International Journal of Bifurcation and Chaos, Vol. 16, No. 12 (2006) 3737–3745

LIMIT CYCLES FOR GENERALIZED ABEL EQUATIONS*

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Received June 13, 2005; Revised January 10, 2006

This paper deals with the problem of finding upper bounds on the number of periodic solutions of a class of one-dimensional nonautonomous differential equations: those with the right-hand sides being polynomials of degree n and whose coefficients are real smooth one-periodic functions. The case $n = 3$ gives the so-called *Abel equations* which have been thoroughly studied and are well understood. We consider two natural generalizations of Abel equations. Our results extend previous works of Lins Neto and Panov and try to step forward in the understanding of the case $n > 3$. They can be applied, as well, to control the number of limit cycles of some planar ordinary differential equations.

Keywords: Abel equation; limit cycles; planar differential equations.

1. Introduction and Main Results

Nonautonomous differential equations of type

$$\frac{dx}{dt} = S(t, x), \quad x \in \Omega \subset \mathbb{R}^n,\ t \in I \subset \mathbb{R}, \tag{1}$$

with additional boundary conditions are encountered in different problems like variational equations of periodic orbits of vector fields, plane autonomous ODE systems (see Sec. 4), control theory (see for instance [Fossas-Colet & Olm-Miras, 2002]), ... One is often interested in particular solutions $x(t)$ of (1) which are defined in a whole interval I (we take $I = [0, 1]$ throughout the paper) and such that $x(0) = x(1)$. In the case when S is one-periodic in t, observe that these solutions, which are closed when we consider (1) on the cylinder $\mathbb{R}^n \times [0, 1]$, can be called *periodic*. A periodic solution which is isolated in the set of all the periodic solutions of (1) is called a *limit cycle* of the differential equation.

One of the most challenging questions for Eq. (1) is the control of the number of limit cycles in families of equations. Is this number finite? Is it bounded?

Despite this interest, the simplest situations are not completely understood yet, as in the one-dimensional "polynomial" case,

$$\frac{dx}{dt} = a_n(t)x^n + a_{n-1}(t)x^{n-1} + \cdots + a_1(t)x + a_0(t), \tag{2}$$

where $x \in \mathbb{R}$, $t \in [0, 1]$ and $a_0, a_1, \ldots, a_n : \mathbb{R} \to \mathbb{R}$, are smooth one-periodic functions. The general

*Partially supported by the DGES grant number BFM2002-04236-C02-2 and CONACIT grant number 2001SGR00173.

problem of studying the number of limit cycles of (2) was proposed by N. G. Lloyd [1973] and C. Pugh (see [Lins Neto, 1980]). Notice that Eq. (2) with $n = 1$ (resp. $n = 2$) is a *linear equation* (resp. a *Riccati equation*). It is well known that linear (resp. Riccati) equations have either a continuum of periodic solutions or at most one limit cycle (resp. two limit cycles), see for instance [Lins Neto, 1980; Lloyd, 1975]. When $n = 3$, Eq. (2) is called *Abel equation*. We will also use the term $(d_1, \ldots, d_r)$*-polynomial*, $d_j \in \mathbb{N}$, to refer to Eq. (2) where $a_j(t) \equiv 0$ if $j \neq d_i$ for all $i = 1, \ldots, r$.

It is known, for instance, that when $a_3(t)$ does not change sign, the (optimal) upper bound for the number of limit cycles of the Abel equation is three, see [Gasull & Llibre, 1990; Lins Neto, 1980; Pliss, 1966]. When $a_3(t) \equiv 1$ this upper bound also holds taking into account complex limit cycles, see [Lloyd, 1973]. Also, when $a_0(t) \equiv 0$ and $a_2(t)$ does not change sign, it is proved in [Gasull & Llibre, 1990] that the maximum number of limit cycles of the Abel equation is again three.

For degrees higher than three, apart from the results of [Lloyd, 1973], three relevant results are those of Lins Neto [1980], Il'yashenko [2000], and Panov [1998]:

(a) In [Lins Neto, 1980], it is proved that there is no upper bound for the number of limit cycles for Abel equations, see also [Panov, 1999]. In particular, it is shown that there are (3, 2)-polynomial equations with at least ℓ solutions, for any natural number ℓ; these examples can be easily extended to $(n, 3, 2)$-polynomial equations. The degree of the polynomials, however, increases with ℓ; this is why Il'yashenko [2004], selected the problem of finding an upper bound in terms of n and the maximum degree of the polynomials $a_j(t)$ as a relevant topic in differential equations.

(b) In [Il'yashenko, 2000], the case of Eq. (2) with $a_n(t) \equiv 1$ is considered and the author is able to give an upper bound (nonrealistic in his own words) for the number of limit cycles of this equation in terms of the bounds of the absolute values of the rest of coefficients of the equation, $a_j(t), j = 0, 1, \ldots, n-1$. This result is coherent with the result of the $(n, 3, 2)$-polynomial equations quoted above; in that case, the systems having ℓ limit cycles are such that when we force the leading coefficient to be one we get that the bounds of the absolute values of the rest of coefficients increase with ℓ.

