

# Study of Perturbed Lotka–Volterra Systems Via Abelian Integrals

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We consider the Lotka–Volterra system as a Hamiltonian one and study a special perturbation of it. For this perturbed system we get results on the number of limit cycles. The main tools used are Abelian integrals and degenerate Hopf bifurcation. © 1996 Academic Press, Inc.

## 1. INTRODUCTION

The classical model of Lotka and Volterra

$$\begin{cases} \dot{x} = x(\alpha - \beta y), \\ \dot{y} = y(\gamma x - m), \end{cases} \quad \alpha, \beta, \gamma, \text{ and } m \in \mathbb{R}^+; x, y \in \mathbb{R}^+, \quad (1)$$

is the source of the evolution of the study of deterministic predator–prey systems using ordinary differential equations. Without quitting the two-dimensional case, we find a wide range of models arising from (1). The first ones evolved from trying to give a more general character to the system, where the positive constants were replaced by more general functions (see [5, 8]). A lot of later models have given more specific shapes to those

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functions, according to the concrete biological problems in which they applied (see [6, 10–12, 16, ...]). However, in most of the cases, the “essence” of the Lotka–Volterra system has not been left behind. In other words, the conditions imposed on the above-mentioned functions imply that a lot of models are not “far” from (1).

Taking into account this fact, the study of perturbations of the Lotka–Volterra system acquires a more global interest. A lot of predator–prey systems can be thought of as perturbations of (1). This approach is used, for instance, in [15, Chap. 3], where the authors start from the transformation of (1) into a Hamiltonian system with Hamiltonian function  $H(x, y) = \delta(e^x - x) + (e^y - y)$ . Then, known results on such systems can be applied to study the number of limit cycles that bifurcate from closed orbits of (1).

In [15] this technique is used in a generalization of (1) consisting of taking a general function  $V(x)$ , putting it in (1) instead of  $\beta x$  and changing  $\gamma x$  by  $kV(x)$ , where  $k \in \mathbb{R}^+$ . This new function has a special biological significance since it represents the amount of prey consumed by one predator in a unit of time. Moreover, the system obtained after introducing  $V(x)$  can be understood as a perturbation of the Hamiltonian version of (1), included in the following general family:

$$\begin{cases} \dot{x} = -\frac{\partial H}{\partial y} - \epsilon \varphi(x)f(y), \\ \dot{y} = \frac{\partial H}{\partial x} + \epsilon \psi(x). \end{cases} \quad (2)$$

One of the most interesting problems for such families is the knowledge of their number of periodic orbits. In this work we will study this problem. Our first result asserts that in the class (2), the number of limit cycles (isolated periodic orbits) is not bounded. In Section 3, we provide examples of (2) with an arbitrary number of limit cycles bifurcating from periodic orbits of (1).

We use Abelian integrals to find out the number of limit cycles in a three-parameter perturbation of (1) of type (2) (see Sections 4 and 5). The meaning of the perturbation function is discussed in Section 2. The use of Abelian integrals in Hamiltonian systems comes from Pontryagin (see [13]) and has taken on an important impulse in recent years. They have been studied mainly when both the Hamiltonian function and the perturbation function are polynomials. Nevertheless, in this paper both functions are not polynomials and the study presents different difficulties. In order to have a more precise description of the bifurcation diagram we relate the approach via Abelian integrals to the study of the degenerate Hopf bifurcation at the origin, see Section 5 and Appendix A.

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## 2. FORMULATION IN TERMS OF ABELIAN INTEGRALS

It is known (see [15, Section 3.6]) that system (1) can be transformed into a Hamiltonian system by means of the following change of variables:

$$T = \alpha t; \quad X = \ln \frac{k\beta}{m} x; \quad Y = \ln \frac{\beta}{\alpha} y, \quad \text{where } k = \frac{\gamma}{\beta}. \quad (3)$$

As we pointed out above, we are interested in the number of limit cycles of predator-prey systems sufficiently close to (1). Then we consider the system

$$\begin{cases} \dot{x} = \alpha x - V(x)y, \\ \dot{y} = y(kV(x) - m), \end{cases} \quad (4)$$

where  $\alpha$ ,  $k$ , and  $m$  are positive parameters  $x, y \in \mathbb{R}^+$ . We may suppose that  $V(x)$  is a  $C^1$ -function.

This is one possible generalization of (1) in which we allow a more precise description of the predator functional response.

If we assume that  $V'(x) \neq 0$ , there exists a unique  $x_0 \in \mathbb{R}^+$  such that  $V(x_0) = m/k$ . Set  $y_0 = \alpha x_0 / V(x_0) = (k\alpha/m)x_0$ . The change of variables

$$T = \alpha t; \quad X = \ln \frac{x}{x_0}; \quad Y = \ln \frac{y}{y_0} \quad (3')$$

transforms (4) into the next system in which we have replaced the notation of capital letters by  $t$ ,  $x$ , and  $y$  again,

$$\begin{cases} \dot{x} = 1 - w(x)e^y, \\ \dot{y} = \delta(w(x)e^x - 1), \end{cases} \quad (5)$$

where  $x, y \in \mathbb{R}$ ,  $\delta = m/\alpha$ , and  $w(x) = (x_0/V(x_0))(V(x_0e^x)/x_0e^x) = V(x_0e^x)/(V(x_0)e^x)$ .

We observe that in the Lotka-Volterra system the function  $V(x)$  is linear and so  $w(x) \equiv 1$ . Instead of doing the generalization of (1) by means of system (4), we can consider a system close to (1) as like (5) having  $w(x) = 1 + \epsilon\varphi(x)$ , where  $\epsilon$  is a small parameter and  $\varphi(x)$  a  $C^1$ -function

such that  $\varphi(0) = 0$ . Then, (5) writes as

$$\begin{cases} \dot{x} = (1 - e^y) - \epsilon\varphi(x)e^y, \\ \dot{y} = \delta(e^x - 1) + \epsilon\delta\varphi(x)e^x, \end{cases} \quad (6_\epsilon)$$

or, equivalently,

$$\begin{cases} \dot{x} = -\frac{\partial H}{\partial y} - \epsilon\varphi(x)e^y, \\ \dot{y} = \frac{\partial H}{\partial x} + \epsilon\delta\varphi(x)e^x. \end{cases} \quad (6'_\epsilon)$$

where  $H(x, y) = \delta(e^x - x) + (e^y - y)$ . Then, when  $\epsilon = 0$ , we have the Lotka–Volterra system obtained applying (3) to (1). For  $\epsilon \neq 0$  we have systems close to it, all of them included in the family (2). Observe that

$$V(x_0 e^x) = (1 + \epsilon\varphi(x))V(x_0)e^x = \frac{m}{k} e^x(1 + \epsilon\varphi(x)),$$

or, similarly,

$$V(x) = \frac{m}{k} \frac{x}{x_0} \left( 1 + \epsilon\varphi\left(\ln \frac{x}{x_0}\right) \right), \quad x > 0.$$

It is easily seen that all the solutions of  $(6_0)$  are periodic. In the following we describe the well-known method of Pontryagin (see [13, 14]) for studying the existence of periodic orbits for

$$\begin{cases} \dot{x} = -\frac{\partial H}{\partial y} - \epsilon f(x, y), \\ \dot{y} = \frac{\partial H}{\partial x} + \epsilon g(x, y), \end{cases} \quad (7)$$

and  $\epsilon$  small enough.

We have a continuum of level curves  $\Gamma := \{\Gamma(h)\}_{h \in J}$ , where  $\Gamma(h)$  is the set of the points  $(x, y)$  such that  $H(x, y) = h$  (note that in  $(6_0)$  the solutions lay on these curves). Taking a transversal section  $\Sigma$  of  $\Gamma$  parameterized by  $h$ , we can compute the return map

$$\begin{aligned} \pi: U \subset \Sigma &\rightarrow \Sigma \\ h &\mapsto \pi(h), \end{aligned}$$

where  $U$  is an open set of  $\Sigma$ .

