

# AN ANALYTIC-NUMERICAL METHOD FOR COMPUTATION OF THE LIAPUNOV AND PERIOD CONSTANTS DERIVED FROM THEIR ALGEBRAIC STRUCTURE.\*

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**Abstract.** We consider the problem of computing the Liapunov and the period constants for a smooth differential equation with a nondegenerate critical point. First, we investigate the structure of both constants when they are regarded as polynomials on the coefficients of the differential equation. Second, we take advantage of this structure to derive a method to obtain the explicit expression of the above-mentioned constants. Although this method is based on the use of the Runge–Kutta–Fehlberg methods of orders 7 and 8 and the use of Richardson’s extrapolation, it provides the real expression for these constants.

**Key words.** center point, Liapunov constants, isochronicity, analytic-numerical method

**AMS subject classifications.** 65L07, 34D20, 34C25

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**1. Introduction and main results.** In this paper we deal with the problem of computing the Liapunov and period constants for the differential equations

$$\begin{cases} \dot{x} = -y + P(x, y), \\ \dot{y} = x + Q(x, y), \end{cases}$$

where  $P(x, y)$  and  $Q(x, y)$  are analytic functions in a neighborhood of the origin and begin, at least, with second order terms. These systems can be expressed in the complex plane using the following notation:

$$(1.1) \quad \dot{z} = iz + F(z, \bar{z}),$$

where  $F(z, \bar{z}) = \sum_{k \geq 2} F_k(z, \bar{z})$ ,  $F_k(z, \bar{z}) = \sum_{j=0}^k f_{k-j,j} z^{k-j} \bar{z}^j$ ,  $f_{k-j,j} \in \mathbf{C}$ , and the dot indicates the derivative with respect to  $t$ , with  $t \in \mathbf{R}$ .

The problem of determining whether (1.1) has a center or a focus at the origin can be solved by studying the Poincaré return map. This study can be done (using the power series of the return map) by means of the computation of infinitely many real numbers,  $v_{2m+1}$ ,  $m \geq 1$ , called the *Liapunov constants*. In fact, we have that if for some  $m$ ,  $v_3 = v_5 = \dots = v_{2m-1} = 0$ , and  $v_{2m+1} \neq 0$ , then the origin is a focus of which the stability is determined by the sign of  $v_{2m+1}$ , while if all  $v_{2m+1}$  are zero, then the origin is a center; see for instance [1].

A closely related problem is the following: Assume that (1.1) has a center at the origin and consider the period of all its periodic orbits. The origin of (1.1) is an *isochronous* center if and only if the period is independent of the orbit. When is the

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origin of (1.1) an isochronous center? It turns out that the solution to this problem can be obtained (using the power series of the period function) by computing infinitely many real numbers,  $P_{2m}, m \geq 1$ , called the *period constants* and by forcing them all to vanish.

If we look at (1.1) as a family of polynomial differential equations, the constants can be seen as functions in the coefficients of the system. Moreover, in the case that (1.1) is a polynomial family of differential equations of fixed degree, the Hilbert basis theorem implies that a finite number of Liapunov constants vanishing is enough to give the characterization of the centers of the family, just as a finite number of Liapunov and period constants vanishing is enough to characterize the isochronous centers of the family.

Many authors have dealt with the problem of computing the constants. Without being exhaustive, see for instance [3], [4], [6], [7], [9], [10], [14], or [16]. All the approaches used to calculate the constants involve a lot of computations. In order to have an a priori estimation of the complexity of the result, it is very useful to know properties of the Liapunov and period constants when they are considered as functions of the coefficients of (1.1).

Before stating our results we need some definitions.

We say that  $M$  is a *monomial* of (1.1) when  $M = \prod_{k,l} f_{k,l}^{m_{k,l}} \overline{f_{k,l}}^{n_{k,l}}$ , with  $m_{k,l}, n_{k,l} \in \mathbf{N}$ , where the product is finite and  $f_{k,l}$  is any coefficient of  $F_{k+l}(z, \bar{z})$ .