(c) In [Panov, 1998], the author proves that differential equations of the form

$$\frac{dx}{dt} = x^{2k+1} + a_2(t)x^2 + a_1(t)x + a_0(t)\,, \quad (3)$$

$k \geq 1$, have at most three limit cycles, taking into account their multiplicities (a nice generalization of the result for Abel equations). See also [Andersen & Sandqvist, 1999].

Observe that the leading coefficient in the last two mentioned results is 1. In most cases they are also valid when $a_n(t)$ does not vanish. Hence, it seems that the sign invariance of some of the functions $a_j(t)$ is crucial in order to get bounds on the number of limit cycles of Eq. (2). In each case, however, the question is choosing the terms for which the sign invariance ensures a bounded number of limit cycles.

In Sec. 3.3, we prove the following extension of the $(n, 3, 2)$-polynomial case given in [Lins Neto, 1980] and stated above in item (a):

Proposition 1. *Given any natural number ℓ and fixed natural numbers $p > n > m \geq 2$, there exist equations of the form*

$$\frac{dx}{dt} = \tilde{\varepsilon}x^p + \varepsilon f(t)x^n + a(t)x^m + \delta x\,, \quad (4)$$

with f and a trigonometrical polynomials; *$|\tilde{\varepsilon}|$ small enough or $\tilde{\varepsilon} = 0$, and $|\delta|$ also small enough or $\delta = 0$, which have at least ℓ limit cycles.*

An interpretation of Proposition 1 tells us that the number of limit cycles of a general $(p, n, m, 1)$-polynomial equation is not bounded if the sign invariance is assumed only on $a_1(t)$ or on $a_p(t)$, with $p \geq 4$. In particular, when $\tilde{\varepsilon} = 0$ and so we consider the $(n, m, 1)$-polynomial equation, this gives a hint that the sign invariance must be imposed either on $a_m(t)$ or $a_n(t)$ (concordant with the above mentioned results on the Abel equation).

According to the exposed background, the present paper tries to step ahead in the understanding of the number of limit cycles of Eq. (2) by considering the natural continuations of the literature: the $(n, m, 1)$-polynomial and $(n, 2, 1, 0)$-polynomial equations. More precisely, the first family that we study is of the form

$$\frac{dx}{dt} = a_n(t)x^n + a_m(t)x^m + a_1(t)x, \quad (5)$$

where $n > m > 1$, and at least one of the functions a_n or a_m does not change sign. The second family is

$$\frac{dx}{dt} = a_n(t)x^n + a_2(t)x^2 + a_1(t)x + a_0(t) \quad n > 2, \tag{6}$$

with the function a_n not changing sign.

The results that we obtain from Eq. (5) mimic the following nice corollary of Budan–Fourier Theorem:

Lemma 2. *For any $a_n, a_m, a_1 \in \mathbb{R}$ and any $n > m > 1$, the polynomial equation*

$$a_n x^n + a_m x^m + a_1 x = 0$$

has at most five real solutions if n is odd and at most four real solutions if n is even. Furthermore, except for $n = 3$, the above upper bounds cannot be improved.

Our results from Eq. (5) are collected in the next theorem, where we establish the maximum number of limit cycles. Notice the parallelisms with Lemma 2 which, in turn, ensures the feasibility of this maximum number.

Theorem 3. *Consider the one-periodic generalized Abel equation (5),*

$$\frac{dx}{dt} = a_n(t)x^n + a_m(t)x^m + a_1(t)x,$$

with $n > m > 1$, and a_n, a_m and a_1 being C^1 functions. Assume that $a_n(t)$ or $a_m(t)$ does not change sign. Then,

(a) *If n is odd, Eq. (5) has at most five limit cycles. Furthermore, apart from the limit cycle $x = x(t) \equiv 0$, in each region $\mathcal{D}^+ := \{x > 0\}$ or $\mathcal{D}^- := \{x < 0\}$ one and only one of the following possibilities can occur*:
   (i) *The differential equation has no limit cycles,*
   (ii) *The differential equation has a unique hyperbolic limit cycle,*
   (iii) *The differential equation has a unique semi-stable limit cycle,*
   (iv) *The differential equation has exactly two hyperbolic limit cycles,*

   and all them are realizable if $n \geq 5$.

(b) *If n is even, Eq. (5) has at most four limit cycles. Furthermore apart from the limit cycle $x = x(t) \equiv 0$, in each region $\mathcal{D}^\pm$ defined above, only one of the above possibilities can occur, taking into account that never more than four limit cycles can coexist, and that the semi-stable limit cycle counts as two limit cycles.*

Remark 4. When we apply Theorem 3 to Abel differential equations, i.e. Eq. (2) with $n = 3$ and $a_0 = 0$, we get that when $a_3(t)$ does not change sign, the Abel equation has at most five limit cycles. As we have already said, this result can be refined to an upper bound of three limit cycles, see [Gasull & Llibre, 1990; Lins Neto, 1980; Pliss, 1966]. See also Theorem 5. Also when $n = 3$ and a_2 does not change sign, the optimal bound is again of three limit cycles, see [Gasull & Llibre, 1990].