Let  $\Gamma_\epsilon(h)$  be a solution curve of the general perturbed system passing through  $\Gamma(h) \cap U$  at time  $t = 0$ . Consider  $F(h) = \pi(h) - h$ . The zeros of  $F(h)$  represent the intersection between  $U$  and the periodic orbits of (7). It can be proved that

$$F(h) = \epsilon M(h) + o(\epsilon),$$

where

$$M(h) := \int_{\Gamma(h)} f(x, y) dy - g(x, y) dx.$$

Observe that in our case,  $f(x, y) = -\varphi(x)e^y$  and  $g(x, y) = \delta\varphi(x)e^x$  and therefore

$$M(h) = - \int_{\Gamma(h)} \varphi(x)(\delta e^x dx + e^y dy).$$

Let  $(x(t), y(t))$  be the time parameterization of  $\Gamma(h)$  and  $\tau_h$  the period of this closed curve. Therefore,  $dx/dt = 1 - e^y$ ,  $dy/dt = \delta(e^x - 1)$ , and so

$$\begin{aligned} M(h) &= - \int_0^{\tau_h} \varphi(x(t))(\delta e^{x(t)}(1 - e^{y(t)}) + e^{y(t)}\delta(e^{x(t)} - 1)) dt \\ &= - \int_0^{\tau_h} \varphi(x(t))(\delta(e^{x(t)} - 1) - \delta(e^{y(t)} - 1)) dt \\ &= - \int_0^{\tau_h} \varphi(x(t))\left(\frac{dy}{dt} - \frac{dx}{dt}\right) dt \\ &= - \int_{\Gamma(h)} \varphi(x) dy. \end{aligned} \tag{8}$$

We also need the following results, see [13]:

**PROPOSITION 2.1.** (a)  $M(h) = 0$  is a necessary condition for  $\Gamma(h)$  to give rise to periodic solution after perturbations. And  $M'(h) \neq 0$  is a sufficient condition. More precisely, if  $M(h_0) = 0$  and  $M'(h_0) \neq 0$ , then there exists  $\epsilon(h_0) > 0$  and a  $C^1$ -function  $F(\epsilon)$  defined for  $|\epsilon| < \epsilon(h_0)$ , with  $F(0) = h_0$  and such that  $(6_\epsilon)$  has a periodic solution cutting the section  $\Sigma$  at the point  $h = F(\epsilon)$ .

(b) If  $M(h_0) = M'(h_0) = \dots = M^{(k-1)}(h_0) = 0$  and  $M^{(k)}(h_0) \neq 0$ , then for small  $\epsilon \neq 0$  we have at most  $k$  limit cycles (multiplicities taken into account) which tend to  $\Gamma(h)$  as  $\epsilon$  tends to 0.

We call  $\varphi(x)$  the *perturbation function*. When both  $H(x, y)$  and  $\varphi(x)$  are polynomials, the integral (7) is called the *Abelian integral*. Our case will be distinct, but we will extend the use of this term to it.

First of all, it will be seen that systems of class (2) can be constructed having an arbitrarily large number of limit cycles.

### 3. SYSTEMS WITH ANY NUMBER OF LIMIT CYCLES

The purpose of this section is to prove:

**PROPOSITION 3.1.** *Given  $k$  an arbitrary positive integer, there are systems of type (2) with at least  $k$  limit cycles.*

*Proof.* First, consider a system of type

$$\begin{cases} \dot{x}_1 = -f_2(x_2), \\ \dot{x}_2 = f_1(x_1). \end{cases} \quad (9)$$

If we take  $H(x_1, x_2) = \int_0^{x_1} f_1(s) ds + \int_0^{x_2} f_2(s) ds =: F_1(x_1) + F_2(x_2)$ , then (9) can be seen as a Hamiltonian system.

Suppose that  $F_j(x)$ ,  $j = 1, 2$ , are concave functions near the origin and monotonically increasing to infinity as  $x \rightarrow \pm\infty$ .

Note that system (6<sub>0</sub>) is a particular case of (9).

Defining  $g_j(x)$  and  $G_j(x)$  in this way (see [17]),

$$(G_j(x))^2 = F_j(x); \quad g_j(x) = G_j^{-1}(x), \quad \text{for } j = 1, 2,$$

and the new variables  $\xi_j$ ,  $j = 1, 2$ , as those such that  $x_j = g_j(\xi_j)$ , we obtain that

$$F_j(g_j(\xi_j)) = \xi_j^2, \quad \text{for } j = 1, 2, \quad (10)$$

$$\frac{dx_j}{dt} = g'_j(\xi_j) \frac{d\xi_j}{dt} = (-1)^{i+1} f_i(x_i), \quad \text{for } i, j = 1, 2, i \neq j. \quad (11)$$

From (10) we have that

$$f_j(g_j(\xi_j))g'_j(\xi_j) = 2\xi_j, \quad \text{for } j = 1, 2,$$

and joining (11) with the last equality:

$$\frac{d\xi_j}{dt} = \frac{(-1)^{i+1} 2\xi_i}{g'_i(\xi_1)g'_2(\xi_2)}, \quad \text{for } i, j = 1, 2, i \neq j.$$

After the change of time  $d\tau/dt = 2/g'_1(\xi_1)g'_2(\xi_2)$ , this equation becomes:

$$\frac{d\xi_j}{d\tau} = (-1)^{i+1} \xi_i, \quad \text{for } i, j = 1, 2, i \neq j.$$

If the variables  $\xi_1$  and  $\xi_2$  are renamed  $u$  and  $v$ , respectively, and a perturbation term,  $\epsilon p(v)u$ , is added to this system, we get

$$\begin{cases} \frac{du}{d\tau} = -v - \epsilon p(v)u, \\ \frac{dv}{d\tau} = u, \end{cases} \quad (12)$$

where  $\epsilon$  is the perturbation parameter,  $p(v) = a_m v^m + a_{m-2} v^{m-2} + \dots + a_2 v^2$ , the power  $m$  is even, and  $a_{2j} \in \mathbb{R}$ ,  $j = 1, \dots, m/2$ .

The bifurcation function (see Section 2) is

$$M(h) = - \int_{u^2+v^2=h} p(v)u \, dv.$$

If we put  $h = \rho^2$  and change to polar coordinates, parameterizing the level curves by the angle, we have

$$\begin{aligned} M(h) &= - \int_0^{2\pi} (a_m \rho^m \sin^m \theta + a_{m-2} \rho^{m-2} \sin^{m-2} \theta + \dots \\ &\quad + a_2 \rho^2 \sin^2 \theta) \rho^2 \cos^2 \theta \, d\theta \\ &= -\rho^4 \sum_{\substack{i=1 \\ i \text{ even}}}^m a_i \rho^{i-2} \int_0^{2\pi} \sin^i \theta \cos^2 \theta \\ &= -\rho^4 \sum_{\substack{i=1 \\ i \text{ even}}}^m a_i \rho^{i-2} I_i = -h^2 \sum_{k=1}^{m/2} a_{2k} h^{k-1} I_{2k}, \end{aligned} \quad (13)$$

where  $I_{2k} = \int_0^{2\pi} \sin^{2k} \theta \cos^2 \theta \, d\theta = 2\pi((2k-1)!!/(2k+2)!!)$ ,  $k \in \mathbb{N}$ ,  $k \geq 1$ , and  $(-1)!! = 1$ .

Obviously,  $a_{2k}$ ,  $k = 1, \dots, m/2$ , can be chosen such that  $M(h)$  has  $m/2 - 1$  simple positive zeros. Hence, system (12) with these parameters and  $\epsilon$  small enough has, by Proposition 2.1(a), at least  $m/2 - 1$  limit cycles.

From (12), unmaking the changes in the particular case when  $u = (\delta(e^{x_1} - x_1))^{1/2}$  and  $v = (e^{x_2} - x_2)^{1/2}$ , we arrive at the following system of type (2):

$$\begin{cases} \frac{dx_1}{dt} = -(e^{x_2} - 1) - \epsilon (\delta(e^{x_1} - x_1))^{1/2} \frac{p((e^{x_2} - x_2)^{1/2})(e^{x_2} - 1)}{(e^{x_2} - x_2)^{1/2}}, \\ \frac{dx_2}{dt} = \delta(e^{x_1} - 1), \end{cases}$$

which proves the proposition. ■

The above example shows the difficulty of a general study of the number of limit cycles when dealing with a general perturbation of (1), like (2). In the next sections, we will come back to the particular case of (2), system  $(6_\epsilon)$ , and look for the bifurcation diagram for some special cases of  $\varphi(x)$ . The purpose is to use Abelian integrals in easy examples to show which are the key difficulties we can meet in general.

#### 4. A SIMPLE PERTURBATION FUNCTION

We start with a simple bifurcation function for  $(6_\epsilon)$ , to give an idea of the troubles that we meet when we try to determine the number of limit cycles arising, after perturbation, from level curves of the Hamiltonian system. We will prove that:

**THEOREM A.** *Suppose that the perturbation function is  $\varphi(x) = ax + bx^2$ , with  $a$  and  $b \in \mathbb{R}$ . Then, for  $\epsilon$  small enough,*

(a) *if  $ab > 0$ , system  $(6_\epsilon)$  has a unique limit cycle bifurcating from the level curves  $\Gamma(h)$ , for  $h \in (1 + \delta, +\infty)$ .*

(b) *if  $ab < 0$ , system  $(6_\epsilon)$  has no limit cycles originated from a periodic orbit of  $(6_0)$ .*

Before giving the proof of this result, we introduce some notation and study technical tools which will be applied afterwards.

