complicated problem. At this point we made ourselves the following question: what about the limit cycles of the normalizer vector field U?

The study of the above question is the main subject of this paper. We consider a planar vector field U, which we assume that is a transversal normalizer of another vector field X. Next result gives a closed formula for the characteristic multiplier of a limit cycle of U in terms of μ. We get:

Theorem 2. *Let γ be a T-periodic orbit of a $\mathcal{C}^1$ planar vector field U, parameterized by $(x(s), y(s))$, $s \in [0, T]$. Assume that in a neighbourhood $\mathcal{V}$ of γ, U is a $\mathcal{C}^1$ transversal normalizer of X, i.e. $[X, U] = \mu X$, for some $\mathcal{C}^1$ function μ. Let $\Sigma = \{\psi(p, t)\colon t \in \mathbb{R}\} \cap \mathcal{V}$, be a cross section of γ, where $\psi(p, t)$ is the solution of $\dot{z} = X(z)$, $z = (x, y)$, such that $\psi(p, 0) = p \in \gamma$. Then, the characteristic multiplier of γ is given by*

$$\pi'(0) = \exp\left(\int_0^T \mu\big(x(s), y(s)\big)\, \mathrm{d}s \right),$$

where π is the Poincaré map on Σ.

Moreover, the time of first crossing of all orbits starting on Σ is T.

Notice that, as we wanted, Theorem 2 is somehow a version of Theorem 1 for the Poincaré return map. The sequel of the paper is devoted to apply our result to control the number of limit cycles of several types of planar vector fields.

1.2. Isochrons of limit cycles

When we try to apply Theorem 2 to some vector field U, the first difficulty is to find another transversal vector field X such that U is its normalizer. The last result of Theorem 2, namely that the Poincaré section Σ, generated by the flow of X, is an "isochronous section", i.e. that all the orbits of U spend the same time for going from Σ to Σ, gives the clue to solve the problem of whether such an X exists or not. Fortunately, although with other interests in mind and by using different notations, the problem of the existence of isochronous section has been already treated in the literature.

In fact in a paper due to Guckenheimer [10], the author discusses the existence of what he calls *isochrons* of a point x lying on a limit cycle γ of a vector field defined on $\mathbb{R}^n$. The problem was posed by Winfree [22], wondering about the features of biological clocks. Guckenheimer was able to clarify and give positive answers to Winfree's questions using tools from dynamical systems. In Winfree's context, it is said that y *is on the isochron of* $x \in \gamma$ if $d(\phi(y,t), \phi(x,t)) \to 0$ as $t \to +\infty$, where $\phi(z,t)$ represents the flow of the vector field U and satisfies that $\phi(z,0) = z$. Notice that, in the planar case, the *isochrons* introduced by Winfree coincide with the *isochronous sections* given by Σ in Theorem 2.

More precisely, Winfree asked:

Question. Do isochrons exist? Is a neighbourhood of a stable limit cycle partitioned into the isochrons of points on the limit cycle?

Guckenheimer is aware of the link between isochrons and stable sets and, using the Invariant Manifold Theorem, he proves:

Theorem 3. (Guckenheimer [10]) *Let* $\phi : M \times \mathbb{R} \to M$ (M *being a smooth manifold*) *be a smooth flow with a hyperbolic, stable limit cycle* γ. *The stable set* $W^s(x)$ *of each* $x \in \gamma$ *is*

(1) *a cross-section of* γ,
(2) *a manifold diffeomorphic to Euclidean space.*

Moreover, the union of the stable manifolds $W^s(x)$, $x \in \gamma$, *is an open neighbourhood of* γ *and the stable manifold of* γ.

As he remarks in his article, the theorem proves the existence of isochrons for hyperbolic stable limit cycles. He also points out the fact that the isochrons are permuted by the flow $(W^s(\phi(x,t)) = \phi(W^s(x),t))$.

In fact, a similar result was already proven in [11, Section VI.2], where the stable limit cycles are labelled as *asymptotically orbitally stable periodic solutions* and the existence of isochrons is described as these periodic solutions having *asymptotic phase*. Moreover, Hale is aware of the existence of asymptotically orbitally stable periodic solutions not having asymptotic phase, see Exercise 2.1 in the same text. Here, for sake of completeness, we also present a two-dimensional example of this situation, see Example 13.

Recently, two different papers which also prove related results in the planar case have appeared. In the notation of our paper, the first result reads as follows:

Theorem 4. (Sabatini [16]) *Let* γ *be a limit cycle of a* $\mathcal{C}^2$ *planar vector field* U. *Then* U *admits a transversal isochron at any point of* γ *if and only if* U *is a non-trivial normalizer of another vector field* X *in a neighbourhood of* γ.

The second result says that for most non-hyperbolic limit cycles isochrons do not exist.

Theorem 5. (Chicone and Liu [3]) *Let* γ *be a generic non-hyperbolic limit cycle, i.e. a double limit cycle of a* $\mathcal{C}^2$ *planar vector field* U. *Then a necessary and sufficient condition for* γ *to admit isochrons at a point* $p \in \gamma$ *is that* $\tau'(p) = 0$, *where* τ *is the time of the first return to* Σ, *a transverse section at* $p \in \gamma$.

It is also worth to say that in [3] a new proof of the existence of isochrons for planar hyperbolic limit cycles is also given.

Gathering Theorems 3, 4 and 5 we get the following corollary.

Corollary 6. *Let γ be a periodic orbit of a $\mathcal{C}^2$ planar vector field U. Then*

(i) *If γ is a hyperbolic limit cycle then there is a neighbourhood of γ such that U is a transversal normalizer of another vector field X.*
(ii) *If γ is a double limit cycle then generically no X exists such that U is its normalizer.*

As a consequence of the above result it could seem that Theorem 2 would only be useful when we consider vector fields having all their limit cycles hyperbolic. This is not true. For instance, in Theorem 7, by using our theorem, we give a result that includes double limit cycles. Notice also that if we substitute the vector field U by BU, where $B : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$ is a positive Dulac function, then the number of periodic solutions of U and BU coincide. By having in mind Theorem 5, it seems natural that this function B can be chosen in such a way that the new vector field BU is a normalizer of some vector field X. In any case, and although we think that it is an interesting problem, we are not devoted, in this paper, to the general question of the existence, in a neighbourhood of any periodic orbit of U, of B, X and μ such that $[X, BU] = \mu X$.