Let  $v_{2m+1}$  be the  $m$ th Liapunov constant. (Respectively, let  $P_{2m}$  be the  $m$ th period constant.) We will also say that a monomial of (1.1),  $M$ , is a *monomial* of  $v_{2m+1}$  (or of  $P_{2m}$ ) if either  $\text{Re}(M)$  or  $\text{Im}(M)$  appears in the expression of the constant.

We define the *degree*,  $\deg(M)$ , the *quasi degree*,  $\text{qd}(M)$ , and the *weight* of  $M$ ,  $w(M)$ , respectively, as

$$\begin{aligned} \deg(M) &= \sum_{k,l} (m_{kl} + n_{kl}), & \text{qd}(M) &= \sum_{k,l} (k + l - 1)(m_{kl} + n_{kl}), & \text{and} \\ w(M) &= \sum_{k,l} (1 - k + l)(m_{kl} - n_{kl}). \end{aligned}$$

We refer the reader to section 3.1 for examples. Finally, we say that a monomial of (1.1) of weight zero,  $M$ , is *basic* if  $M'|M$  and  $w(M') = 0$  imply that  $M' = \pm M$ . Roughly speaking, the basic monomials are the prime factors of the monomials of weight zero.

With the above notation, the following result is well known; see [2], [12], [13], [17], and [18].

**THEOREM 1.** *Let  $M$  be a monomial of  $v_{2m+1}$  or  $P_{2m}$ . Then,  $\text{qd}(M) = 2m$  and  $w(M) = 0$ .*

The property  $w(M) = 0$  is derived from the fact that the constants are invariant under rotations of the vector field. Property  $\text{qd}(M) = 2m$  comes from the effect of homoteties in the vector field.

Theorem 1 gives some information about the monomials that appear in the Liapunov and period constants. In our main result, we improve Theorem 1 by describing how these monomials are distributed according to their degrees.

**THEOREM A.** *The following statements hold:*

- (i) *Let  $M_1, M_2, \dots, M_k$  (respectively,  $M_{k+1}, M_{k+2}, \dots, M_{k+l}$ ) be monomials of*

TABLE 1  
Estimations of the number of monomials.

Const.	Polynomials of deg = $2m$	From Thm. 1	From Thms. 1 and A	Actual number
$v_3$	119	7	2	2
$v_5$	91.389	79	28	22
$v_7$	156.238.907		310	259
$P_2$	119	7	5	4
$P_4$	91.389	79	40	29

$v_{2m+1}$  with even degree (respectively, odd degree). Then

$$v_{2m+1} = \sum_{i=1}^k \alpha_i \operatorname{Im}(M_i) + \sum_{i=k+1}^{k+l} \beta_i \operatorname{Re}(M_i)$$

for some  $\alpha_i, \beta_i \in \mathbf{R}$ .

(ii) Let  $M_1, M_2, \dots, M_p$  (respectively,  $M_{p+1}, M_{p+2}, \dots, M_{p+q}$ ) be monomials of  $P_{2m}$  with even degree (respectively, odd). Then

$$P_{2m} = \sum_{i=1}^p \gamma_i \operatorname{Re}(M_i) + \sum_{i=p+1}^{p+q} \delta_i \operatorname{Im}(M_i)$$

for some  $\gamma_i, \delta_i \in \mathbf{R}$ .

The proof of Theorem A appears in section 2. It is an improvement of Theorems 1 and 2 of [2]. Observe that for any vector field,  $v_{2m+1}$  and  $P_{2m}$  are real numbers. Hence, if  $M$  is a monomial of (1.1),  $v_{2m+1} = \alpha M + \bar{\alpha} \bar{M} + N$ , where  $N$  denotes the sum of the other monomials appearing in its expression, and so  $v_{2m+1} = 2\operatorname{Re}(\alpha)\operatorname{Re}(M) - 2\operatorname{Im}(\alpha)\operatorname{Im}(M) + N$  (an analogous argument is valid for  $P_{2m}$ ). Therefore, Theorem A reduces by half the estimation of the length of the Liapunov and period constants obtained using only Theorem 1. To give an idea of this reduction, in Table 1 we display different estimations of the number of terms of  $v_{2m+1}$  and  $P_{2m}$ , considered polynomials on the monomials  $M_i$ . These estimations have been obtained by imposing progressively the following constraints:

Second column:  $v_{2m+1}$  and  $P_{2m}$  are real-valued polynomials of degree  $2m$ .