According to (7), we look for the zeroes of  $M(h) = \int_{\Gamma(h)} \varphi(x) dy$ , the bifurcation function.

Set  $I_1(h) = \int_{\Gamma(h)} x dy$  and  $I_2(h) = \int_{\Gamma(h)} x^2 dy$ . Then

$$M(h) = aI_1(h) + bI_2(h) = I_1(h)(a + bp(h)), \quad (14)$$

where  $p(h) := I_2(h)/I_1(h)$ , for  $1 + \delta < h < +\infty$ . Observe that  $I_1(h) = \int_{\Gamma(h)} x dy = \int_{\text{int } \Gamma(h)} dx dy > 0$ .

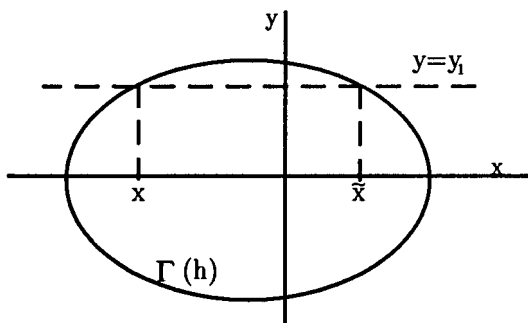
From now on, we set  $J = (1 + \delta, +\infty)$ .

Observe that for any  $h \in J$  and any fixed  $y_1$ ,  $\Gamma(h)$  has two intersection points  $(x \leq 0, \tilde{x} \geq 0)$  with the line  $y = y_1$  (see Fig. 1) satisfying  $e^x - x = e^{\tilde{x}} - \tilde{x}$ .

By considering the auxiliary function  $h(x) = e^x - x$ , we can see that  $h(x) - h(-x) > 0$  if  $x > 0$ . Then, it must be  $|\tilde{x}| \leq |x|$ . This fact is stated in the following remark:

**Remark 4.1.** If  $e^x - x = e^{\tilde{x}} - \tilde{x}$ ,  $x \leq 0$ ,  $\tilde{x} \geq 0$ , then  $|\tilde{x}| \leq |x|$  (Fig. 1).



FIG. 1. Definition of  $\tilde{x}(x)$ .

We also introduce two technical lemmas:

**LEMMA 4.2.** *Let  $f(x, y)$  be a  $C^1$ -function on  $\Gamma(h) = \{(x, y): H(x, y) = \delta(e^x - x) + (e^y - y) = h\}$ . Then*

$$\frac{d}{dh} \int_{\Gamma(h)} f(x, y) dy = \int_{\Gamma(h)} \frac{f_x(x, y)}{H_x(x, y)} dy.$$

*Proof.* We parameterize  $\Gamma(h)$  by  $t \in [0, t(h)]$ . Then

$$\begin{aligned} \frac{d}{dh} \int_{\Gamma(h)} f(x, y) dy &= \frac{d}{dh} \int_0^{t(h)} f(x(t, h), y(t, h)) y_t(t, h) dt \\ &= f(x(t(h), h), y(t(h), h)) y_t(t(h), h) t'(h) \\ &\quad + \int_0^{t(h)} \frac{d}{dh} (f(x(t, h), y(t, h)) y_t(t, h)) dt \\ &= f(x(t(h), h), y(t(h), h)) y_t(t(h), h) t'(h) \\ &\quad + \int_0^{t(h)} \left( (f_x(x(t, h), y(t, h)) x_h(t, h) \right. \\ &\quad \left. + f_y(x(t, h), y(t, h)) y_h(t, h)) y_t(t, h) \right. \\ &\quad \left. + f(x(t, h), y(t, h)) y_{ht}(t, h) \right) dt. \end{aligned}$$

After integration by parts and taking into account that  $x(0, h) = x(t(h), h)$  and  $y(0, h) = y(t(h), h)$ , we reach

$$\begin{aligned} \frac{d}{dh} \int_{\Gamma(h)} f(x, y) dy &= \int_0^{t(h)} f_x(x(t, h), y(t, h)) (x_h(t, h) y_t(t, h) \\ &\quad - x_t(t, h) y_h(t, h)) dt. \end{aligned}$$

The case in which  $t$  is exactly the time of the system  $(6_0)$  gives that

$$x_h(t, h)y_t(t, h) - x_t(t, h)y_h(t, h) = \frac{y_t(t, h)}{H_x(x(t, h), y(t, h))},$$

and so

$$\begin{aligned} \frac{d}{dh} \int_{\Gamma(h)} f(x, y) dy &= \int_0^{t(h)} \frac{f_x(x(t, h), y(t, h))}{H_x(x(t, h), y(t, h))} y_t(t, h) dt \\ &= \int_{\Gamma(h)} \frac{f_x(x, y)}{H_x(x, y)} dy. \end{aligned}$$

We remark that in the transformation  $t \leftrightarrow y$  we have removable singularities in the two points in which  $x = 0$ , since  $H_x(x, y) = \delta(e^x - 1)$ . ■

**LEMMA 4.3.** *With the notation given in (14) the following statements hold.*

(a)  $I_1(h) > 0$ ,  $(d/dh)I_1(h) = T(h) > 0$  and  $(d^2/dh^2)I_1(h) > 0$ , for all  $h \in J$ , where  $T(h)$  is the period of  $\Gamma(h)$ .

(b)  $I_2(h) < 0$ ,  $(d/dh)I_2(h) < 0$  and  $(d^2/dh^2)I_2(h) < 0$ , for all  $h \in J$ .

(c)  $\lim_{h \rightarrow (1+\delta)^+} p(h) = 0$ .

*Proof.* (a) We already know that  $I_1(h) > 0$ . Using the link  $H(x, y) = h$  and Lemma 4.2, we have that  $(d/dh)I_1(h) = \int_{\Gamma(h)} (dy/\delta(e^x - 1))$ . The integrand always has a positive sign because the orbits of  $(6_0)$  turn counterclockwise.

Observe that  $(d^2/dh^2)I_1(h) = (1/\delta^2) \int_{\Gamma(h)} (-e^x/(e^x - 1)^3) dy$ . From this expression, it is not easy to conclude that it is positive. Nevertheless, it is clear that  $(d/dh)I_1(h)$  is, in fact, the period of  $\Gamma(h)$ . Since it is well-known (see [7, Theorem 2; 17, Theorem 2]) that the period in the Lotka–Volterra system is strictly increasing, it follows that  $(d^2/dh^2)I_1(h) > 0$ .

(b) By Remark 4.1,  $I_2(h) = \int_{\Gamma(h)} x^2 dy$  is always negative.

From Lemma 4.2,

$$\frac{d}{dh} I_2(h) = 2 \int_{\Gamma(h)} \frac{x}{\delta(e^x - 1)} dy.$$

For  $x < 0$  we know that  $-x > \tilde{x}$ , and so  $x/(e^x - 1) > \tilde{x}/(e^{\tilde{x}} - 1)$ . This produces a negative sign in the first derivative. Observe also that the integrand is a derivable function on  $\Gamma(h)$ . Hence, we can easily do the second derivative:

$$\frac{d^2}{dh^2} I_2(h) = \frac{2}{\delta^2} \int_{\Gamma(h)} \frac{e^x - 1 - xe^x}{(e^x - 1)^3} dy.$$

Since the numerator is negative for  $x \neq 0$ , this integral becomes negative.

(c) Recall that  $\Gamma(1 + \delta) = \{(0, 0)\}$ . Using the Green Theorem and the Mean Value Theorem,

$$\begin{aligned} p(h) &= \frac{\int_{\Gamma(h)} x^2 dy}{\int_{\Gamma(h)} x dy} = \frac{\int_{D_h} 2x dx dy}{\int_{D_h} dx dy} = \frac{2\bar{x}(h) \int_{D_h} dx dy}{\int_{D_h} dx dy} \\ &= 2\bar{x}(h) \xrightarrow{h \rightarrow 1+\delta} 0, \end{aligned}$$

where in the above expression  $\bar{x}(h)$  is a point in  $D_h$ . ■

The decrease of  $p(h)$  will play a key role in the final proof. We state that:

**PROPOSITION 4.4.** *The function  $p(h)$  introduced in (14) is strictly decreasing and, moreover,  $\lim_{h \rightarrow \infty} p(h) = -\infty$ .*

In order to prove this proposition, we have to see first of all that  $p'(h) < 0$  for all  $h \in J$ . Our proof is inspired in the ideas introduced in [3]. Take  $h_0 \in J$  such that  $p'(h_0) = 0$ . If we prove that  $p(h) - p(h_0) < 0$  when  $0 < |h - h_0| \ll 1$ , we will get a contradiction, and so  $p'(h) < 0$  for all  $h \in J$ .