Let us describe, then, the applications of Theorem 2 that we develop in this paper.

Theorem 2, as well as Theorem 4, can be thought of as a way of finding the isochrons of a given vector field possessing a limit cycle. For the sake of illustration, in Section 2, and after the proof of our theorem, we reproduce and explain in a general framework an example by Winfree about the isochrons of a concrete family of integrable systems. The Lie symmetries approach allows to unveil the hidden virtues of Winfree's example.

1.3. Number of limit cycles

Section 3 is devoted to study the number of limit cycles of several families of planar vector fields. Inspired in the so called *rigid systems*, see for instance [4], and also in the systems treated in [16], we consider families of vector fields of the form

$$U = Z + F(x, y)X, \tag{2}$$

where $[Z, X] = 0$.

For rigid systems,

$$Z = \begin{pmatrix} -y \\ x \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x \\ y \end{pmatrix},$$

and our main result is the following theorem.

Theorem 7. *Consider the vector field*

$$U := \begin{cases} \dot{x} = -y + xF(x, y), \\ \dot{y} = x + yF(x, y), \end{cases} \tag{3}$$

with $F(x, y) = F_0 + F_m(x, y) + F_n(x, y)$, being $F_i(x, y)$ homogeneous polynomials of degree i in x and y and $0 < m < n$. If either the function $F_m(\cos\theta, \sin\theta)$ or the function $F_n(\cos\theta, \sin\theta)$ do not change sign, then (3) *has at most two limit cycles and, more precisely, it can happen only one the following possibilities*:

(i) *the vector field U has no limit cycles*;

(ii) *the vector field U has a unique hyperbolic limit cycle*;
(iii) *the vector field U has a unique semi-stable limit cycle*;
(iv) *the vector field U has exactly two hyperbolic limit cycles*;

and all them are realizable.

We want to stress that the above result adds a new example to the not very large list of systems for which an upper bound of two limit cycles can be given.

Remark 8. In [9], it is proved that there are vector fields of the form (3) with $m=2$ (resp. $m=3$) and $n=4$ having at least 3 (resp. 4) limit cycles. Hence, one cannot avoid hypotheses on F_n or F_m to get criteria that give an upper bound of 2 limit cycles for systems of this type.

For more general Z and X in (2), we consider $F(x,y)=\varphi(V(x,y))$ where V is a two variable function. Notice that if $[X,Z]=0$, the U given in (2) is a normalizer of X and in fact $[X,U]=\mu X$ with $\mu(x,y)=\varphi'(V(x,y))\dot{V}$, being $\dot{V}$ the derivative of V along the orbits of X ($\dot{V}=\nabla V\cdot X$). As we have noted, the orbits of X are the isochrons of the limit cycles of U; in addition, if the Z considered has a centre, it must be an isochronous centre. In this case, we give a result about existence and upper bound of limit cycles of (2), adding some other hypotheses on this vector field, see Theorem 14. We also study with more detail a vector field constructed with Z being a Loud isochronous centre, also inspired by [16], see Section 3.3.

1.4. Strong hyperbolicity of limit cycles

Last section compares the method given in Theorem 2 with other methods to study the stability of a limit cycle and focuses on the problem of the strong hyperbolicity of a limit cycle. Recall that the concept of *strong hyperbolicity* is used in the literature when the divergence does not change sign on the limit cycle (see for instance [2] and the references therein); as the limit cycle is usually unknown explicitly, this concept can be crucial to prove the hyperbolicity. The term itself would not be misleading if the computation of the divergence was the only tool to prove hyperbolicity but, as Theorem 2 of this paper shows, there exist other independent techniques that crop up the "essence" of strong hyperbolicity. As far as we know, the book of Ye et al., see [24], provides the third different (and more geometrical) way to prove the stability of a limit cycle of the vector field U, by using the curvature of the orbits of the orthogonal vector field $U^{\perp}$, $K^{\perp}(U)$. Roughly speaking, this criterion says that the sign of $\int_\gamma K^{\perp}(U)$ gives the stability character of the limit cycle. Recall that if $U(x,y)=(P(x,y),Q(x,y))$, then $K^{\perp}(U)=(Q_yP^2-(P_y+Q_x)PQ+P_xQ^2)/(P^2+Q^2)^{3/2}$. In the computations, we will use $\tilde{K}^{\perp}(U):=Q_yP^2-(P_y+Q_x)PQ+P_xQ^2$. An extension to any Riemannian metrics and examples of independence of this method with respect to the divergence one were shown in [8].

With these three methods in mind we propose a refinement of the concept of strong hyperbolicity:

Definition 9. Given a periodic orbit γ of a $\mathcal{C}^1$ planar vector field U, and X a vector field transversal to U in a neighbourhood of the limit cycle such that $[X,U]=\mu X$, we say that

- γ is *strongly hyperbolic via divergence* if $\operatorname{div} U$ does not change sign on γ and only vanishes on isolated points on it.

- γ is *strongly hyperbolic via the orthogonal curvature* if $\tilde{K}^{\perp}(U)$ does not change sign on γ and only vanishes on isolated points on it.
- γ is *strongly hyperbolic via the Lie symmetry of X* if μ does not change sign on γ and only vanishes on isolated points on it.

In the next result we collect three examples, each of them being optimal for each of the three methods (divergence, Lie symmetries and orthogonal curvatures), respectively, to stress the independence of each one. The second and third ones are taken from [8] but we include them for the sake of clarity and completeness. Its proof is done in Section 4.