Our main result about the second family we consider is the following:

Theorem 5. *Consider the one-periodic generalized Abel equation*

$$\frac{dx}{dt} = a_n(t)x^n + a_2(t)x^2 + a_1(t)x + a_0(t),$$

with a_n, a_2, a_1, a_0 being $\mathcal{C}^1$ functions. Assume that $a_n(t)$ does not change sign. Then,

(a) *If $n \geq 3$ is odd, Eq. (6) has at most three limit cycles taking into account their multiplicities.*
(b) *If $n \geq 4$ is even, for any $\ell \in \mathbb{N}$, there is an equation of type (6) having at least ℓ limit cycles.*

Part (a) of Theorem 5 is essentially the same result as that proved by Panov [1998]. Our proof is different from the one given there. Part (b) is a new result. A smart suggestion of Colin Christopher allowed us to unblock the proof of this result for $n > 4$.

Let us end this introduction pointing out a couple of remarks:

Remark 6. Although the above theorems are stated for differential equations of the form (2) that are one-periodic in t, it is easy to see that they also hold when we remove the periodicity hypothesis and we consider differential equations of the same form defined in a neighborhood of the strip $\{(t, x) \in [0, 1] \times \mathbb{R}\}$.

Remark 7. Proposition 1 with $\delta = 0$ and Theorem 3 show that the generalized one-periodic Abel equation

$$\frac{dx}{dt} = x^n + a_m(t)x^m + a_q(t)x^q, \tag{7}$$

where q, m, n are natural numbers satisfying $q < m < n$ and $n > 3$, can have an arbitrary number of limit cycles if $q \geq 2$, but cannot have more than five limit cycles if $q = 1$. Although the problem is beyond the aim of this paper, it seems interesting to elucidate the bifurcation phenomena that can occur when n, m, a_m and a_q are fixed and q varies along the interval $[1, 2]$.

The rest of the paper is organized as follows. In Sec. 2 we give some preliminary results addressed to prove Theorems 3 and 5. The main results are proved in Sec. 3. Finally, Sec. 4 is devoted to remark how these results can be used to study the maximum number of limit cycles of several families of autonomous planar polynomial vector fields.

2. Preliminary Results

Consider a smooth one-periodic ordinary differential equation of the form

$$\frac{dx}{dt} = S(t, x), \tag{8}$$

defined in a neighborhood of the strip $\{(t, x) \in [0, 1] \times \mathbb{R}\}$. Let L_0 and L_1 be the straight lines $\{(t, x) : t = 0\}$ and $\{(t, x) : t = 1\}$, respectively. Whenever it is defined, we can consider the *return map* $h : L_0 \to L_1$ given as follows: if $y \in L_0$, then

$$h(y) = \bar{x}(1; y), \tag{9}$$

where $x = \bar{x}(t; y)$ denotes the solution of (8) such that $\bar{x}(0; y) = y$. Notice that the periodic solutions of the differential equation correspond to the solutions having initial conditions of the form $(t, x) = (0, y)$, being y fixed points of h. The *multiplicity of a periodic solution* $x = \bar{x}(t; y)$ is defined by the multiplicity of y as a zero of the function $h(y) - y$. Simple solutions are called *hyperbolic* limit cycles.

Next two lemmas will be useful to prove Theorem 3. The first one takes advantage of the structure of systems of type (5) to simplify the expression of $h'(y)$. The second one says that, in some sense, a specific one parametric family of equations of the form (8) behaves as a rotatory family of planar vector fields, see Sec. 4.6 in [Perko, 2001].

Lemma 8. *Consider the functions* $\Phi, \Psi : I \subset \mathbb{R} \to \mathbb{R}$, *defined as*

$$\Phi(y) = (n - m) \int_0^1 a_n(t)\bar{x}^{n-1}(t; y)dt + (1 - m) \int_0^1 a_1(t)dt, \tag{10}$$

$$\Psi(y) = (m - n) \int_0^1 a_m(t)\bar{x}^{m-1}(t; y)dt + (1 - n) \int_0^1 a_1(t)dt, \tag{11}$$

where $\bar{x}(t; y)$ *denotes the solution of the differential equation* (5) *such that* $\bar{x}(0; y) = y$. *Then,*

$$h'(y) = \begin{cases} \exp(\Phi(y)) = \exp(\Psi(y)), & \text{if } y \neq 0, \\ \exp\left(\int_0^1 a_1(t)dt\right) = \exp\dfrac{\Phi(0)}{1 - m} \\ \qquad = \exp\dfrac{\Psi(0)}{1 - n}, & \text{if } y = 0, \end{cases} \tag{12}$$

where h *is the return map given in* (9).