Third column: The monomials of  $v_{2m+1}$  and  $P_{2m}$  satisfy the restrictions of Theorem 1.

Fourth column: The monomials of  $v_{2m+1}$  and  $P_{2m}$  satisfy the restrictions of Theorem 1 and Theorem A.

Fifth column: The actual number of monomials of  $v_{2m+1}$  or  $P_{2m}$  computed in [6].

On the other hand, when we want to compute some exact Liapunov or period constant, from the above results we know that it is a polynomial in which the only unknowns are the coefficients of its monomials. It is easy to see from section 24 of [1] that these coefficients are rational multiples of  $\pi$  and so, multiplying by an appropriate factor (which can be upper bounded), they can be reduced to integer numbers. The way to compute these coefficients is a key point in our approach since we obtain them from numerical integrations of some particular cases of the ordinary differential equation. The process, based on a combination of the Runge–Kutta–Fehlberg method with variable steps and Richardson’s extrapolation method, is developed in section 3. In contrast to most of the other methods used to compute the Liapunov and period

constants, our process does not require the use of computer algebra systems. In our method we need only to list all the monomials of  $v_{2m+1}$  or  $P_{2m}$ , choose appropriate systems that can provide their coefficients, and carry out the numerical process to obtain them effectively. The last step consists only of solving a system of linear equations.

In section 3.1, Theorem 3, we illustrate our approach by computing the Liapunov constant  $v_5$  for a general system of type (1.1). In section 3.2, as an example of how the numerical method works and how the round-off errors increase if the method is applied to the computation of  $v_7$ , we apply our method to compute  $v_7$  for a particular system. These computations are obtained by using arithmetic of double precision. In general, to compute higher order Liapunov or period constants, one should work with quadruple or even higher precision.

Finally, observe that in order to apply our method to a given polynomial system of the form (1.1), we need to know all their monomials of weight zero. To list them it would be useful to find a kind of “finite system of generators.” In the appendix we prove that those generators are the basic monomials defined above. In our opinion, the use of basic monomials in systems with few coefficients provides an elegant and compact form with which to present the Liapunov (and the period) constants (see, for instance, Remark 4).

**2. Proof of Theorem A.** We need a preliminary result and some notation.

DEFINITION. *We denote the following:*

- (i)  $\mathcal{P}_0$  is the set of functions of the form  $\operatorname{Re}(P(\theta))$ , where  $P(\theta)$  writes as

$$P(\theta) = \sum_l p_l(\theta) M_l i^{\deg(M_l)};$$

$M_l$  denotes any monomial of (1.1), and  $p_l(\theta)$  are trigonometric polynomials with real coefficients, that is, elements of  $\mathbf{R}[e^{i\theta}]$ .

- (ii)  $\mathcal{P}_1$  is the set of functions of the form  $\operatorname{Re}(i P(\theta))$  such that  $\operatorname{Re}(P(\theta)) \in \mathcal{P}_0$ .

When we have an operation  $*$  between the elements of two sets  $A$  and  $B$ , we denote  $A * B = \{a * b : a \in A, b \in B\}$ .

LEMMA 2. *The following relations are satisfied:*

- (i)  $\mathcal{P}_0 + \mathcal{P}_0 \subset \mathcal{P}_0$ ,  $\mathcal{P}_1 + \mathcal{P}_1 \subset \mathcal{P}_1$ .  
(ii)  $\mathcal{P}_i \cdot \mathcal{P}_j \subset \mathcal{P}_{i+j}$ , where  $i, j \in (\mathbf{Z}_2, +)$ .