Example 10. The three methods to prove strong hyperbolicity stated above (via divergence, via orthogonal curvature and via Lie symmetries) are independent. In particular, considering the auxiliary vector field $X(x, y) = (-x, -y)$, we have:

(i) the unique limit cycle of system

$$U := \begin{cases} \dot{x} = -y + x(ax^2 + by^2 + c), \\ \dot{y} = x + y(ax^2 + by^2 + c), \end{cases}$$

with $a, b > 0$ and $c < 0$, is strongly hyperbolic via a Lie symmetry of X. In this case, we are able to find a ring-shaped domain containing the limit cycle in which the function μ of the Lie symmetry does not change sign. On the other hand, since both the divergence and the orthogonal curvature change sign over this domain, we cannot decide if the limit cycle is strongly hyperbolic neither via divergence nor via orthogonal curvature;

(ii) the closed curve $\gamma = \{(x, y)\colon\ R(x, y) = 0\}$, where $R(x, y) = x^2 + y^2 - 1$, is a strongly hyperbolic limit cycle of

$$U := \begin{cases} \dot{x} = -y + x - x^2y + xy^2 - 2y^3 - x^5 - 3x^3y^2 - 2xy^4, \\ \dot{y} = x + x^3 + 2xy^2 \end{cases}$$

via orthogonal curvature but not strongly hyperbolic via divergence nor via a Lie symmetry of X;

(iii) the closed curve $\gamma = \{(x, y)\colon\ R(x, y) = 0\}$, where $R(x, y) = x^2 + 4y^2 - 1$, is a strongly hyperbolic limit cycle of

$$U := \begin{cases} \dot{x} = -4y + 2x - 2x^3 - 8xy^2, \\ \dot{y} = x \end{cases}$$

via divergence but not strongly hyperbolic via orthogonal curvature nor via a Lie symmetry of X.

A result of Amel'kin (see [2]) says that any hyperbolic limit cycle can be made strongly hyperbolic via divergence by means of a suitable Dulac function. At the same time, changing the metrics of the plane we could also convert a limit cycle into a strongly hyperbolic via orthogonal curvature one, since the topology remains unaltered after a change to another Riemannian metrics. Finally, if a specific Lie symmetry ($[X, U] = \mu X$) does not provide strong hyperbolicity on a limit cycle, we can eventually obtain new symmetries ($[X', U] = \mu' X'$) such that μ' has a definite sign on the limit cycle; for instance, by taking $X' = BX$ for some non-negative scalar function B. It seems, then, that the three methods share a kind of degree of freedom to arrive to strong hyperbolicity. In fact, it is known that $\operatorname{div}(\frac{X}{\|X\|}) = K^{\perp}(X)$, and the link between Lie symmetries and Dulac functions will be stated in Remark 11.

2. Proof of Theorem 2 and construction of isochrons

Proof of Theorem 2. Let $\gamma = \{\gamma(t), t \in \mathbb{R}\}$ be a periodic orbit of $\dot{x} = U(x)$ of period T. Denote by $p = \gamma(0)$ and consider

$$Y(t) := \exp\left(\int_0^t \mu(\gamma(s))\,\mathrm{d}s\right) X(\gamma(t)).$$

We next prove that $Y(t)$ is a solution of the first variational equation associated to $\dot{x} = U(x)$. We use that $DUX - DXU = \mu X$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} Y(t) &= \exp\left(\int_0^t \mu(\gamma(s))\,\mathrm{d}s\right)\left(\mu(\mathbf{x})X(\mathbf{x}) + DX(\mathbf{x})U(\mathbf{x})\right)\Big|_{\mathbf{x}=\gamma(t)} \\
&= \exp\left(\int_0^t \mu(\gamma(s))\,\mathrm{d}s\right)\left(\mu(\mathbf{x})X(\mathbf{x}) + DU(\mathbf{x})X(\mathbf{x}) - \mu(\mathbf{x})X(\mathbf{x})\right)\Big|_{\mathbf{x}=\gamma(t)} \\
&= \exp\left(\int_0^t \mu(\gamma(s))\,\mathrm{d}s\right) DU(\gamma(t))X(\gamma(t)) = DU(\gamma(t))Y(t).
\end{aligned}$$

Finally, observe that $Y(0) = X(p)$ and $Y(T) = \exp(\int_0^T \mu(\gamma(s))\,\mathrm{d}s)X(p)$.

On the other hand, $\frac{\mathrm{d}}{\mathrm{d}t}U(t) = DU(\gamma(t))U(\gamma(t))$ and $U(\gamma(0)) = U(\gamma(T)) = U(p)$. Then, the monodromy matrix of the variational equation of the return map in the basis $\{U(p), X(p)\}$, where $p \in \gamma$, is

$$\begin{pmatrix} 1 & 0 \\ 0 & \exp\left(\int_0^T \mu(\gamma(s))\,\mathrm{d}s\right) \end{pmatrix}.$$

That is, the characteristic exponent of the periodic orbit γ is $\int_0^T \mu(\gamma(s))\,\mathrm{d}s$.

The assertion that says that the time of first crossing of all orbits starting at Σ is T is an straightforward consequence of the fact that the flow of U sends orbits of X to orbits of X. Hence, in particular, the flow of U after time T sends Σ to Σ, as we wanted to prove. □

Remark 11.

1. In [5], it was already proved that the function $V(x, y) = X(x, y)^{\perp} \cdot U(x, y)$ satisfies

   $$\mu = \operatorname{div} U - \frac{\nabla V^t U}{V}, \tag{4}$$

   provided that $[X, U] = \mu X$, and being X and U two transversal planar vector fields. In fact, this is a particular case of the formula

   $$[X, U] = \left(\frac{\nabla V^t X}{V} - \operatorname{div} X\right)U - \left(\frac{\nabla V^t U}{V} - \operatorname{div} U\right)X,$$

   given by S. Walcher in [21], when $1/V$ is an integrating factor of X.
   Notice that equality (4) can be used to prove the theorem in an alternative way because $(\nabla V^t U)/V$ vanishes when integrated along a T-periodic orbit $\gamma(t)$ of U. Then,

$\int_0^T \mu(\gamma(t))\,\mathrm{d}t = \int_0^T \mathrm{div}(\gamma(t))\,\mathrm{d}t$, and it is well known that the latest integral gives the characteristic exponent of the periodic orbit.
On the other hand, since $\mathrm{div}(U/V) = \mu/V$, we can obtain strong hyperbolicity via divergence from strong hyperbolicity via Lie symmetries just considering $B(x,y) := |1/V(x,y)|$ as the Dulac function on the limit cycle.