To this end, we define

$$L_p(h) := I'_2(h) - I'_1(h)p(h_0); \quad g_p(x) := \frac{2x - p(h_0)}{e^x - 1}. \quad (15)$$

These two functions satisfy some special properties:

**LEMMA 4.5.** *With the definitions of (14) and (15) the following statements hold.*

(a) *Suppose that for all  $h$  such that  $0 < |h - h_0| \ll 1$  we have  $(h - h_0)L_p(h) < (>)0$ . Then,  $p(h) - p(h_0) < (>)0$ , for all  $h$ :  $0 < |h - h_0| \ll 1$ .*

(b)  $L_p(h) = (1/\delta) \int_{\Gamma^-(h)} (g_p(x) - g_p(\tilde{x})) dy$ , where  $\Gamma^-(h) = \Gamma(h) \cap \{x \leq 0\}$ .

(c)  $g'_p(x) < 0$ , for all  $x \neq 0$ .

(d)  $(d/dx)(g_p(x) - g_p(\tilde{x}(x))) < 0$ , for all negative  $x$ .

(e) *There exists a unique  $x^* < 0$  such that  $g_p(x^*) - g_p(\tilde{x}(x^*)) = 0$ .*

*Proof of Lemma 4.5.* (a) Define  $\xi(h) := I_2(h)I_1(h_0) - I_2(h_0)I_1(h)$ . Observe that  $\xi(h_0) = 0$ . Hence,

$$\begin{aligned} p(h) - p(h_0) &= \frac{I_2(h)}{I_1(h)} - \frac{I_2(h_0)}{I_1(h_0)} = \frac{I_2(h)I_1(h_0) - I_2(h_0)I_1(h)}{I_1(h_0)I_1(h)} \\ &= \frac{\xi(h) - \xi(h_0)}{I_1(h)I_1(h_0)} = \frac{\xi'(c_h)}{I_1(h_0)} \frac{h - h_0}{I_1(h)} = L_p(c_h) \frac{h - h_0}{I_1(h)}, \end{aligned}$$

for some  $c_h$  between  $h_0$  and  $h$ .

(b) By Lemma 4.3 and definitions (15), we know that

$$\begin{aligned} L_p(h) &= I'_2(h) - I'_1(h)p(h_0) = \int_{\Gamma(h)} \frac{2x - p(h_0)}{\delta(e^x - 1)} dy = \frac{1}{\delta} \int_{\Gamma(h)} g_p(x) dy \\ &= \frac{1}{\delta} \int_{\Gamma^-(h)} (g_p(x) - g_p(\tilde{x}(x))) dy. \end{aligned}$$

(c)  $g'_p(x) = (2(e^x - 1) + e^x(p(h_0) - 2x))/(e^x - 1)^2 := N(x)/(e^x - 1)^2$ . Observe that  $N'(x) = e^x(p(h_0) - 2x)$ . Then,  $N(x)$  has its absolute maximum at  $x = p(h_0)/2$ . Since  $N(p(h_0)/2) < 0$  we have that  $g'_p(x) < 0$  for all  $x$ .

(d)

$$\frac{d}{dx}(g_p(x) - g_p(\tilde{x}(x))) = g'_p(x) - g'_p(\tilde{x}(x))\tilde{x}'(x).$$

Recall that  $e^x - x = e^{\tilde{x}} - \tilde{x}$ . From this equality, we get that  $\tilde{x}'(x) = (e^x - 1)/(e^{\tilde{x}(x)} - 1)$ . Then, since  $x < 0$  and  $\tilde{x}(x) > 0$ ,  $\tilde{x}'(x) < 0$ .

It follows (using also (c)) that  $(d/dx)(g_p(x) - g_p(\tilde{x}(x))) < 0$ , for all negative  $x$ .

(e) Adding to (d) the facts

$$\begin{aligned} \lim_{x \rightarrow -\infty} g_p(x) &= \lim_{x \rightarrow -\infty} \frac{2x - p(h_0)}{e^x - 1} = +\infty; \\ \lim_{x \rightarrow 0^-} g_p(x) &= -\infty; \end{aligned}$$

and

$$g_p(x) > 0, \quad \text{for all } x > 0,$$

the assertion follows. ■

*Proof of Proposition 4.4.* Take  $h > h_0$ . By Lemma 4.5(b), and using the fact that  $L_p(h_0) = 0$ ,

$$\begin{aligned} L_p(h) &= L_p(h) - L_p(h_0) \\ &= \frac{1}{\delta} \left( \int_{\Gamma^-(h)} (g_p(x) - g_p(\tilde{x})) dy - \int_{\Gamma^-(h_0)} (g_p(x) - g_p(\tilde{x})) dy \right). \end{aligned}$$

Let  $D_h$  be the region bounded by  $\Gamma^-(h)$ ,  $\Gamma^-(h_0)$ , and  $x = 0$ . Define  $D_1$ ,  $D_2$ , and  $D_3$  as in Fig. 2. Then, from the above equality,

$$\begin{aligned} L_p(h) &= \frac{1}{\delta} \left( \int_{\partial D_1 \cup \partial D_3} (g_p(x) - g_p(\tilde{x})) \frac{-\delta(e^x - 1)}{e^y - 1} dx \right. \\ &\quad \left. + \int_{\partial D_2} (g_p(x) - g_p(\tilde{x})) dy \right) \\ &= \frac{1}{\delta} \left( \int \int_{D_1 \cup D_3} (g_p(x) - g_p(\tilde{x})) (-\delta(e^x - 1)) \frac{e^y}{(e^y - 1)^2} dx dy \right. \\ &\quad \left. + \int \int_{D_2} \frac{d}{dx} (g_p(x) - g_p(\tilde{x})) dx dy \right) \\ &=: \frac{1}{\delta} (J_1 + J_2). \end{aligned}$$

By using Lemma 4.5(e) we know that  $g_p(x) - g_p(\tilde{x}) < 0$  in  $D_1 \cup D_3$ . Moreover,  $e^x - 1 < 0$  in this region. So,  $J_1 < 0$ .

On the other hand, applying Lemma 4.5(d),  $J_2 < 0$ .

Hence,  $L_p(h) < 0$  and, by Lemma 4.5(a),  $p(h) - p(h_0) < 0$ .

The proof when  $h < h_0$  consists of similar computations. Finally, we obtain  $L_p(h) > 0$  due to the clockwise orientation around  $D_h$  in this case (see Fig. 2). Consequently, again by Lemma 4.5(a) the decrease of  $p(h)$  is proved.

It remains to see that  $p(h)$  tends to  $-\infty$  when  $h$  tends to  $+\infty$ .

We already know that  $p(h) = (2 \int_{D_h} x dx dy) / S(h)$ , where  $D_h$  is the region surrounded by  $\Gamma_h$  and  $S(h)$  is the area of  $D_h$ .

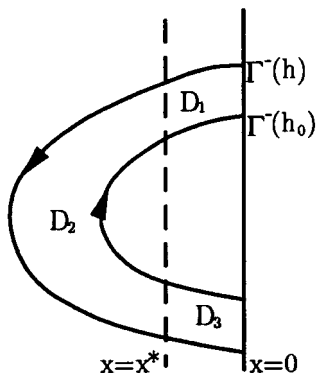


FIG. 2. Regions used in the proof of the decrease of  $p(h)$  in Proposition 4.4.

We will prove that  $\lim_{h \rightarrow \infty} |\int_{D_h} x dx dy|/S(h) = +\infty$ .

Let  $(\alpha, 0)$ ,  $(\beta, 0)$ ,  $(0, \xi)$ ,  $(0, \eta)$ ,  $(-\beta, \xi')$ ,  $(-\beta, \eta')$  be the intersection points of  $\Gamma_h$  with the two axes and with the straight line  $x = -\beta$ ; see Fig. 3. All the coordinates of these points are functions of  $h$  and satisfy

$$\begin{aligned}\alpha < 0 < \beta, \quad \xi < \xi' < 0 < \eta' < \eta, \\ e^\alpha - \alpha = e^\beta - \beta = \frac{h-1}{\delta}, \quad e^\xi - \xi = e^\eta - \eta = h - \delta, \\ e^{\xi'} - \xi' = e^{\eta'} - \eta' = h - \delta(e^{-\beta} + \beta).\end{aligned}$$

From the last equations it can be deduced easily that  $-\alpha$  and  $\beta$  tend to infinity with  $h$ . Moreover, applying Hôpital's rule we get:

$$\begin{aligned}\lim_{h \rightarrow +\infty} \frac{-\alpha}{h/\delta} &= \lim_{h \rightarrow +\infty} \frac{\beta}{\ln(h/\delta)} = \lim_{h \rightarrow +\infty} \frac{-\xi}{h} = \lim_{h \rightarrow +\infty} \frac{\eta}{\ln h} \\ &= \lim_{h \rightarrow +\infty} \frac{-\xi'}{h} = \lim_{h \rightarrow +\infty} \frac{-\eta'}{\ln h} = 1.\end{aligned}$$