Proof. For the general Eq. (8) it is known (see [Lloyd, 1979]) that

$$h'(y) = \exp \int_0^1 \frac{\partial S}{\partial x}(t, \bar{x}(t; y))dt,$$

where $\bar{x}(t; y)$ denotes the solution of the differential equation such that $\bar{x}(0, y) = y$. In our case, we obtain

$$h'(y) = \exp\left\{\int_0^1 na_n(t)\bar{x}^{n-1}(t; y)dt + \int_0^1 ma_m(t)\bar{x}^{m-1}(t; y)dt + \int_0^1 a_1(t)dt\right\},$$

from which the expression of $h'(0)$ in (12) is directly obtained.

Notice also that, using definitions (10) and (11), $h'(y)$ can be written as

$$\begin{aligned} h'(y) &= \exp\{\Phi(y) + mZ(y)\} \\ &= \exp\{\Psi(y) + nZ(y)\}, \end{aligned}$$

where

$$Z(y) := \int_0^1 a_n(t)\bar{x}^{n-1}(t; y)dt + \int_0^1 a_m(t)\bar{x}^{m-1}(t; y)dt + \int_0^1 a_1(t)dt.$$

On the other hand, if $x = \bar{x}(t; y)$ is a nonzero periodic solution of (5) we get

$$\frac{1}{\bar{x}(t; y)}\left(\frac{\partial \bar{x}(t; y)}{\partial t}\right) = a_n(t)\bar{x}^{n-1}(t; y) + a_m(t)\bar{x}^{m-1}(t; y) + a_1(t),$$

and, integrating from 0 to 1, we obtain

$$Z(y) = \int_0^1 a_n(t)\bar{x}^{n-1}(t;y)dt + \int_0^1 a_m(t)\bar{x}^{m-1}(t;y)dt + \int_0^1 a_1(t)dt = 0.$$

∎

Lemma 9. *Consider the one-parametric family of one-periodic nonautonomous equations*

$$\frac{dx}{dt} = f(t,x) + \varepsilon\, x, \tag{13}$$

where $f(t,0) \equiv 0$. Assume that for $\varepsilon = 0$ it has a nonzero semi-stable limit cycle $x = \bar{x}(t;y^*)$. Then for $|\varepsilon|$ small enough and with the suitable sign, Eq. (13) has at least two limit cycles in a small neighborhood of $x = \bar{x}(t;y^*)$.

Proof. Define $D_\varepsilon(y) = h_\varepsilon(y) - y$ as the displacement map associated with the solutions $\bar{x}_\varepsilon(t;y)$ of (13) such that $\bar{x}_\varepsilon(0;y) = y$.

Let us suppose for instance that $x = \bar{x}_0(t;y^*) > 0$ is a semi-stable limit cycle, stable from below and unstable from above. The other possible cases can be studied in a similar way.

This means that $D_0(y)$ is positive in a punctured neighborhood of y^*. Take, then, two numbers $y_1 \lesssim y^* \lesssim y_2$ such that $D_0(y_i) > 0$ for $i = 1,2$. By continuity of the solutions of (13) with respect to parameters, for $|\varepsilon|$ small enough, $D_\varepsilon(y_i) > 0$ for $i = 1,2$.

Since the curve $x = \bar{x}_0(t;y^*)$ is a solution of the differential equation when $\varepsilon = 0$, when we consider the flow of (13) for $\varepsilon \neq 0$, we get that the flow crosses it upwards (resp. downwards) when ε is positive (resp. negative). So, when $\varepsilon < 0$ we get that $D_\varepsilon(y^*) < 0$, and from Bolzano's Theorem, D_ε has at least two zeros near $y*$; that is to say, the differential equation (13) has at least two limit cycles near the limit cycle, as we wanted to prove. ∎

3. Proof of the Main Results

3.1. *Proof of Theorem 3*

Consider first the case n odd. We can restrict our attention to the region $\mathcal{D}^+ := \{(t,x) : 0 \leq t \leq 1, x > 0\}$ and to the case $a_n(t) \geq 0$. This restriction can be achieved by means of one of the following changes of variables: $(t,x) \to (t,-x)$, $(t,x) \to (1-t,x)$ or $(t,x) \to (1-t,-x)$.

We will prove that $\Phi(y)$ is an increasing function for $y > 0$ and we will see how this fact excludes the possibility of having three limit cycles in $\mathcal{D}^+$.

To prove the increase of $\Phi(y)$ for $y > 0$, take $0 < y_1 < y_2$. Of course, the solutions of (5) with these initial conditions will satisfy $0 < \bar{x}(t;y_1) < \bar{x}(t;y_2)$ and, as a consequence,

$$\int_0^1 a_n(t)\bar{x}^{n-1}(t;y_1)dt < \int_0^1 a_n(t)\bar{x}^{n-1}(t;y_2)dt.$$

From the expression of Φ given in (10) we get that $\Phi(y_1) < \Phi(y_2)$.

By using $h'(y) = \exp(\Phi(y))$ from Lemma 8, and the fact that two hyperbolic consecutive limit cycles must have different stability, it turns out that we could have three limit cycles starting at the points y_1, y_2 and y_3, with $0 < y_1 < y_2 < y_3$, only if $\Phi(y_1) < \Phi(y_2) = 0 < \Phi(y_3)$; that is, only if $h'(y_1) < h'(y_2) = 1 < h'(y_3)$. Thus, in case that the three limit cycles exist the middle one is semi-stable.