2. Notice that the proof of the theorem (and, hence, the statement) can be extended to higher dimensions. Any vector field X_1 for which U is a transversal normalizer ($[X_1, U] = \mu_1 X_1$) provides a characteristic exponent $p_1 := \int_0^T \mu_1(\gamma(t))\,\mathrm{d}t$ of the T-periodic orbit $\gamma(t)$. We can obtain different characteristic exponents $p_1, \ldots, p_k$ provided that the corresponding vector fields $X_1, \ldots, X_k$ are independent (that is, $\dim\langle X_1, \ldots, X_k, U\rangle = k+1$) in a neighbourhood of $\gamma(t)$. In particular, if $k = n-1$, we get all the characteristic exponents.

Let us see how our approach using Lie symmetries leads to a general constructive procedure to get the isochrons associated to a hyperbolic limit cycle of a given integrable system. Next proposition includes all the examples considered by Winfree in [23, Chapter 6]. As we will see it also gives the intuition to construct an example of asymptotically orbitally stable limit cycle without asymptotic phase, i.e. without isochrons, see Example 13.

Proposition 12. *Consider the $\mathcal{C}^1$ system*

$$U := \begin{cases} \dot r = r a(r), \\ \dot\theta = b(r), \end{cases}$$

where r and θ are the polar coordinates, $r_0 > 0$, $a(r_0) = 0$, $a'(r_0) \neq 0$ and $b(r_0) \neq 0$. Then, $\{r = r_0\}$ is a hyperbolic limit cycle of U and its isochrons satisfy the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\tilde b(r)}{r\tilde a(r)},$$

where $\tilde b(r) = (b(r) - b(r_0))/(r - r_0)$ and $\tilde a(r) = a(r)/(r - r_0)$.

Proof. We can think of U as

$$U = a(r)\begin{pmatrix} x \\ y \end{pmatrix} + b(r)\begin{pmatrix} -y \\ x \end{pmatrix}$$

and look for functions α and β to build a vector field

$$X = \alpha(r)\begin{pmatrix} x \\ y \end{pmatrix} + \beta(r)\begin{pmatrix} -y \\ x \end{pmatrix}$$

and a function μ satisfying $[X, U] = \mu X$.

A straightforward computation gives

$$[X, U] = \big(a'(r)\alpha(r) - a(r)\alpha'(r)\big) r \begin{pmatrix} x \\ y \end{pmatrix} + \big(b'(r)\alpha(r) - a(r)\beta'(r)\big) r \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Forcing $[X, U] = \mu X$, we get that

$$\frac{a'\alpha - a\alpha'}{\alpha} = \frac{b'\alpha - a\beta'}{\beta}, \tag{5}$$

since one must have $\mu = r(a' - a\alpha'/\alpha) = r(\alpha b'/\beta - a\beta'/\beta)$. Notice that there is a high freedom to choose $\alpha(r)$. In order to simplify the above equation, we try with $\alpha(r) \equiv 1$. Observe that this choice guarantees the transversality of X and U as well, because $X^\perp \cdot U|_{r=r_0} = b(r_0) r_0^2 \neq 0$.

When $\alpha = 1$, from (5) we get that β satisfies the linear differential equation $a\beta' + a'\beta = b'$ and, then, $a(r)\beta(r) = b(r) + k$. Since $a(r_0) = 0$, then $k = -b(r_0)$. Hence,

$$\beta(r) = \frac{b(r) - b(r_0)}{a(r)} = \frac{\tilde{b}(r)}{\tilde{a}(r)},$$

and the vector field X writes as

$$\begin{cases} \dot{r} = r, \\ \dot{\theta} = \frac{\tilde{b}(r)}{\tilde{a}(r)}. \end{cases}$$

By using Theorem 2 we know that the orbits of X are, in fact, the isochrons of $r = r_0$, as we wanted to prove.

Finally, observe that, with this choice of α, the stability of the limit cycle is determined (as we could expect) by the sign of $a'(r_0)$ since $\mu(r) = ra'(r)$. □

To conclude with the discussion of Section 1.2 about the existence of isochrons, in the next example we give an instance of asymptotically orbitally stable periodic solution without asymptotic phase:

Example 13. The orbit $\{r = 1\}$ is an asymptotically orbitally stable periodic solution without asymptotic phase for system

$$\begin{cases} \dot{r} = -r(r-1)^3, \\ \dot{\theta} = r. \end{cases} \tag{6}$$

Proof. We call $P(r_0)$ the value of the Poincaré return map of the solution of (6) starting at $(r, \theta) = (r_0, 0)$. Simple computations give

$$P(r_0) = 1 - \frac{1 - r_0}{\sqrt{1 + 4\pi(1 - r_0)^2}}.$$

On the other hand,

$$T(r_0) = \int_0^{T(r_0)} \mathrm{d}t = \int_{r_0}^{P(r_0)} \frac{\mathrm{d}r}{r(1-r)^3},$$

and so,

$$T'(r_0) = -\frac{\sqrt{1 + 4\pi - 8\pi r_0 + 4\pi r_0^2} - 1}{r_0(r_0 - 1)^2\left(\sqrt{1 + 4\pi - 8\pi r_0 + 4\pi r_0^2} + r_0 - 1\right)}.$$

Taking limits to approach to the limit cycle,

$$\lim_{r_0 \to 1} T'(r_0) = -2\pi \neq 0.$$

Hence, from the result by Chicone and Liu, see [3] or Theorem 5, we know that the limit cycle has no asymptotic phase. Notice that, we are not under the hypotheses of previous proposition because $a'(1) = 0$. In fact the“possible” isochrons would be the solutions of $\dot{r} = r$, $\dot{\theta} = -(r-1)^{-2}$, which are not well defined on the limit cycle $\{r = 1\}$. □

3. Examples

In this section we consider two different families of examples. The first subsection is devoted to study some rigid systems. The two other sections deal with systems of the form (2). Section 3.2 studies a quite general family of systems and gives a result providing both upper and lower bounds for its number of limit cycles. On the other hand, Section 3.3 considers a system constructed from an isochronous quadratic Loud system, and proves (in Proposition 17) the uniqueness and hyperbolicity of its limit cycle. It is worth to say that we have tried to prove the uniqueness of the limit cycle studied in Proposition 17 by other methods, but they have failed.