As an example consider the second limit. First observe that an implicit derivation of  $\beta(h)$  gives that  $d\beta(h)/dh = 1/\delta(e^{\beta(h)} - 1)$ . Now, assuming

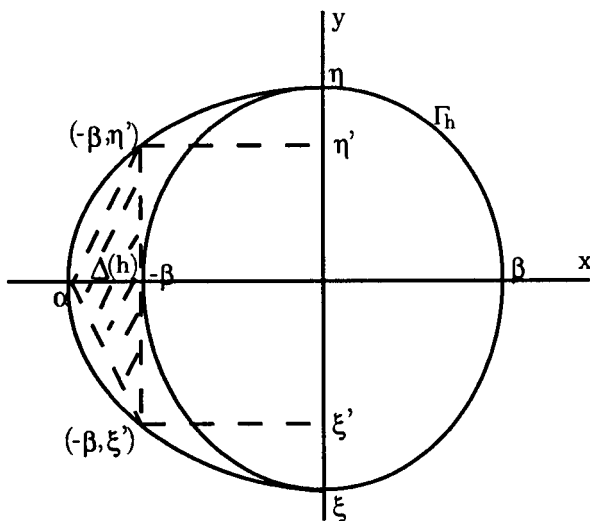


FIG. 3. The construction given to see that  $|p(h)|$  tends to infinity.

that all the limits exist (as is proved a posteriori from Hôpital's rule) we have:

$$\begin{aligned} \lim_{h \rightarrow +\infty} \frac{\beta}{\ln(h/\delta)} &= \lim_{h \rightarrow +\infty} \frac{d\beta(h)/dh}{1/h} = \lim_{h \rightarrow +\infty} \frac{h}{\delta(e^{\beta(h)} - 1)} \\ &= \lim_{h \rightarrow +\infty} \frac{1}{\delta e^{\beta(h)}(d\beta(h)/dh)} = \lim_{h \rightarrow +\infty} \frac{e^{\beta(h)} - 1}{e^{\beta(h)}} = 1. \end{aligned}$$

At this point, we divide  $D_h$  into  $D_l(h)$  and  $D_r(h)$  which are defined by:

$$D_l(h) = \{(x, y) \in D_h : x \leq -\beta\} \text{ and } D_r(h) = \{(x, y) \in D_h : x \geq -\beta\}.$$

We call  $S_l(h)$  and  $S_r(h)$  their respective areas. By the symmetry, it is clear that  $\int_{D_r(h)} x dx dy < 0$ . Furthermore, by construction,  $\int_{D_l(h)} x dx dy < -\beta \int_{D_l(h)} dx dy = -\beta S_l(h)$ . Therefore,

$$\frac{|\int_{D_h} x dx dy|}{S(h)} \geq \frac{\beta S_l(h)}{S(h)} = \beta \left(1 - \frac{S_r(h)}{S(h)}\right).$$

Since  $D_r(h)$  is contained in the rectangle  $[-\beta, \beta] \times [\xi, \eta]$  and  $D_h$  contains the triangle  $\Delta(h)$  depicted in Fig. 3, we finally have that:

$$|p(h)| = \frac{2|\int_{D_h} x dx dy|}{S(h)} \geq 2\beta \left(1 - \frac{4\beta(|\xi| + \eta)}{(|\alpha| - \beta)(|\xi'| + \eta')}\right).$$

Then, from the above limits it is easy to prove that

$$\lim_{h \rightarrow +\infty} \left( \frac{\beta(|\xi| + \eta)}{(|\alpha| - \beta)(|\xi'| + \eta')} \right) \bigg/ \left( \frac{\ln(h/\delta)}{h/\delta} \right) = 1,$$

and hence  $\lim_{h \rightarrow +\infty} |p(h)| = \infty$ . ■

Now, we can easily prove the main result of this section.

*Proof of Theorem A.* Remember that  $M(h) = bI_1(h)(a/b + p(h))$  and that  $I_1(h) > 0$  for all  $h \in J$ . Furthermore, Lemma 4.3(c) states that  $\lim_{h \rightarrow (1+\delta)^+} p(h) = 0$  and Proposition 4.4 gives that  $p'(h) < 0$  for all  $h \in J$  and  $\lim_{h \rightarrow \infty} p(h) = -\infty$ . From the above considerations and Proposition 2.1(a), the theorem follows. ■

Although the ideas and the structure of the last proofs are not easy to extend to general perturbation functions, in some particular cases they work. One of them is shown in the next result. In it, we change the quadratic part of  $\varphi(x)$  by a function bounded for  $x > 0$ .

**THEOREM B.** *Suppose that the perturbation function is  $\varphi(x) = ax + c(e^{-x} - 1)$ , with  $a$  and  $c \in \mathbb{R}$ . Then, for  $\epsilon$  small enough,*

(a) *if  $a/c > 1$ , system  $(6_\epsilon)$  has at most one limit cycle bifurcating from the level curves  $\Gamma(h)$ , for  $h \in (1 + \delta, +\infty)$ .*

(b) *if  $a/c < 1$ , system  $(6_\epsilon)$  has no limit cycles originated from a periodic orbit of  $(6_0)$ .*

The proof of this theorem follows the same structure and type of preliminary results as the proof of Theorem A, and so we omit it. However, we should stress that it strongly involves the study of the function  $Q(h) = I_3(h)/I_1(h)$ , where  $h \in J$  and  $I_3(h) = \int_{\Gamma(h)} (e^{-x} - 1) dy$ , which plays an important role, similar to that of  $p(h)$  in the proof of Theorem A. In particular, we remark two properties of  $Q(h)$  which will be used later on:

$$\lim_{h \rightarrow 1+\delta} Q(h) = -1; Q'(h) < 0, \quad \text{for all } h \in J. \quad (16)$$

## 5. PERTURBATION FUNCTION WITH LINEAR, QUADRATIC, AND EXPONENTIAL TERMS

In this section we give an example which generalizes those studied in the previous section. The new perturbation function will be  $\varphi(x) = ax + bx^2 + c(e^{-x} - 1)$ , with  $c \neq 0$ . We can choose, without lack of generality,  $c = 1$ .

Although positive answers for the previous perturbations have been obtained, here we find additional obstacles to overcome. Until now, for  $c = 0$  or  $b = 0$ , it was enough to know the sign of  $p'(h)$  or  $Q'(h)$ , to solve the respective problems. However, in this general perturbation, we need to link the information about the decrease of  $p(h)$  and  $Q(h)$ . This fact requires a deeper knowledge of these functions. Actually, the key point is that there is an essential difference between  $I_1(h)$  and  $I_2(h)$  on the one hand, and  $I_3(h)$  on the other; otherwise, it might be possible to find some kind of Picard–Fuchs equations for these integrals, and  $I_1(h)$ , and go further in their study (see for instance [4]).

As a result of the above reasons, we are not able to provide a full description of the bifurcation diagram, but the partial results we have obtained and some numerical tests allow us to give an approach to this diagram.

We divide our study into several steps.

### 5.1. Reduction to the Variable $p = p(h)$

Rescaling the three parameters, we may suppose that  $c = 1$ . Then, taking into account the bijectivity given by Proposition 4.4, we may take  $p$



as the parameter of the Hamiltonian function, instead of  $h$ . In these terms, we write the bifurcation function as:

$$M(h(p)) = I_1(h(p))(a + bp + \tilde{Q}(p)) =: I_1(h(p))M_1(p), \quad (17)$$

where  $\tilde{Q}(p) = Q(h(p))$ . The zeroes in  $(-\infty, 0)$  of  $M(h(p))$  obviously coincide with those of  $M_1(p)$ .

## 5.2. Geometrical Approach to the Perturbation Function

According to Proposition 2.1, we have to look for the zeroes of the sequence  $M_1(p)$ ,  $M_1'(p)$ ,  $M_1''(p)$ ,  $\dots$ , where

$$M_1(p) = a + bp + \tilde{Q}(p),$$

$$M_1'(p) = b + \tilde{Q}'(p),$$

$$M_1''(p) = \tilde{Q}''(p),$$

$M_1(p) = M_1(a, b; p)$ , and the prime denotes the derivative with respect to  $p$ . Now, suppose that we want to know which are the  $p$ -levels where we might find two limit cycles after a small perturbation of the system  $(6_0)$ . To obtain this, it must be held that:

$$M_1(p) = 0, \quad (18.1)$$

$$M_1'(p) = 0. \quad (18.2)$$

Observe that the solution of the above system  $\{(18.1), (18.2)\}$  on the  $(a, b)$ -plane can be thought of as the envelope of the family  $\{R(p)\}_{p < 0}$  of straight lines  $R(p) = \{(a, b): M_1(a, b; p) = 0\}$ . We call this envelope  $\mathcal{E}$ .