*Proof.* The proof of (i) is trivial. Here we prove for instance the case  $\mathcal{P}_0 \cdot \mathcal{P}_0 \subset \mathcal{P}_0$  of (ii).

Consider  $a(\theta), b(\theta) \in \mathcal{P}_0$ , that is,

$$a(\theta) = \operatorname{Re} \left( \sum_j p_j(\theta) M_j i^{\deg(M_j)} \right),$$

$$b(\theta) = \operatorname{Re} \left( \sum_k q_k(\theta) N_k i^{\deg(N_k)} \right),$$

where  $M_j, N_k$  denote monomials of (1.1) and  $p_j(\theta), q_k(\theta) \in \mathbf{R}[e^{i\theta}]$ . Then,

$$\begin{aligned}
a(\theta) b(\theta) &= \operatorname{Re} \left[ \frac{1}{2} \sum_j p_j(\theta) M_j i^{\deg(M_j)} \left( \sum_k q_k(\theta) N_k i^{\deg(N_k)} + \sum_k \bar{q}_k(\theta) \bar{N}_k (-i)^{\deg(\bar{N}_k)} \right) \right] \\
&= \operatorname{Re} \left[ \sum_{j,k} \left( \frac{1}{2} p_j(\theta) q_k(\theta) M_j N_k i^{\deg(M_j \cdot N_k)} \right. \right. \\
&\quad \left. \left. + \frac{1}{2} p_j(\theta) \bar{q}_k(\theta) (-1)^{\deg(\bar{N}_k)} M_j \bar{N}_k i^{\deg(M_j \cdot \bar{N}_k)} \right) \right],
\end{aligned}$$

where we have used that  $\operatorname{Re}(z) \operatorname{Re}(w) = \operatorname{Re}(z(w + \bar{w}))/2$  for any  $z, w \in \mathbf{C}$ . Thus the lemma follows.  $\square$

*Proof of Theorem A.* First observe that proving (i) is equivalent to proving that  $v_{2m+1} \in \mathcal{P}_1$  (where the trigonometric polynomials involved are constants). To achieve this result we study to which set,  $\mathcal{P}_0$  or  $\mathcal{P}_1$ , belong the functions appearing in the algorithm of computation of the constants given in [1]. We briefly recall it here.

Equation (1.1) can be written in the polar coordinates  $r^2 = z\bar{z}$  and  $\theta = \arctan \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}$ , as

$$(2.1) \quad \frac{dr}{d\theta} = \frac{r^2 \operatorname{Re}(S_2(\theta)) + r^3 \operatorname{Re}(S_3(\theta)) + \dots}{1 + r \operatorname{Im}(S_2(\theta)) + r^2 \operatorname{Im}(S_3(\theta)) + \dots} = \sum_{k \geq 2} r^k(\theta) R_k(\theta),$$

where  $S_k(\theta) = e^{-i\theta} F_k(e^{i\theta}, e^{-i\theta})$ .

Observe that  $S_k(\theta) = \sum_{j=0}^k f_{k-j,j} e^{i(k-2j-1)\theta} = \sum_{j=0}^k e^{i(k-2j-1)\theta} (-f_{k-j,j})$ ,  $i^{\deg(f_{k-j,j})+1}$ , because  $\deg(f_{k-j,j}) = 1$  for all  $j$  and  $k$ . Hence, it is clear that  $\operatorname{Re}(S_k(\theta)) \in \mathcal{P}_1$ .

The functions  $R_k(\theta)$  can be computed using the recursive formula for the quotient of series given in [8] as

$$(2.2) \quad R_k(\theta) = \operatorname{Re}(S_k(\theta)) - \sum_{j=1}^{k-2} \operatorname{Im}(S_{j+1}(\theta)) R_{k-j}(\theta).$$

We also have that  $\operatorname{Im}(S_{j+1}(\theta)) = \operatorname{Re}(-iS_{j+1}(\theta)) \in \mathcal{P}_0$ . By using Lemma 2, it is easy to prove by induction that  $R_k(\theta) \in \mathcal{P}_1$ .