3.1. Rigid systems

Recall that rigid systems are planar systems with constant angular speed. They can be written as

$$U := \begin{cases} \dot{x} = -y + xF(x,y), \\ \dot{y} = x + yF(x,y). \end{cases} \tag{7}$$

Our main result, already stated in the Introduction, is Theorem 7. It provides a subfamily of systems of type (7) (more precisely, $F(x,y) = F_0 + F_m(x,y) + F_n(x,y)$, being $F_i(x,y)$ homogeneous polynomials of degree i) having at most two limit cycles.

Proof of Theorem 7. Taking

$$X := \begin{cases} \dot{x} = x, \\ \dot{y} = y, \end{cases}$$

it follows that $[X,U] = \mu X$, with $\mu(x,y) = xF_x(x,y) + yF_y(x,y)$. In virtue of Euler's formula, $\mu(x,y) = mF_m(x,y) + nF_n(x,y)$ and, by Theorem 2, the stability of an eventual limit cycle $r = \bar{r}(\theta) > 0$ is given by the sign of

$$h(\bar{r}) := mI_m(\bar{r}) + nI_n(\bar{r}), \tag{8}$$

where $I_k(\bar{r}) = \int_0^{2\pi} \bar{r}(\theta)^k F_k(\cos\theta, \sin\theta)\,\mathrm{d}\theta$, for $k = 0, m, n$. Notice that $I_0(\bar{r}) \equiv I_0 = 2\pi F_0$. Since the expression in polar coordinates of (7) is

$$\begin{cases} \dot{r} = rF(r\cos\theta, r\sin\theta), \\ \dot{\theta} = 1, \end{cases} \tag{9}$$

on one hand,

$$\int_0^{2\pi} F\big(\bar{r}(\theta)\cos\theta, \bar{r}(\theta)\sin\theta\big)\,\mathrm{d}\theta = \int_0^{2\pi} \frac{\bar{r}'(\theta)}{\bar{r}(\theta)}\,\mathrm{d}\theta = \ln\left(\frac{\bar{r}(2\pi)}{\bar{r}(0)}\right) = 0, \tag{10}$$

and on the other hand

$$\int_0^{2\pi} F\big(\bar{r}(\theta)\cos\theta, \bar{r}(\theta)\sin\theta\big)\,\mathrm{d}\theta = I_0 + I_m(\bar{r}) + I_n(\bar{r}). \tag{11}$$

Let us assume, for instance, that $F_n(\cos\theta, \sin\theta) \geqslant 0$. The case in which this function is less or equal than zero can be treated in a similar way. Gathering (8), (10) and (11) we get that

$h(\bar r) = -mI_0 + (n-m)I_n(\bar r)$. The hypothesis on F_n implies that if $r = \bar r_1(\theta)$ and $r = \bar r_2(\theta)$ are two periodic solutions of (9) satisfying $0 < \bar r_1(\theta) < \bar r_2(\theta)$ then $I_n(\bar r_1) < I_n(\bar r_2)$ and hence

$$h(\bar r_1) < h(\bar r_2). \tag{12}$$

By using inequality (12) and the fact the origin is the only critical point of system (9) we get that it has at most three limit cycles, because two nested consecutive hyperbolic limit cycles, can never have the same stability. Furthermore, if they are given by $r = \bar r_i(\theta)$, $i = 1, 2, 3$, and satisfy $0 < \bar r_1(\theta) < \bar r_2(\theta) < \bar r_3(\theta)$, then we must have $h(\bar r_1) < h(\bar r_2) = 0 < h(\bar r_3)$. This last inequality says in particular that the inner and outer limit cycle are hyperbolic and that the middle one is a semi-stable one. As usual, we use the theory of rotated vector fields [13, Section 4.6], to prove that this semi-stable limit cycle cannot exist.

Notice that system (9) is a rotated family with respect the parameter F_0. Hence, by moving a little bit this parameter in the suitable direction, we can assure that the two hyperbolic limit cycles remain as hyperbolic limit cycles for the new system and, at the same time, the semi-stable limit cycle breaks giving rise to two more limit cycles. This new vector field would have four limit cycles and $F_n(\cos\theta, \sin\theta) \geqslant 0$, a contradiction with the result proved above. Thus, only the two hyperbolic limit cycles could exist.

If only the semi-stable limit cycle and one on the hyperbolic limit cycles would exist a similar reasoning using again the rotatory parameter F_0 would give a new contradiction. Hence, the results (i)–(iv) follow when F_n does not change sign. To prove the same results when F_m does not change sign, we can combine again (8), (10) and (11), giving rise to the expression $h(\bar r) = -nI_0 + (m-n)I_m(\bar r)$. From it, we can follow again the argument of the previous case.

It is not difficult to construct simple examples under the hypotheses of the theorem presenting each one on the possibilities (i)–(iv); it suffices to consider functions F of the form $F = F_0 + a(x^2+y^2)^{m/2} + b(x^2+y^2)^{n/2}$ for suitable constants a and b. Hence the proof is ended.

We only remark that taking into account that the stability of the origin is governed by the sign of F_0 and considering also the parity of m (resp. n) when F_n (resp. F_m) does not change sign, in some cases the upper bound of two limit cycles can be reduced. $\square$

3.2. *A family of systems of the form $U = Z + F(x, y)X$*

Let Z have a centre at p_0 with period annulus $\mathcal{P}$ and let X be a vector field transversal to Z, that is, $Z^\perp \cdot X > 0$ (or < 0) in some region $\mathcal{D}$, $\mathcal{D} \subset \mathcal{P}$, assuming that $\{Z, Z^\perp\}$ is a positively oriented orthogonal basis. Suppose that $X(p_0) = 0$.

Let $V \in \mathcal{C}^1(\mathcal{D}')$, $\mathcal{D} \subseteq \mathcal{D}'$, such that all the level curves $\{(x, y)\colon\ V(x, y) = a > 0\}$ included in $\mathcal{D}$ are closed and connected.