We can suppose that  $\mathcal{E}$  is parameterized by  $p$  as  $(a(p), b(p))$ . Then, from (18.1) and (18.2) we can compute

$$\left. \frac{da}{db} \right|_{M(p)=M'(p)=0} = -p - b \frac{dp}{db} - \tilde{Q}'(p) \frac{dp}{db} = -p,$$

$$\left. \frac{d^2a}{db^2} \right|_{M(p)=M'(p)=0} = -\frac{dp}{db} = \frac{1}{\tilde{Q}''(p)},$$

using in the last expression that  $db/dp = -\tilde{Q}''(p)$ , obtained by derivating (18.2).

We recall that from Lemma 4.3(c) and Proposition 4.4 we know that  $p(h) \leq 0$  and  $p'(h) < 0$ ; moreover, from (16) we know that  $Q(h) \leq -1$  and  $Q'(h) < 0$ . Consequently,  $\tilde{Q}(p) \leq -1$  and  $\tilde{Q}'(p) > 0$ .

As a first result, we have that  $da/db > 0$ , and so we can also parameterize the envelope as  $a(b)$ . Some aspects about the straight lines  $R(p)$  and the envelope are established in the following statements:

(i)  $R(p) \cap \{(a, b): a < 0, b > 0\} = \emptyset$ , since the equation of  $R(p)$  is  $a = -bp - \tilde{Q}(p)$ , which has a positive slope  $-p$  and cuts the  $b$ -axis at a negative point  $b = -\tilde{Q}(p)/p$ .

(ii) The envelope  $\mathcal{E}$  must lay on the half-plane  $b < 0$ , since (18.2) implies that  $b = -\tilde{Q}'(p) < 0$  on  $\mathcal{E}$ .

(iii)  $\lim_{p \rightarrow 0^-} a(b(p)) = 1$  because  $a = -bp - \tilde{Q}(p)$  and  $\lim_{p \rightarrow 0^-} \tilde{Q}(p) = -1$ .

(iv)  $a(b)$  is an increasing function of  $b$  and  $\lim_{p \rightarrow 0^-} (da(b)/db) = 0$ , since  $(da/db)|_{M(p)=M'(p)=0} = -p$ .

These remarks give a first approximation of the shape of  $\mathcal{E}$ . This shape is an important factor because it determines how many limit cycles bifurcate from  $(6_0)$ , for each value of  $(a, b)$ . In fact, observe that, given  $\mathcal{E}$  and a point  $p \in \mathbb{R}^2$ , the number of tangent lines to  $\mathcal{E}$  passing through  $p$  coincides, by (18), with the number of limit cycles bifurcating from periodic orbits of  $(6_0)$ . Then, the necessity of knowing  $\tilde{Q}''(p)$  is also manifest because it fixes the sign of  $d^2a/db^2$  and so the convexity of  $\mathcal{E}$ .

Summing up, there remain two important aspects to look into in order to know the exact graph of  $\mathcal{E}$ :

(a) Which is the point on  $a = 1$  where the envelope ends? This would imply, from (18.2), knowing the  $\lim_{p \rightarrow 0^-} \tilde{Q}'(p)$ , but we cannot solve it directly. We compute this limit in Section 5.4.

(b) The sign of  $\tilde{Q}''(p)$ , as it is indicated above, would give the shape of  $\mathcal{E}$ . Numerical experiments, performed using Simpson's method, suggest that  $\tilde{Q}(p)$  is convex upward, as shown in Fig. 4(a). Under this situation, the bifurcation diagram would be like Fig. 4(b).

### 5.3. Hopf Bifurcations in System $(6_\epsilon)$

Apart from the "large" limit cycles that could arise from the level curves  $\Gamma(h)$ , we are also interested in the number of small amplitude limit cycles. Since  $(0, 0)$  is always a critical point of  $(6_\epsilon)$ , we wonder about the local bifurcation that occur near the origin.

Until this point, we have only taken care of what happens near  $\Gamma(h)$ , when  $h > 1 + \delta$ . In fact, the local bifurcations can be interpreted as the perturbations of  $\Gamma(1 + \delta) = (0, 0)$ . However, in this subsection we will not take into account that  $\epsilon$  is the parameter of the perturbation. The main interest point is not the study of the perturbation of the Hamiltonian

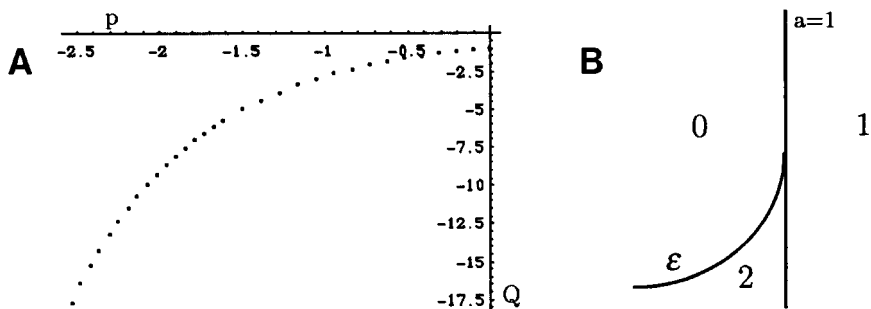


FIG. 4. (a) Numerical approach for  $\tilde{Q}(p)$ , defined in (17). (b) Envelope of (18),  $\mathcal{E}$ , and number of limit cycles for  $(6_\epsilon)$  for  $\epsilon$  small enough, in the  $(a, b)$ -plane.

system, but the stability of the origin as a critical point and the determination of the number of limit cycles near it. Then, we regard system  $(6_\epsilon)$  simply as a four-parametric system (depending on the parameters  $a, b, \epsilon, \delta$ ).

We apply the change of variables

$$\bar{x} = \sqrt{\delta}x; \quad \bar{y} = y; \quad \bar{t} = \sqrt{\delta}t$$

to the system  $(6_\epsilon)$ , with  $\varphi(x) = ax + bx^2 + (e^{-x} - 1)$ . Omitting the bars, we can write:

$$\begin{cases} \dot{x} = 1 - e^y - \epsilon\varphi\left(\frac{x}{\sqrt{\delta}}\right)e^y, \\ \dot{y} = \sqrt{\delta}\left(e^{x/\sqrt{\delta}} - 1 + \epsilon\varphi\left(\frac{x}{\sqrt{\delta}}\right)e^{x/\sqrt{\delta}}\right). \end{cases} \quad (19)$$

The origin is an elementary critical point and  $\text{div } X(0) = (1/\delta)\epsilon(1 - a)$ . Therefore it is a weak focus if and only if  $a = 1$  (of course, we do not consider the case  $\epsilon = 0$ ). In other words,  $\alpha_1 = (\epsilon/\delta)(1 - a)$  is the first Liapunov value.

By using Taylor's expansion of (19) when  $a = 1$  and using the general expressions of the second and third Liapunov values (see Appendix A), we obtain

$$\alpha_3 = \frac{\pi\epsilon}{8\delta^{3/2}}(2(1 + b) + \epsilon(1 + 2b)^2);$$

and when  $\alpha_3 = 0$ ,

$$\alpha_5 = \frac{\pi}{1152\delta^{5/2}}(15 - 21\epsilon - 20\epsilon^2 \pm \sqrt{1 - 4\epsilon}(31\epsilon - 15)),$$

where the sign “ $\pm$ ” depends on which solution of  $\alpha_3 = 0$ , for a fixed  $\epsilon \leq \frac{1}{4}$ , that we take. These solutions are  $b_{\pm}(\epsilon) = (-2\epsilon - 1 \pm \sqrt{1 - 4\epsilon})/4\epsilon$ . As a function of  $\epsilon$ ,  $b_{+}(\epsilon)$  is a  $C^1$ -function at  $\epsilon = 0$ , with  $\lim_{\epsilon \rightarrow 0} b_{+}(\epsilon) = -1$ , whilst  $b_{-}(\epsilon)$  has asymptotic behavior at  $\epsilon = 0$ . Both functions take the same value at  $\epsilon = \frac{1}{4}$  ( $b_{\pm}(\frac{1}{4}) = -\frac{3}{2}$ ).

The fourth Liapunov value  $\alpha_7$  must be computed where  $\alpha_5$  vanishes. This happens only once and this case is found to be approximately when  $\epsilon^* \approx -12.683$  and  $b^* := b_{-}(\epsilon^*) \approx -0.339$ . Since we do not know any expression of  $\alpha_7$  for the general case, we decided to take a numerical approach for this Liapunov value. Our main interest was to determine the role of the parameter  $\delta$ , which has not been significant up to now. In the previous Liapunov values  $\alpha_i$ ,  $i = 1, 3, 5$ , this parameter appeared to the power of  $-i/2$ . So, we decided to search for an  $\alpha_7$  of the form  $M \cdot \delta^k$ . The numerical experiment we carried out, which is explained with more detail in Appendix A, leads to the approximations  $M \approx -3.22296$ ,  $k \approx -3.49998$ . So, assuming that  $\alpha_7 = M/\delta^{7/2}$ , where  $M < 0$ , we can finish the description of the bifurcations near the systems having weak focus at the origin.