At this point, we can change our equation by adding the parameter ε as in (13). From Lemma 9, the new equation, which is again of the form (5) with the same $a_n(t)$, has four limit cycles. This fact is a contradiction with the upper bound proved above. Hence, the upper bound of two limit cycles is proved, the closest to the origin being stable and the other unstable; consequently, the upper bounds given in the theorem follow when n is odd.

In the case of n even, the main change in the proof is that, when $a_n(t) \geq 0$, the function Φ is increasing for all $y \in \mathbb{R}$ at which $h(y)$ is defined (in the case n is odd it has a parabola shape with a minimum at $y = 0$). This difference forces the existence of at most four limit cycles starting at the points $y_1 < 0 < y_2 < y_3$. Notice that Φ is negative at the points $y_1, 0$ and y_2, but, by Lemma 8, the stability of the origin is given by the sign of $-\Phi(0)$, which provides the possibility of sign alternance and so, the existence of alternate stable/unstable limit cycles.

It is not difficult to construct examples under the hypotheses of the theorem presenting each one of the different configurations of limit cycles stated in it. It suffices to take into account Lemma 2 and consider functions a_n, a_m and a_1 with suitable constant values.

The proof when $a_m(t)$ does not change sign is similar. The main difference is that when we use Lemma 8, we deal with Ψ instead of Φ. ∎

3.2. *Proof of Theorem 5*

Following the same arguments as in the proof of Theorem 3, in this case we can assume that $a_n(t) \geq 0$.

Part (a) can be proved by using (see [Lloyd, 1979]) the fact that the third derivative of the return map h of a differential equation of type (8) satisfies

$$h'''(y) = h'(y)\left[\frac{3}{2}\left(\frac{h''(y)}{h'(y)}\right)^2 + \int_0^1 \frac{\partial^3 S}{\partial x^3}(t, \bar{x}(t;y)) \times \exp\left\{2\int_0^t \frac{\partial S}{\partial x}(s, \bar{x}(s;y))ds\right\}dt\right]$$
$$= n(n-1)(n-2)\int_0^1 a_n(t)\, x(t;y)^{n-3}dt > 0.$$

Notice that the solutions satisfying $x(0) = x(1)$ correspond to zeroes of $h(y) - y$. If $h(y)$ had four zeroes (taking into account multiplicities), applying Rolle's Theorem successively to $h(y) - y$, $h'(y) - 1$ and $h''(y)$, we would infer that $h'''(y)$ would vanish at least once. This would contradict the above inequality and so, Eq. (6) with odd n can have at most three solutions satisfying $x(0) = x(1)$, taking into account their multiplicities.

Notice the importance of the even power in x^{n-3}, which differentiates part (a) from part (b).

The starting point to prove part (b) is considering an equation of the form

$$\dot{z} = \tilde{f}(t)z^3 + \tilde{a}(t)z^2, \tag{14}$$

with $\tilde{f}(t)$ and $\tilde{a}(t)$ trigonometrical polynomials and having, at least, ℓ hyperbolic limit cycles. The existence of this equation is proved in [Lins Neto, 1980], see also Proposition 1 with $\tilde{\varepsilon} = \delta = 0$, $m = 3$ and $n = 2$. As a second step in the proof, we assert that given any even number n, we can construct another equation of the form

$$\dot{z} = \binom{n}{3} f(t)^{n-3}z^3 + a(t)z^2, \tag{15}$$

having also, at least, ℓ hyperbolic limit cycles and having the functions $f(t)$ and $a(t)$ of class $\mathcal{C}^M$, for any $M \in \mathbb{N}$. Indeed if we only impose f to be continuous, the result is trivial because it suffices to take

$$f(t) = \sqrt[n-3]{\tilde{f}(t)\Big/\binom{n}{3}}.$$

The regularization of the function f will be achieved through a scaling of the time which increases the order of the zeroes of $\tilde{f}$.

Let us prove the above assertion on the existence of such Eq. (15). Denote by $t_1, \ldots, t_r$ the zeroes of $\tilde{f}(t)$. Fix any natural odd number $N \geq 3$ and an integer $k \geq N$. Consider also a $\mathcal{C}^k$ increasing function $\psi(t)$ in $[0,1]$, satisfying $\psi(0) = 0, \psi(1) = 1$, $\psi(t_j) = t_j$ and $\psi^{(d)}(t_j) = 0$ for all $j = 1, \ldots, r$ and all $d = 1, \ldots, N-1$. We also assume that ψ' only vanishes in $[0,1]$ at the values $t_1, \ldots, t_r$.