Let also $\varphi(x)$ be a $\mathcal{C}^1$ function in a subset $I \subseteq \mathbb{R}$ with exactly $(m+1)$ simple zeroes $(a_0, \ldots, a_m)$. We define, for convenience, $a_{-1} := V(p_0)$ and denote $I_j := (a_{j-1}, a_j)$, for $j = 0, \ldots, m$. Observe that $I = \bigcup_{j=0}^{m+1} I_j \cup \bigcup_{j=0}^{m} \{a_j\}$. Suppose that φ does not change sign in I_0. Of course, φ will not change sign in $I_{m+1} := I \cap (a_m, +\infty)$.

Given any pair γ, γ' of closed orbits, we say that $\gamma \trianglelefteq \gamma'$ if $\operatorname{Int}\gamma \subseteq \operatorname{Int}\gamma'$. This operation defines an ordering in $\mathcal{P}$ that allows to distinguish the following orbits of Z:

$$\Gamma_{\mathrm{in}}^j := \max_{\gamma\in\mathcal{P}}\bigl\{\gamma \trianglelefteq \{V = a_j\}\bigr\}, \quad \Gamma_{\mathrm{out}}^j := \min_{\gamma\in\mathcal{P}}\bigl\{\gamma \trianglerighteq \{V = a_j\}\bigr\}, \quad \text{for } j = 0, \ldots, m.$$

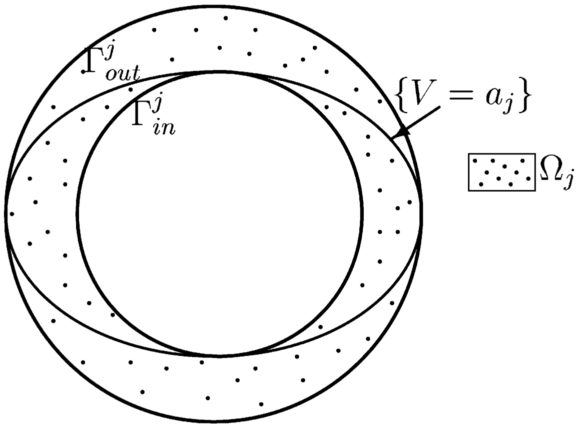

Fig. 1. Formation of a region Ω_j. Γ^j_{out} (Γ^j_{in}) is the outmost (inmost) tangency of an orbit of Z with $\{V = a_j\}$ (symbolized here by an ellipse). Ω_j is the region filled in with scattered dots.

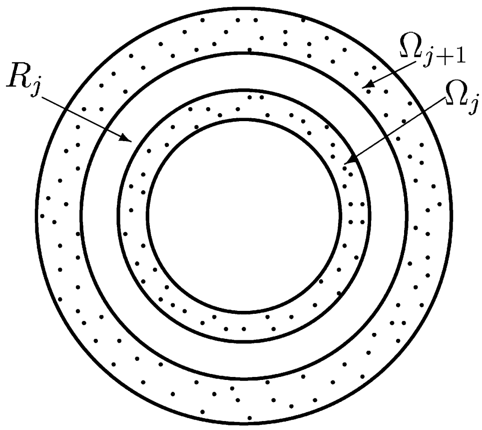

Fig. 2. The j-*inclusion*: two consecutive Ω-regions (Ω_j and Ω_{j+1}) filled in with scattered dots that do not overlap. The white region in between is defined as R_j.

We also define Ω_j as the ring bounded by Γ^j_{in} and Γ^j_{out}, for any $j = 0, \ldots, m$, see Fig. 1. We will also say that the j-*inclusion* is fulfilled whenever $\Gamma^{j-1}_{\mathrm{out}} \trianglelefteq \Gamma^j_{\mathrm{in}}$. In this case, we define R_j as the ring bounded by $\Gamma^{j-1}_{\mathrm{out}}$ and Γ^j_{in}, $j \in \{1, \ldots, m\}$, see Fig. 2.

Theorem 14. *Taking into account all the above definitions, consider the vector field $U = Z + \varphi(V(x, y))X$. Then,*

(a) *If the 1-inclusion holds, then U has at least one limit cycle in the ring Ω_0. If the j-inclusion and the $(j + 1)$-inclusion hold for some $j \in \{1, \ldots, m - 1\}$, then U has at least one limit cycle in the ring Ω_j. If the m-inclusion holds, then U has at least one limit cycle in the region Ω_m.*

(b) *Assume, in addition, that X is a commutator of Z ($[Z, X] \equiv 0$ in $\mathcal{D}$) and V is a strict Liapunov function for X in $\mathcal{D}$.*
Then, U has exactly $m + 1$ limit cycles (which are hyperbolic) provided that φ is monotone in every Ω_j, for $j = 0, \ldots, m$. In fact, U has exactly one limit cycle in each Ω_j, which is hyperbolic and stable (unstable) if $\varphi > 0$ (< 0) in (a_j, a_{j+1}).

Remark 15. A practical way to ensure the monotonicity of φ in Ω_j, for a particular $j \in \{1, \dots, m-1\}$, is that φ has a unique maximum or minimum both in I_j (we denote it by a'_j) and in I_{j+1}, and that $\{V = a'_j\} \subseteq R_j$ and $\{V = a'_{j+1}\} \subseteq R_{j+1}$. In Ω_0, a sufficient condition is that a'_1 exists, with $\{V = a'_1\} \subseteq R_1$, plus φ monotone in I_0. Similarly, in Ω_m, a sufficient condition is that a'_m exists, with $\{V = a'_m\} \subseteq R_m$, plus φ monotone in I_{m+1}.

Proof. Without loss of generality, we can assume that the flow of Z is counterclockwise and write the transversality hypothesis as $Z^{\perp} \cdot X > 0$. Consequently, $Z^{\perp} \cdot U = (\varphi \circ V)(Z^{\perp} \cdot X)$ and, then, the sign of $Z^{\perp} \cdot U$ is exactly the sign of $\varphi \circ V$.

For the proof, we fix $j \in \{1, \dots, m-1\}$ such that $\varphi'(a_j) > 0$ (as a consequence, $\varphi(x) < 0$ in I_j and $\varphi(x) > 0$ in I_{j+1}). The special cases $j = 0$ and $j = m$ are proved separately.