Let  $r(\theta, \rho) = \rho + \sum_{j \geq 2} u_j(\theta) \rho^j$  be the solution of (2.2) for which  $r(0, \rho) = \rho$ . Consider the Poincaré map given by  $\Pi(\rho) = r(2\pi, \rho) = \rho + \sum_{j \geq 2} u_j(2\pi) \rho^j$ . It is well known that if  $\Pi(\rho)$  is not the identity, the first nonvanishing term in the power series corresponds to an odd order term. When  $u_2(2\pi) = u_3(2\pi) = \dots = u_{2m}(2\pi) = 0$  and  $u_{2m+1}(2\pi) \neq 0$ , the  $m$ th Liapunov constant is  $v_{2m+1} = u_{2m+1}(2\pi)$ .

Recall also that when  $u_2(2\pi) = u_3(2\pi) = \dots = u_{n-1}(2\pi) = 0$ , the functions  $u_j(\theta)$  are trigonometric polynomials for  $j \leq n-1$  and

$$(2.3) \quad u'_n(\theta) = \sum_{k=2}^n R_k(\theta) \left( \sum_{\mathbf{a} \in D_n^k} \frac{k!}{a_1! a_2! a_3! \dots a_{n-1}!} u_2^{a_2}(\theta) u_3^{a_3}(\theta) \dots u_{n-1}^{a_{n-1}}(\theta) \right),$$

where  $D_n^k$  is the following subset of indices (see [2] for more details):

$$\begin{aligned}
D_n^k &= \{\mathbf{a} = (a_1, a_2, \dots, a_{n-1}) \in \mathbf{N}^{n-1} \text{ such that } a_1 + \dots + a_{n-1} = k, \\
&\quad a_1 + \dots + j a_j + \dots + (n-1) a_{n-1} = n\}.
\end{aligned}$$

We claim that

$$u_n(\theta) = Q_n(\theta) + P_n \theta,$$

where  $Q_n(\theta) \in \mathcal{P}_0$  and  $P_n(\theta) \equiv P_n \in \mathcal{P}_1$ . Furthermore,

$$P_n = \operatorname{Re} \left( \sum_l p_l M_l i^{\deg(M_l)+1} \right), \quad Q_n(\theta) = \operatorname{Re} \left( \sum_l q_l(\theta) M_l i^{\deg(M_l)} \right),$$

where  $M_l$  denote monomials of (1.1),  $p_l \in \mathbf{R}$ , and  $q_l(\theta) \in \mathbf{R}[e^{i\theta}]$ .

Observe that if this claim is proved, then Theorem A follows because when  $n = 2m + 1$ , since  $v_{2m+1} = u_{2m+1}(2\pi) = \int_0^{2\pi} u'_{2m+1}(\theta) d\theta$ , we obtain that  $v_{2m+1} \in \mathcal{P}_1$ , as we wanted to prove.

We now prove the claim by induction. We have that for  $n = 2$

$$\begin{aligned} u_2(\theta) &= \operatorname{Im} \left[ f_{20} e^{i\theta} - f_{11} e^{-i\theta} - \frac{f_{02}}{3} e^{-3i\theta} - f_{20} + f_{11} + \frac{f_{02}}{3} \right] \\ &= \operatorname{Re} \left[ (1 - e^{i\theta}) f_{20} i + (e^{-i\theta} - 1) f_{11} i + \frac{1}{3} (e^{-3i\theta} - 1) f_{02} i \right] \in \mathcal{P}_0. \end{aligned}$$

Suppose that the claim is true for  $u_j(\theta)$  with  $j = 2, \dots, n-1$ . Since the functions  $u_j$ , for  $j \leq n-1$ , are trigonometric polynomials (that is, they do not contain terms with the factor  $\theta$ ), applying the induction hypothesis we get that  $u_j(\theta) \in \mathcal{P}_0$ , for  $j \leq n-1$ . Since  $u'_n(\theta)$  is obtained from (2.3), considering Lemma 2 and the structure of  $R_k(\theta)$ , we obtain that  $u'_n(\theta) \in \mathcal{P}_1$ .