If we apply the change of time $t = \psi(\tau)$, Eq. (14) is written as

$$z' = \tilde{f}(\psi(\tau))\psi'(\tau)z^3 + \tilde{a}(\psi(\tau))\psi'(\tau)z^2.$$

Notice that the zeroes of $\hat{f}(\tau) := \tilde{f}(\psi(\tau))\,\psi'(\tau)$, are also $\tau = t_j, j = 1, 2, \ldots, r$. Furthermore, for each j, $\tau = t_j$ has at least multiplicity $2N - 1$ for $\hat{f}(\tau)$. By defining

$$f(\tau) := \sqrt[n-3]{\hat{f}(t)\Big/\binom{n}{3}}$$

we get a function whose regularity is at least $\mathcal{C}^M$, M being the integer part of $(2N-1)/(n-3)$. Note that, since N is arbitrary, M can also be chosen arbitrarily.

Finally we consider the following perturbation of Eq. (15):

$$\dot{z} = \sum_{k=4}^{n} \epsilon^{k-3}\binom{n}{k} f(t)^{n-k}z^k + \binom{n}{3} f(t)^{n-3}z^3 + a(t)z^2. \tag{16}$$

Equation (16) maintains the ℓ limit cycles for small $|\epsilon| \neq 0$ and, moreover, carrying out the change of variables

$$x(t) = z(t) + \frac{f(t)}{\epsilon},$$

it is transformed into

$$\dot{x} = \epsilon^{n-3}x^n + C_{n,2}(\epsilon, t)x^2 + C_{n,1}(\epsilon, t)x + C_{n,0}(\epsilon, t), \tag{17}$$

with $C_{n,j}(\epsilon, t)$, $j = 1, 2, 3$ being polynomial expressions in $f(t), f'(t), a(t)$ and $1/\epsilon$. Equation (17) gives the searched example because this equation is of the form $\dot{x} = a_n(t)x^n + a_2(t)x^2 + a_1(t)x + a_0(t)$, with n even and $a_n(t) \neq 0$ for all t. ∎

3.3. *Proof of Proposition 1*

This proof is an adaptation of the one given in [Lins Neto, 1980] for the case $n = 3$, $m = 2$.

Consider the equation

$$\frac{dx}{dt} = \varepsilon f(t)x^n + a(t)x^m, \quad t \in [0, 1]. \tag{18}$$

Define $A(t) = \int_0^t a(s)ds$ and take $a(t)$ such that $A(1) = 0$. Call $\overline{A} = \max_{t\in[0,1]} |A(t)|$. Denoting by $\varphi_\varepsilon(t, x, \varepsilon)$ the solution of (18), for $\varepsilon = 0$ and $|x| \le ((m-1)\overline{A})^{1/(1-m)}$ we get that

$$\varphi_0(t,x) = x\left(\frac{1}{1-(m-1)A(t)x^{m-1}}\right)^{1/(m-1)}. \tag{19}$$

We write $\varphi_\varepsilon(t, x, \varepsilon)$ in powers of ε as

$$\varphi_\varepsilon(t,x,\varepsilon) = \varphi_0(t,x) + \varepsilon W(t,x) + \varepsilon^2 R(t,x,\varepsilon),$$

where

$$W(t,x) = \frac{\partial\varphi}{\partial\varepsilon}(t,x,0), \quad W(0,x) = 0.$$

Following typical perturbative arguments, one can see that if $W(x) := W(1, x)$ has a simple root at x_0, then for small values of ε, the function $\varphi_\varepsilon(t, x, \varepsilon)-x$ has also a simple root close to x_0. The question, now, is choosing $f(t)$ and $a(t)$ such that W has an arbitrary number of different simple solutions. We wish to compute W in terms of f and a; by (18),

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial\varepsilon}\right)
&= \frac{\partial\dot\varphi}{\partial\varepsilon} = \frac{\partial}{\partial\varepsilon}\left(\varepsilon f(t)\varphi^n + a(t)\varphi^m\right)\\
&= f(t)\left(\varphi_0 + \varepsilon W + \varepsilon^2 R\right)^n + \varepsilon f(t) n(\varphi_0\\
&\quad + \varepsilon W + \varepsilon^2 R)^{n-1}(W + 2\varepsilon R) + a(t)m(\varphi_0\\
&\quad + \varepsilon W + \varepsilon^2 R)^{m-1}(W + 2\varepsilon R)\\
&= f(t)\varphi_0^n + a(t)m\varphi_0^{m-1}W + \mathcal{O}(\varepsilon).
\end{aligned} \tag{20}$$

Restricting to $\varepsilon = 0$ and using (18), we get:

$$\frac{dW(t,x)}{dt} = f(t)\varphi_0^n + m\frac{\dot\varphi_0}{\varphi_0^m}\varphi_0^{m-1}W, \tag{21}$$

which can be written as:

$$\frac{d}{dt}\left(\frac{W}{\varphi_0^m}\right) = f(t)\varphi_0^{n-m}. \tag{22}$$

Integrating on both sides of Eq. (22), we get:

$$W(x) = x^n\int_0^1 \frac{f(t)}{(1-(m-1)A(t)x^{m-1})^{(n-m)/(m-1)}}dt. \tag{23}$$