(a) Since $\{V = a_{j-1}\} \trianglelefteq \Gamma_{\text{out}}^{j-1} \trianglelefteq \Gamma_{\text{in}}^{j} \trianglelefteq \{V = a_j\}$, the j-inclusion implies that $(\varphi \circ V) < 0$ on Γ_{in}^{j}. Then, $Z^{\perp} \cdot U < 0$ on Γ_{in}^{j} also and, since it is an orbit of Z, then $\operatorname{Int} \Gamma_{\text{in}}^{j}$ is a negatively invariant region for the flow of $U = Z + \varphi(V(x, y))X$. Similar arguments (through $(j+1)$-inclusion) provide that $\operatorname{Int} \Gamma_{\text{out}}^{j}$ is positively invariant. Then, using the Poincaré–Bendixson theorem, one must encounter at least one (attracting) limit cycle in the region Ωj.

The fact that φ does not change sign both in I_0 and in I_{m+1} provides, respectively, the appropriate invariance to Γ_{in}^{0} and Γ_{out}^{m}.

Then, a suitable φ such that the j-inclusion holds for any $j = 1, \dots, m-1$, forces U to have at least $m+1$ limit cycles. It remains the question of hyperbolicity and uniqueness of the limit cycles.

(b) Since $[Z, X] = 0$, it is easy to see that $[X, U] = ((\nabla\varphi)^{\perp} \cdot X)U$; in other terms:

$$[X, U] = \mu X, \quad \text{with } \mu(x, y) = \varphi'\big(V(x, y)\big)\dot{V},$$

where $\dot{V}$ is the derivative of V along the orbits of X.

The fact that V is a strict Liapunov function for X, ensures that $\operatorname{sgn}(\mu) = -\operatorname{sgn}(\varphi')$. Since φ is monotone in Ω_j, then $\varphi' \neq 0$ and, hence, applying Theorem 2, the hyperbolicity and, as a consequence, the uniqueness of the limit cycle in each Ω_j are proven. It is stable (resp. unstable) if $\varphi' > 0$ (resp. < 0) on Ω_j. □

We keep assuming that the flow of Z is counterclockwise and writing the transversality hypothesis as $Z^{\perp} \cdot X > 0$. A corollary of Theorem 14 is the following:

Corollary 16. *If V is a first integral of Z, then, for each $j = 0, \dots, m$, $\Gamma_{\text{in}}^{j} = \Gamma_{\text{out}}^{j}$ is a limit cycle of U. The limit cycles are always hyperbolic (assuming that the zeroes of φ are simple) and the sign of the stability is provided by $\varphi'(a_j)$.*

3.3. *Limit cycles from a Loud system*

The system studied in the following result is similar to a system considered in [16]. In fact, the only difference is that in Sabatini's paper the function $1 - \alpha V(x, y)$ below is replaced by $V(x, y) - 4B^2 V(x, y)^2$. In that paper, the author uses a procedure to prove the existence of at least one limit cycle for the system, that in fact was our main inspiration to state and prove part (a) of Theorem 14. Unfortunately, part (b) of our theorem does not work to prove the uniqueness of the limit cycle of Sabatini's example. On the other hand, we can prove the existence and uniqueness of the limit cycle of the system studied in the next proposition.

Proposition 17. *Consider the vector field $U = Z + (1 - \alpha V(x, y))Z^{\perp}$, where*

$$Z = \begin{cases} \dot{x} = -y + Bxy, \\ \dot{y} = x - \frac{1}{2}Bx^2 + \frac{1}{2}By^2, \end{cases}$$

$V(x, y) = x^2 + y^2$, $B > 0$, $\alpha > B^2 > 0$ and Z, $Z^{\perp}$ form a positive oriented basis. Then, U has a unique limit cycle which is hyperbolic and unstable.

Proof. Notice that Z has an isochronous centre at the origin, as it was proved by Loud, see [12]. A transversal commutator of Z is its orthogonal vector field

$$X := Z^{\perp} = \begin{cases} \dot{x} = -x + \frac{1}{2}Bx^2 - \frac{1}{2}By^2, \\ \dot{y} = -y + Bxy. \end{cases}$$

To see that $V = x^2 + y^2$ is a Liapunov function for X notice that

$$\dot{V} = 2x\left(-x + \frac{1}{2}Bx^2 - \frac{1}{2}By^2\right) + 2y(-y + Bxy) = (x^2 + y^2)(-2 + Bx).$$

Since the period annulus of the centre of Z is $\mathcal{P} = \{(x, y)\colon\ x < 1/B\}$, we have that V is a strict Liapunov function for X on $\mathcal{P}$.

Consider now the function $\varphi(V) = 1 - \alpha V$. In the notation of Theorem 14, $m = 0$ and $a_0 = 1/\alpha$. Moreover, $\varphi'(a_0) = -\alpha < 0$; $\varphi(v) > (<) 0$ if $v < (>) a_0$. We are then on the track of proving that $U = Z + (1 - \alpha V(x, y))X$ has a unique limit cycle which is hyperbolic and unstable. This is true because $\mu = \varphi'(V)\dot{V} = -\alpha\dot{V} > 0$.

To have a more precise idea of the location of the limit cycle, we can compute the Γ^0_{out} and Γ^0_{in} given in Theorem 14.

Using that $H(x, y) = \frac{x^2+y^2}{-1+Bx}$ is a first integral of Z, we seek for k_1 and k_2 such that the circle $H(x, y) = k_1$ is the greatest integral curve completely included in the region bounded by $\{V = -1/\alpha\}$ and the circle $H(x, y) = k_2$ is the smallest one including completely $\{V = -1/\alpha\}$ in its finite Jordan component. An easy computation shows that $k_1 = \frac{1}{-B\sqrt{\alpha}-\alpha}$ and $k_2 = \frac{1}{B\sqrt{\alpha}-\alpha}$.