We distinguish two types of terms in  $u'_n(\theta)$ : those of the form  $\operatorname{Re}(C M i^{\deg(M)+1})$  and those of the form  $\operatorname{Re}(C e^{i\alpha\theta} M i^{\deg(M)+1})$ , with  $C, \alpha \in \mathbf{R}$ ,  $\alpha \neq 0$ , and  $M$  a monomial of (1.1).

By integrating between 0 and  $\theta$ , the terms of the first form are transformed into

$$\operatorname{Re}(C M i^{\deg(M)+1}) \theta,$$

and so they are terms of the form  $p\theta$ , with  $p \in \mathcal{P}_1$ , as we wanted to prove.

The integration of the second form terms between 0 and  $\theta$  leads to expressions like

$$\begin{aligned} &\operatorname{Re} \left( \frac{C}{i\alpha} e^{i\alpha\theta} M i^{\deg(M)+1} - \frac{C}{i\alpha} M i^{\deg(M)+1} \right) \\ &= \operatorname{Re} \left( \frac{-C}{\alpha} e^{i\alpha\theta} M i^{\deg(M)+2} - \frac{-C}{\alpha} M i^{\deg(M)+2} \right) \\ &= \operatorname{Re} \left( \frac{C}{\alpha} e^{i\alpha\theta} M i^{\deg(M)} - \frac{C}{\alpha} M i^{\deg(M)} \right), \end{aligned}$$

and so they belong to  $\mathcal{P}_0$ , as we wanted to prove. Hence, the claim is proved and, as a consequence, the proof of (i) is finished.

The proof of (ii) is similar and we just establish the differences. Assume that (1.1) has a center at the origin. Consider (1.1) expressed in polar coordinates. The period function, which gives the period of the orbit of (2.1) for which  $r(0, \rho) = \rho$ , can be

expressed as

$$\begin{aligned} P(\rho) &= 2\pi + \sum_{k \geq 1} \int_0^{2\pi} H_k(\theta) r(\theta, \rho)^k d\theta \\ &= 2\pi + \sum_{k \geq 1} t_k(2\pi) \rho^k, \end{aligned}$$

where the functions  $H_k(\theta)$  can be calculated using the recursive formula

$$(2.4) \quad H_k(\theta) = - \sum_{j=1}^k \operatorname{Im}(S_{j+1}(\theta)) H_{k-j}(\theta)$$

and the functions  $t_k(\theta)$  satisfy the following recurrence:

$$(2.5) \quad t'_k(\theta) = H_1(\theta) u_k(\theta) + \sum_{m=2}^k H_m(\theta) \left[ \sum_{\mathbf{a} \in D_k^m} \frac{m!}{a_1! a_2! a_3! \cdots a_{k-1}!} u_2^{a_2}(\theta) u_3^{a_3}(\theta) \cdots u_{k-1}^{a_{k-1}}(\theta) \right].$$

See [2] for more details.

It is well known that the first nonvanishing term in the power series of  $P(\rho)$  corresponds to an even order term. Therefore, if  $t_2(2\pi) = t_3(2\pi) = \cdots = t_{2m-1}(2\pi) = 0$  and  $t_{2m}(2\pi) \neq 0$ , the  $m$ th period constant is  $P_{2m} = t_{2m+1}(2\pi)$ .

In the above notation, the proof of (ii) follows from arguments similar to those in (i) but considering the recursive formula (2.4) instead of (2.2) and considering (2.5) instead of (2.3).  $\square$

**3. The analytic-numerical method with applications.** Here we present a method to compute the general formula of the constants. Let us suppose, for instance, that we want to find the expression for  $v_{2m+1}$ . We proceed as follows:

Step 1. By using Theorems 1 and A and the appendix, we list all the monomials involved in  $v_{2m+1}$ . That is, we write  $v_{2m+1}$  as a linear function of products of basic monomials (see Remark 4).