Set $y = x^{m-1}$, $\alpha := (n-m)/(m-1)$ and $a(t) = (2\pi/(m-1))\cos(2\pi t)$. Instead of studying the zeroes of $W(x)$ we consider the function

$$\begin{aligned}
H_f(y) &:= \int_0^1 \frac{f(t)}{(1-(m-1)A(t)y)^\alpha}dt\\
&= \int_0^1 \frac{f(t)}{(1-\sin(2\pi t)y)^\alpha}dt.
\end{aligned}$$

Fix $\ell \in \mathbb{N}$ and an arbitrary polynomial of degree ℓ, $p(y)$; we will prove that there exists $f(t)$ of the form

$$f(t) = \sum_{j=0}^{\ell} \beta_j f_j(t),$$

where

$$f_j(t) = \sin^j(2\pi t), \quad \beta_j \in \mathbb{R},$$

such that

$$H_f(y) = p(y) + O(y^{\ell+1}). \tag{24}$$

From the above result, the fact that $H_f(y)$ has at least ℓ simple zeroes in a small neighborhood of the origin is a straightforward consequence of Lemmas 1 and 2 of [Lins Neto, 1980]. From the definition of $H_f(y)$ the same result holds for $W(x)$, as we wanted to see.

Hence it remains to prove (24). We strongly use that H_f is linear with respect to f. Notice that

$$\begin{aligned}
H_{f_j}(y) &= \sum_{k=0}^{\ell}\left[\binom{\alpha+k-1}{k}\int_0^1 \sin^{j+k}(2\pi t)dt\right]y^k\\
&\quad + O(y^{\ell+1})\\
&=: \sum_{k=0}^{\ell} c_{j,k}y^k + O(y^{\ell+1}).
\end{aligned}$$

Observe also that if we define the matrix $C = (c_{j,k})_{j,k=0}^{\ell}$, then

$$\det C = \prod_{k=0}^{\ell}\binom{\alpha+k-1}{k}\det G,$$

where G is a new matrix, $G = (g_{j,k})_{j,k=0}^{\ell}$, with $g_{j,k} = \int_0^1 \sin^{j+k}(2\pi t)dt$. Since the matrix G is the matrix of the inner product in $\langle f_0, f_1, \ldots, f_\ell\rangle$ defined by $f \cdot g = \int_0^1 f(t)g(t)dt$ we get that $\det G \neq 0$. Thus from the linearity of H_f it is clear that given any polynomial $p(y)$ there exist unique values $\beta_0, \ldots, \beta_\ell$ such that its associated f satisfies (24), as we wanted to prove.

To end the proof of Proposition 1, it suffices to consider the following perturbation of (18):

$$\frac{dx}{dt} = \tilde{\varepsilon} x^p + \varepsilon f(t) x^n + a(t) x^m + \delta x. \qquad (25)$$

Take then a system with $\tilde{\varepsilon} = \delta = 0$ and ℓ simple limit cycles. For $|\tilde{\varepsilon}|$ and $|\delta|$ small enough, we can ensure that all of them persist. ■

4. Consequences for Planar Vector Fields

The Abel equation has been specially used to study the maximum number of limit cycles of several families of autonomous planar polynomial vector fields. In particular, the study of the number of limit cycles surrounding the origin of systems with homogeneous nonlinearities (see [Carbonell & Llibre, 1988; Cherkas, 1976; Lins Neto, 1980]), or the so-called quadratic-like cubic systems (see [Gasull & Prohens, 1996] and [Lloyd *et al.*, 1997]),

$$\begin{cases} \dot{x} = -y + \lambda x + p(x,y) + x f(x,y), \\ \dot{y} = x + \lambda y + q(x,y) + y f(x,y), \end{cases}$$

with p, q and f homogeneous quadratic polynomials, can be reduced to the study of the number of limit cycles of (2), with $n = 3$.

The study of the limit cycles of other families of planar polynomial systems can also be reduced to the study equation of (2). We describe some of these families in the sequel:

- The family of systems
$$\begin{cases} \dot{x} = P_{n+m}(x,y) + x f(x,y), \\ \dot{y} = Q_{n+m}(x,y) + y f(x,y), \end{cases}$$
where $f(x,y) = f_{n-1}(x,y) + \sum_{i=2}^{k} f_{n+im-1}(x,y)$, $n, m, k \in \mathbb{N}$, $k \geq 2$, P_{n+m} and Q_{n+m} being homogeneous polynomials of degree $n+m$ and the functions $f_j(x,y)$ also homogeneous polynomials of degree j. This family is an extension of the one considered in [Giné & Llibre, 2004].
- The so-called polynomial rigid systems (see [Conti, 1994]),
$$\begin{cases} \dot{x} = -y + x f(x,y), \\ \dot{y} = x + y f(x,y), \end{cases} \qquad (26)$$
where f is an arbitrary polynomial function.
- The cubic polynomial systems studied in [Devlin *et al.*, 1998].