Again from easy computations, we conclude that the limit cycle must lie in the region bounded by the circles $C_1 := \Gamma^0_{\text{out}}$ and $C_2 := \Gamma^0_{\text{in}}$, where C_i is given by

$$\left(x - \frac{k_i B}{2}\right)^2 + y^2 = k_i\left(\frac{k_i B^2}{4} - 1\right). \qquad \square$$

4. Proof of Example 10

(i) In [1, p. 215] (Example 12), it is proved that the following system

$$U := \begin{cases} \dot{x} = -y + x(ax^2 + by^2 + c), \\ \dot{y} = x + y(ax^2 + by^2 + c), \end{cases} \tag{13}$$

has a unique limit cycle for $a = 3$ and $b = 2$. The authors use the generalized Dulac criterion ensuring that the divergence does not change sign in a (negatively) invariant region homotopic to a ring and including the limit cycle. Thus, it can be called a *strongly hyperbolic* limit cycle via divergence.

This result can be somehow improved. Indeed, the ring

$$\Omega = \left\{(x, y)\colon\ |c|/\max(a, b) \leqslant x^2 + y^2 \leqslant |c|/\min(a, b)\right\}$$

is negatively invariant for any pair $(a,b) \in \mathbb{R}^{2^+}$ and free of critical points, and $\operatorname{div}(U) = 2(2ax^2 + 2by^2 + c)$. Thus, the ring Ω is included in the region of positive divergence if and only if $2\min(a,b) \geqslant \max(a,b)$. So, we can assert that the limit cycle is unique and strongly hyperbolic via divergence if $2\min(a,b) \geqslant \max(a,b)$.

On the other hand, if we use our approach, we are able to prove the uniqueness and strong hyperbolicity via Lie symmetries of the limit cycle for any pair $(a,b) \in \mathbb{R}^{2^+}$. Let us show how:

Consider

$$Z = \begin{cases} \dot{x} = -y, \\ \dot{y} = x, \end{cases} \quad \text{and} \quad X = \begin{cases} \dot{x} = -x, \\ \dot{y} = -y. \end{cases}$$

Let also be $V(x,y) = ax^2 + by^2$ (it is easy to see that it is a Liapunov function for X) and $\varphi(v) = -v - c$. In the notation of Theorem 14, $a_{-1} = 0$ and $a_0 := -c = |c|$.

The ellipse $\{(x,y)\colon\ V(x,y) = |c| = a_0\}$ defines the curves $\Gamma^0_{\text{in}} = \{x^2 + y^2 = |c|/\max(a,b)\}$ and $\Gamma^0_{\text{out}} = \{x^2 + y^2 = |c|/\min(a,b)\}$.

Since $\varphi' < 0$ everywhere and $\dot{V} = -2V < 0$, then, applying Theorem 14(b), $\mu > 0$ and so there is a unique limit cycle which is strongly hyperbolic via Lie symmetries and unstable.

Once the elements of the symmetry are known, using Remark 11 for system (13), we obtain the Dulac function $B(x,y) = (x^2 + y^2)^{-1}$, for which $\operatorname{div}(BU) = 2\frac{ax^2+by^2}{x^2+y^2}$.

Finally, the numerator of the curvature of $U^\perp$ is $\tilde{K}^\perp = C(x,y)(x^2 + y^2)$, where

$$\begin{aligned} C(x,y) = {} & a^3x^6 + 3a^2bx^4y^2 + 3ab^2x^2y^4 + b^3y^6 + 3ca^2x^4 \\ & + (2a^2 - 2ba)x^3y + 6abcx^2y^2 + (2ba - 2b^2)xy^3 + 3b^2cy^4 \\ & + (3a + 3c^2a)x^2 + (2ac - 2bc)xy + (3b + 3bc^2)y^2 + c + c^3. \end{aligned}$$

Since we just want to show here that this limit cycle is not strongly hyperbolic via orthogonal curvature for all (a,b,c), we only show an instance where C changes sign in Ω: in the case $a = 3$, $b = 2$, $C(x,x)$ has always one zero in the interval $(\sqrt{|c|}/\sqrt{6}, \sqrt{|c|}/2)$ (that is, inside Ω) if $c \in (12\sqrt{10} - 66, 8\sqrt{10} - 44)$. Then, we cannot ensure that the limit cycle lies on a region with constant sign of $\tilde{K}^\perp$.

(ii) First of all, it can be easily seen that $\gamma = \{x^2 + y^2 - 1 = 0\}$ is a periodic orbit of the system

$$U := \begin{cases} \dot{x} = -y + x - x^2y + xy^2 - 2y^3 - x^5 - 3x^3y^2 - 2xy^4, \\ \dot{y} = x + x^3 + 2xy^2, \end{cases}$$

just checking that $R_x\dot{x} + R_y\dot{y}|_{\{R=0\}} = 0$.

If we compute $\tilde{K}^\perp$ and afterwards we restrict it to γ, we obtain $\tilde{K}^\perp(x,y)|_\gamma = 2x^2(x^2 - 3)^3$. Then, γ is strongly hyperbolic via orthogonal curvature.

On the other hand, the divergence of U is $\operatorname{div} U = 1 + 2xy + y^2 - 5x^4 - 9x^2y^2 - 2y^4$, which takes the value $\operatorname{div} U|_\gamma = -6x^2 + 2xy + 2x^4$ evaluated on the limit cycle. This function changes its sign on γ and hence, it is not strongly hyperbolic via divergence.

Both in this example and the next one, the system U is not a normalizer of the "trivial" vector field $X(x,y) = (-x,-y)$, as in the first example. Of course, it could exist another vector field X' such that $[X', U] = \mu' X$ with μ' non-vanishing on the limit cycle.

(iii) As in the previous example, it can be easily checked that $\gamma = \{x^2 + 4y^2 - 1 = 0\}$ is a periodic orbit for the system

$$U := \begin{cases} \dot{x} = -4y + 2x - 2x^3 - 8xy^2, \\ \dot{y} = x. \end{cases}$$

Here, $\tilde{K}^{\perp}(x, y)|_{\gamma} = -4x(-3x^3 + 4x + 3y)$, which changes sign on the limit cycle. On the other hand, it turns out that $\operatorname{div} X|_{\gamma} = -4x^2$ and so, the limit cycle is strongly hyperbolic via divergence. □

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