Step 2. Once the monomials are listed, we look for all the undetermined coefficients by computing the constant for some particular systems. To do this, we use the Runge–Kutta–Fehlberg 7–8 method to calculate the Poincaré return map. Afterward, we apply the Richardson's extrapolation method in order to reduce the error.

The above procedure will be followed in the next subsection to compute the expression of  $v_5$  for a general system of type (1.1). Step 2 can also be used to obtain numerically the  $m$ th Liapunov constant for a particular differential equation of type (1.1). This has already been done in [5], but here we also study the behavior of the round-off errors. In subsection 3.2 we present how to compute the third Liapunov constant,  $v_7$ , for a quadratic system.

First, let us describe Step 2 more carefully in the case of the computation of  $v_5$ .

It is known that when  $v_3 = 0$ , the Poincaré map  $\Pi(x)$  near the origin is given by

$$\Pi(x) - x = v_5 x^5 + o(x^5),$$

where  $x$  is the first coordinate of a point on the semiaxis  $\{(x, 0) : x > 0\}$ . Then,

$$F(x) = \frac{\Pi(x) - x}{x^5} = v_5 + o(x) = v_5 + a_1 x^{r_1} + a_2 x^{r_2} + \cdots,$$

TABLE 2

Degree	Monomials of quasi-degree 4 and weight 0
1	$O$
2	$D\bar{D}, F\bar{F}, G\bar{G}, DF, A\bar{I}, AJ, BI, B\bar{J}, CH, C\bar{K}, E^2, E\bar{E}$
3	$A^2F, A^2\bar{D}, B^2D, B^2\bar{F}, A\bar{B}F, A\bar{B}\bar{D}, ACD, AC\bar{F}, B\bar{C}F, B\bar{C}\bar{D}, A\bar{C}G, B\bar{C}\bar{G}, A\bar{A}E, B\bar{B}E, C\bar{C}E, ABE, AB\bar{E}$
4	$A^3C, A^2\bar{B}C, A\bar{B}^2C, B^3\bar{C}, A^2\bar{A}^2, AB\bar{A}\bar{B}, AC\bar{A}\bar{C}, A^2\bar{A}\bar{B}, B^2\bar{B}^2, BC\bar{B}\bar{C}, AB^2\bar{B}, C^2\bar{C}^2, ABC\bar{C}, A^2B^2$

where  $1 \leq r_1 < r_2 < r_3 < \dots$ .

The constant  $v_5$  could be approximated by a direct computation of  $F(x)$  for  $x$  small enough, but the factor  $x^{r_i}$  might not be small enough near  $x = 0$ . We can obtain a better precision by increasing the powers  $r_i$ . This can be done by the Richardson's extrapolation method, described next.

From a sequence of values of  $F(x)$ , namely  $F(x_1), F(x_2), \dots, F(x_m)$ , such that  $x_{i+1} = q x_i$ ,  $x_1 = x$ ,  $i = 1, \dots, m-1$ , and  $q > 1$ , we define

$$\begin{cases} F_1(x) = F(x), \\ F_{j+1}(x) = F_j(x) + \frac{F_j(x) - F_j(qx)}{q^{r_j} - 1} \quad \text{for all } j \geq 1. \end{cases}$$

Then, it can be proved that

$$F_k(x) = v_5 + a_k^{(k)} x^{r_k} + a_{k+1}^{(k)} x^{r_{k+1}} + \dots, \quad k \leq m.$$

Therefore,  $F_k(x)$  is a better approximation for  $v_5$  than  $F(x)$ , for  $x$  small enough. (See [11] for more details.)

The images  $F(x_1), \dots, F(x_m)$  are obtained by using the Runge–Kutta–Fehlberg 7–8 method, with tolerance  $\epsilon_1 = 10^{-13}$ , and an initial step  $h_i = 10^{-5}$ , with maximum and minimum steps  $h_M = 10^{-1}$  and  $h_m = 10^{-16}$ , respectively, and a precision of  $10^{-16}$ . Observe that this is a three-parameter method, with parameters  $x$ ,  $q$ , and  $m$ .