For each one of the above families our results on (2) can be applied to get a criterion for an upper bound, under some additional hypotheses, of the number of limit cycles of the corresponding planar differential system. Instead of carrying this out case by case we only present a representative example:

Consider the particular case of system (26) with $f(x,y) = f_0 + f_{m-1}(x,y) + f_{n-1}(x,y)$, $f_i(x,y)$ being homogeneous polynomials of degree i and $0 < m < n$. In this case, in polar coordinates, the system is written as

$$\begin{aligned} \frac{dr}{d\theta} = {} & f_{n-1}(\cos\theta, \sin\theta) r^n \\ & + f_{m-1}(\cos\theta, \sin\theta) r^m + f_0 r, \end{aligned} \qquad (27)$$

which is a differential equation of the form (5). Hence, Theorem 3 implies that when either $f_{n-1} \times (\cos\theta, \sin\theta)$ or $f_{m-1}(\cos\theta, \sin\theta)$ does not change sign, the differential equation (26) has at most two limit cycles, a result already proved in [Freire *et al.*, 2006]. Notice that the upper bound of two limit cycles comes from Theorem 3 because $\mathbb{R}^2$ corresponds to the region $r \geq 0$. Hence the limit cycles appearing in the region $r \leq 0$ for (27) do not correspond with real limit cycles of (26). It can be easily seen that this bound is indeed attained for some planar polynomial systems.

Acknowledgment

The authors wish to express their gratitude to Colin Christopher for his useful comments.

References

Andersen, K. M. & Sandqvist, A. [1999] "Existence of closed solutions of an equation $\dot{x} = f(t,x)$, where $f'_x(t,x)$ is weakly convex or concave in x," *J. Math. Anal. Appl.* **229**, 480–500.

Carbonell, M. & Llibre, J. [1988] "Limit cycles of a class of polynomial systems," *P. Roy. Soc. Edinb. A* **109**, 187–199.

Cherkas, L. A. [1976] "Number of limit cycles of an autonomous second-order system," *Diff. Eq.* **5**, 666–668.

Conti, R. [1994] *Uniformly Isochronous Centres of Polynomial Systems in* $\mathbb{R}^2$, Lecture Notes in Pure and Applied Mathematics, Vol. 152 (M. Dekker, NY), pp. 21–31.

Devlin, J., Lloyd, N. G. & Pearson, J. M. [1998] "Cubic systems and Abel equations," *J. Diff. Eqs.* **147**, 435–454.

Fossas-Colet E. & Olm-Miras, J. M. [2002] "Asymptotic tracking in dc-to-dc nonlinear power converters E," *Discr. Cont. Dyn. Syst. B* **2**, 295–307.

Freire, E., Gasull, A. & Guillamon, A. [2006] "Limit cycles and Lie symmetries," *Bull. Sci. Math.*, to appear.

Gasull, A. & Llibre, J. [1990] "Limit cycles for a class of Abel equations," *SIAM J. Math. Anal.* **21**, 1235–1244.

Gasull, A. & Prohens, R. [1996] "Quadratic and cubic systems with degenerate infinity," *J. Math. Anal. Appl.* **198**, 25–34.

Giné, J. & Llibre, J. [2004] "Darboux integrability and limit cycles for a class of polynomial differential systems," in *Differential Equations with Symbolic Computation*, Trends in Mathematics, pp. 55–65.

Il'yashenko, Y. [2000] "Hilbert-type numbers for Abel equations, growth and zeros of holomorphic functions," *Nonlinearity* **13**, 1337–1342.

Il'yashenko, Y. [2004] "Selected topics in differential equations with real and complex time," in *Normal Forms, Bifurcations and Finiteness Problems in Differential Equations*, NATO Sci. Ser. II Math. Phys. Chem., Vol. 137 (Kluwer Acad. Publ., Dordrecht), pp. 317–354.

Lins Neto, A. [1980] "On the number of solutions of the equation $dx/dt = \sum_{j=0}^{n} a_j(t)x^j$,$0 \leq t \leq 1$ for which $x(0) = x(1)$," *Invent. Math.* **59**, 67–76.

Lloyd, N. G. [1973] "The number of periodic solutions of the equation $\dot{z} = z^N + p_1(t)z^{N-1} + \cdots + p_N(t)$," *P. Lond. Math. Soc.* **27**, 667–700.

Lloyd, N. G. [1975] "On a class of differential equations of Riccati type," *J. Lond. Math. Soc.* **10**, 1–10.

Lloyd, N. G. [1979] "A note on the number of limit cycles in certain two-dimensional systems," *J. Lond. Math. Soc.* **20**, 277–286.

Lloyd, N. G., Christopher, C. J., Devlin, J., Pearson, J. M. & Yasmin, N. [1997] "Quadratic-like cubic systems," *Diff. Eq. Dyn. Syst.* **5**, 329–345.

Panov, A. A. [1998] "The number of periodic solutions of polynomial differential equations," *Math. Notes* **64**, 622–628.

Panov, A. A. [1999] "On the diversity of Poincaré mappings for cubic equations with variable coefficients," *Funct. Anal. Appl.* **33**, 310–312.

Perko, L. [2001] *Differential Equations and Dynamical Systems*, 3rd edition, Texts in Applied Mathematics, Vol. 7 (Springer-Verlag, NY).

Pliss, V. A. [1966] *Non Local Problems of the Theory of Oscillations* (Academic Press, NY).