Before continuing we observe that the case  $\alpha_5 = 0$  occurs far from the Hamiltonian case. Anyway, we follow our study because it shows that system (6 <sub>$\epsilon$</sub> ) can have for this perturbation function at least three limit cycles.

Table I shows the signs of the Liapunov constants depending on the values of the parameters and supposing that  $a = 1$ .

TABLE I

$\epsilon, \beta$	$\alpha_3$	$\alpha_5$	$\alpha_7$
$\epsilon > 1/4$	$> 0$		
$\epsilon = 1/4$	$= 0$	$> 0$	
$0 < \epsilon < 1/4, b \notin [b_{-}(\epsilon), b_{+}(\epsilon)]$	$> 0$		
$0 < \epsilon < 1/4, b \in (b_{-}(\epsilon), b_{+}(\epsilon))$	$< 0$		
$0 < \epsilon < 1/4, b = b_{-}(\epsilon), b_{+}(\epsilon)$	$= 0$	$> 0$	
$\epsilon < 0, b \notin [b_{-}(\epsilon), b_{+}(\epsilon)]$	$< 0$		
$\epsilon < 0, b \in (b_{-}(\epsilon), b_{+}(\epsilon))$	$> 0$		
$\epsilon < 0, b = b_{+}(\epsilon)$	$= 0$	$> 0$	
$\epsilon^* < \epsilon < 0, b = b_{-}(\epsilon)$	$= 0$	$> 0$	
$\epsilon < \epsilon^*, b = b_{-}(\epsilon)$	$= 0$	$< 0$	
$\epsilon = \epsilon^*, b = b^* = b_{-}(\epsilon)$	$= 0$	$= 0$	$< 0$

We summarize the results obtained as follows:

**PROPOSITION 5.1.** *Assume that the Liapunov constant  $\alpha_7 < 0$ . Define the following subsets of the  $(a, b, \epsilon)$ -parameter space:*

$$\Omega_1 = \{(a, b, \epsilon) : a = 1, b \neq b_-(\epsilon), b_+(\epsilon), \epsilon \neq 0\} \subset \mathbb{R}^3;$$

$$\Omega_2^+ = \{(a, b, \epsilon) : a = 1, b = b_+(\epsilon), \epsilon \neq 0\} \subset \mathbb{R}^3;$$

$$\Omega_2^- = \{(a, b, \epsilon) : a = 1, b = b_-(\epsilon), \epsilon \neq 0, \epsilon^*\} \subset \mathbb{R}^3;$$

$$\Omega_2 = \Omega_2^+ \cup \Omega_2^-;$$

$$\Omega_3 = (1, b^*, \epsilon^*).$$

*Then, in any neighborhood of  $\Omega_k$  in the space of parameters there exist triplets  $(a_0, b_0, \epsilon_0)$  such that the corresponding system has  $k$  limit cycles.*

We will give more details about the conclusion of the above proposition in the next paragraphs.

In a first approximation, we can say that, in a neighborhood of  $\Omega_1$ , only the signs of  $\alpha_1$  and  $\alpha_3$  affect the number of limit cycles. Figures 5(a) and 5(b) show this number for planes  $a = \tilde{a}$ , with  $\tilde{a}$  slightly greater and less than 1, respectively.

In a neighborhood of  $\Omega_2$ , we must also take into account the sign of  $\alpha_5$ . Consider the following subsets of the plane  $a = 1$ :

$$R_1 = \{(b, \epsilon) : b = b_-(\epsilon), \epsilon^* < \epsilon < 0\};$$

$$R_2 = \{(b, \epsilon) : b = b_-(\epsilon), \epsilon < \epsilon^*\};$$

$$R_3 = \{(b, \epsilon) : b = b_+(\epsilon), \epsilon < 0\};$$

$$R_4 = \{(b, \epsilon) : b \in \{b_+(\epsilon), b_-(\epsilon)\}, \epsilon > 0\}.$$

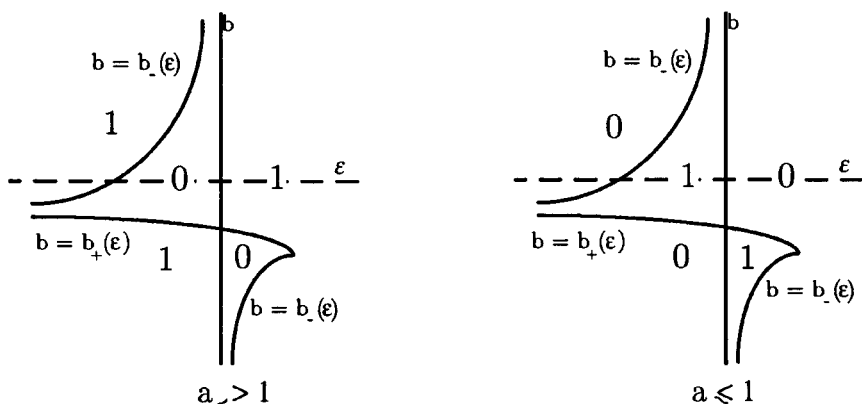


FIG. 5. Limit cycles of  $(6_\epsilon)$  near the origin when  $\alpha_3 \neq 0$ .

Call  $U_i$ , for  $i = 1, \dots, 4$ , a neighborhood of  $R_i$  in the  $(a, b, \epsilon)$ -space. We carry out a graphical description of the number of limit cycles in  $\tilde{U}_i := U_i \cap \{\epsilon = k, \text{ with } k \text{ constant}\}$ . In Fig. 6 (left-hand side) we show the distribution of the number of limit cycles in  $\tilde{U}_4$ , and in Fig. 6 (right-hand side) we add the information given in Fig. 5 to give a complete description near  $a = 1$  for a fixed  $\epsilon > 0$ . In the next subsection, these results will play a key role.

#### 5.4. Relation between Abelian Integrals and Hopf Bifurcation

The previous study about the Hopf bifurcations in system  $(6_\epsilon)$  can give local information on the envelope  $\mathcal{E}$  given by (18.1) and (18.2). Indeed, we will prove which is the point on  $a = 1$  where  $\mathcal{E}$  ends.

**PROPOSITION 5.2.** *The envelope  $\mathcal{E}$  of the family  $\{R(p)\}_{p < 0}$  given by the system  $\{(18.i)\}_{i=1,2}$  is tangent to  $a = 1$  at the point  $(a, b) = (1, -1)$ .*

*Proof.* Note that  $(1, -1)$  is the limit of the codimension 2 Hopf bifurcation curves near  $b_+(\epsilon)$  as  $\epsilon \rightarrow 0^+$ , as described in Fig. 6. We want to use this fact to prove our result.

Note that from system (18) and the fact that  $\lim_{p \rightarrow 0^-} a(b(p)) = 1$  and  $db/da = -1/p > 0$ , we know that on  $\mathcal{E}$ , the envelope of  $\{R_p\}_{p < 0}$ , where  $R(p) = \{(a, b): a + bp + \tilde{Q}(p) = 0\}$ , we have:

$$\lim_{a \rightarrow 1^-} b'(a) = +\infty \quad \text{and} \quad \lim_{a \rightarrow 1^-} b(a) = L, \quad L = \mathbb{R} \cup \{+\infty\}.$$

We will prove that  $L = -1$ . To this end, we observe that the above equalities imply that for each  $p < 0$ , there exist at least two pairs  $(a_1, b_1)$ , with  $a_1 < 1$  and  $b_1 < L$ , and  $(a_2, b_2)$ , with  $a_2 > 1$  and  $b_2 > L$ , such that  $(a_i, b_i) \in R(p)$  for  $i = 1, 2$ .

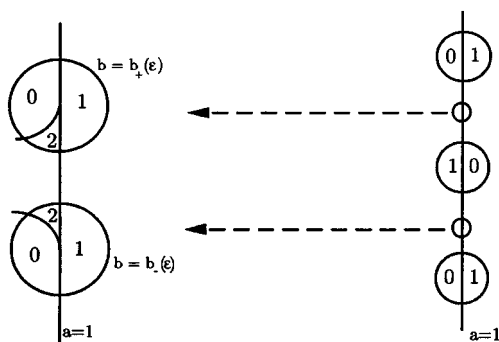


FIG. 6. Limit cycles of  $(6_\epsilon)$  near the origin when  $\alpha_5 \neq 0$ .

In terms of the number of limit cycles, this result means that we can always find systems  $(a_i, b_i)$  having at least one limit cycle as close to the origin as we desire.

However, if  $L \neq -1$ , this last assertion is impossible, according to what we saw in Fig. 6 of Section 5.3. Effectively, if  $L > -1$ , the contradiction comes from the points like  $(a_1, b_1)$  such that  $b_1 > b_+(\epsilon)$  (similarly when  $L = +\infty$ ); if  $L < -1$ , we reach the contradiction from the points like  $(a_2, b_2)$  such that  $b_-(\epsilon) < b_2 < b_+(\epsilon)$ .

## APPENDIX A: COMPUTATION OF THE LIAPUNOV VALUES

Recall that Poincaré gave an algorithmic way to study the stability of the origin for a system of type  $\dot{x} = -y + f(x, y)$ ,  $\dot{y} = x + g(x, y)$ , with  $f(x, y)$  and  $g(x, y)$  analytic functions, starting with degree 2 terms. The method consists of computing the so-called Liapunov values  $\alpha_1, \alpha_3, \alpha_5, \dots$ .

A general expression for the Liapunov values has only been obtained for the first ones: the first Liapunov value is the trace of the Jacobian matrix, and so is very easy to compute; the expression for the second one ( $\alpha_3$ ) can be found in many references (see for instance [1]); for the third value ( $\alpha_5$ ), a large number of computations is required, but even so general formulas have been given; however, for the fourth value ( $\alpha_7$ ), as far as we know, there is no explicit general expression.

The shortest way that we know to write  $\alpha_3$  and  $\alpha_5$  comes from the expression of the system in complex variables:

PROPOSITION ([2]). Set  $\dot{z} = iz + F(z, \bar{z})$ , with  $F = F_2 + F_3 + F_4 + \dots$  and  $F_2(z, \bar{z}) = Az^2 + Bz\bar{z} + C\bar{z}^2$ ,  $F_3(z, \bar{z}) = Dz^3 + Ez^2\bar{z} + Fz\bar{z}^2 + G\bar{z}^3$ ,  $F_4(z, \bar{z}) = Hz^4 + Iz^3\bar{z} + Jz^2\bar{z}^2 + Kz\bar{z}^3 + L\bar{z}^4$ ,  $F_5(z, \bar{z}) = Mz^5 + Nz^4\bar{z} + Oz^3\bar{z}^2 + Pz^2\bar{z}^3 + Qz\bar{z}^4 + R\bar{z}^5$ . Then:

- (i)  $\alpha_3 = 2\pi[\operatorname{Re}(E) - \operatorname{Im}(AB)]$ .
- (ii)  $\alpha_5 = (\pi/3)[6\operatorname{Re}(O) + \operatorname{Im}(3E^2 - 6DF + 6A\bar{I} - 12BI - 6B\bar{J} - 8CH - 2C\bar{K}) + \operatorname{Re}(-8C\bar{C}E + 4AC\bar{F} + 6A\bar{B}F + 6B\bar{C}F - 12B^2D - 4ACD - 6A\bar{B}D + 10B\bar{C}D + 4A\bar{C}G + 2BC\bar{G}) + \operatorname{Im}(6A\bar{B}^2C + 3A^2B^2 - 4A^2\bar{B}C + 4\bar{B}^3C)]$ .

Finally, we describe how we get numerically the fourth Liapunov value,  $\alpha_7$ , used in Section 5.3. It is known that the expression of the Poincaré

return map  $\pi(x)$  near the origin and when  $\alpha_1$ ,  $\alpha_3$  and  $\alpha_5$  vanish is given by

$$\pi(x) - x = \alpha_7 x^7 + o(x^7),$$

where  $x$  is the first coordinate of a point on the half-line  $\{(x, 0): x > 0\}$ . Then,

$$F(x) = \frac{\pi(x) - x}{x^7} = \alpha_7 + o(x) = \alpha_7 + a_1 x^{r_1} + a_2 x^{r_2} + \cdots,$$

where  $r_1 < r_2 < r_3 < \cdots$  and  $r_i \geq 1$ , for all  $i \in \mathbb{N}$ .

Trying to do a direct numerical computation of  $F(x)$  for small  $x$  could be an approximate way to find  $\alpha_7$ . However, the factor  $x^{r_i}$  can still have a significant weight near  $x = 0$ . The precision will be greater if we increase the value of the powers  $r_i$  in some way. This purpose can be achieved by using extrapolation procedures, like Richardson's method, described below; see [9, pp. 436–441].

From a sequence of values of  $F(x)$ , say  $F(x_1), F(x_2), \dots, F(x_m)$ , such that  $x_{i+1} = qx_i$ ,  $x_1 = x$ ,  $i = 1, \dots, m - 1$ , and  $q > 1$ , define

$$\begin{cases} F_1(x) &= F(x) \\ F_{j+1}(x) &= F_j(x) + \frac{F_j(x) - F_j(qx)}{q^{r_j} - 1}, \quad \text{for all } j \geq 1. \end{cases}$$

Then, it can be proved that  $F_k(x)$  can be written as

$$F_k(x) = \alpha_7 + a_k^{(k)} x^{r_k} + a_{k+1}^{(k)} x^{r_{k+1}} + \cdots, \quad k \leq m,$$

and hence  $F_k(x)$ , for small  $x$ , is a better approximation of  $\alpha_7$  than  $F(x)$ .

The images  $F(x_1), \dots, F(x_m)$  are obtained by using the Runge–Kutta–Fehlberg method with orders 7 and 8, a tolerance  $10^{-13}$ , an initial step  $10^{-5}$ , maximum and minimum steps  $10^{-1}$  and  $10^{-16}$ , respectively, and an accuracy of  $10^{-16}$ .

Observe that the above descriptions give a three-parameter method, depending on  $x$ ,  $q$ , and  $m$ , to approximate  $\alpha_7$ . Of course, the same idea can be used to get numerical approximations of  $\alpha_1$ ,  $\alpha_3$ , and  $\alpha_5$ .

We implement our method in the computer for the system  $(6_\epsilon)$  taking  $a = 1$ ,  $b \approx b^*$ ,  $\epsilon \approx \epsilon^*$  (recall that this was the case for which  $\alpha_1 = \alpha_3 = \alpha_5 = 0$ ), and  $\delta$  as a parameter. For instance, in this case and for  $\delta = 1$ , we obtained that  $\alpha_1 \approx 10^{-7}$ ,  $\alpha_3 \approx 10^{-6}$ , and  $\alpha_5 \approx 10^{-4}$ . This information about the previous Liapunov values is very useful to indicate a suitable range of parameters where we can get the best approximations, in other words, what the proper  $x$ ,  $q$ , and  $m$  to be chosen in our method are.



TABLE II

$\delta$ :	0.5	0.6	0.7	0.8	0.9
$\alpha_7(\delta)$ :	-36.4552	-19.2687	-11.2295	-7.03735	-4.66008
$\delta$ :	1	1.1	1.2	1.3	1.4
$\alpha_7(\delta)$ :	-3.22276	-2.30898	-1.70259	-1.28658	-0.992595

The procedure described up to now allows us to compute numerically  $\alpha_7(\delta)$ , i.e., the fourth Liapunov value for a concrete value of  $\delta$ . The last step consists of finding a general expression of  $\alpha_7$  as a function of  $\delta$ . From the observation of the role of this parameter in  $\alpha_1$ ,  $\alpha_3$ , and  $\alpha_5$ , one can expect that  $\alpha_7(\delta) = M \cdot \delta^k$ , where  $M, k \in \mathbb{R}$  (in fact, it seems that  $k$  should be  $-\frac{7}{2}$ ). We get Table II, where the value of  $\alpha_7(\delta)$  chosen is the mode of the results obtained for several selected values of  $x, q$ , and  $m$ , satisfying  $x^m < 10^{-10}$ .

In order to know the numerical values of  $M$  and  $k$ , we take the results given in Table II and solve a least-squares problem with the entries  $(\ln \delta, \ln |\alpha_7(\delta)|)$ . The slope of the regression line is an approximation of  $k$ , while the constant term is approximately  $\ln |M|$ , because

$$\ln |\alpha_7(\delta)| = \ln |M| + k \ln \delta.$$

In this way, we obtain that  $M \approx -3.22296$  and  $k \approx -3.49998$ . To determine the sign of  $M$  we only have to observe the sign of  $\alpha_7(\delta)$  which is always negative.

Finally, we observe that

$$\sum_{i=1}^{10} (\ln |\alpha_7(0.5 + (i-1)0.1)| - \ln |-3.22296| - 3.49998 \ln(0.5 + (i-1)0.1))^2 \leq 2 \cdot 10^{-7}.$$

Therefore, it seems reasonable to conclude that

$$\alpha_7 \approx -3.22296 \delta^{-3.49998} \approx -3.223/\delta^{7/2}.$$

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