**3.1. Computation of the second Liapunov constant.** In this subsection, by using the above procedure we compute the expression of the second Liapunov constant  $v_5$ . This gives the following well-known result (see [1], [2], [4], [7], [9], and [15]).

**THEOREM 3.** *Consider the equation  $\dot{z} = iz + F(z, \bar{z})$  with  $F_2(z, \bar{z}) = Az^2 + Bz\bar{z} + C\bar{z}^2$ ,  $F_3(z, \bar{z}) = Dz^3 + Ez^2\bar{z} + Fz\bar{z}^2 + G\bar{z}^3$ ,  $F_4(z, \bar{z}) = Hz^4 + Iz^3\bar{z} + Jz^2\bar{z}^2 + Kz\bar{z}^3 + L\bar{z}^4$ , and  $F_5(z, \bar{z}) = Mz^5 + Nz^4\bar{z} + Oz^3\bar{z}^2 + Pz^2\bar{z}^3 + Qz\bar{z}^4 + R\bar{z}^5$ . Then*

- (i)  $v_3 = 2\pi [\operatorname{Re}(E) - \operatorname{Im}(AB)]$ ,
- (ii)  $v_5 = \frac{\pi}{3} [6\operatorname{Re}(O) + \operatorname{Im}(3E^2 - 6DF + 6A\bar{I} - 12BI - 6B\bar{J} - 8CH - 2C\bar{K}) + \operatorname{Re}(-8C\bar{C}E + 4AC\bar{F} + 6A\bar{B}F + 6B\bar{C}F - 12B^2D - 4ACD - 6A\bar{B}\bar{D} + 10B\bar{C}\bar{D} + 4A\bar{C}G + 2B\bar{C}\bar{G}) + \operatorname{Im}(6A\bar{B}^2C + 3A^2B^2 - 4A^2\bar{B}C + 4\bar{B}^3C)]$ .

By using the appendix, we start listing the monomials satisfying Theorem 1. They are given in Table 2.

By Theorem A, the real monomials of even degree ( $D\bar{D}$ ,  $F\bar{F}$ ,  $G\bar{G}$ ,  $E\bar{E}$ ,  $A^2\bar{A}^2$ ,  $AB\bar{A}\bar{B}$ ,  $AC\bar{A}\bar{C}$ ,  $B^2\bar{B}^2$ ,  $BC\bar{B}\bar{C}$ ,  $C^2\bar{C}^2$ ) do not appear in  $v_5$ . Moreover, imposing that  $v_3 = 0$ , i.e.,  $\operatorname{Re}E = \operatorname{Im}(AB)$ , we can consider that  $\operatorname{Im}(ABC\bar{C}) = C\bar{C}\operatorname{Im}(AB) =$



TABLE 3  
Determination of the coefficients of  $v_5$ .

	Equation $\dot{z} = iz + F(z, \bar{z})$ , where $F(z, \bar{z})$ is:	Linear equation in $\alpha_k$
1	$z^3 \bar{z}^2$	$\alpha_1 = 6$
2	$z^3 - iz \bar{z}^2$	$-\alpha_2 = 6$
3	$z^2 - iz^3 \bar{z}$	$\alpha_3 = 6$
4	$z^2 + iz^2 \bar{z}^2$	$\alpha_4 = 0$
5	$z \bar{z} - iz^3 \bar{z}$	$-\alpha_5 = 12$
6	$z \bar{z} + iz^2 \bar{z}^2$	$-\alpha_6 = 6$
7	$\bar{z}^2 - iz^4$	$-\alpha_7 = 8$
8	$\bar{z}^2 + iz \bar{z}^3$	$-\alpha_8 = 2$
9	$z^2 + z \bar{z}^2$	$\alpha_{10} = 0$
10	$z \bar{z} - z^3$	$-\alpha_{11} = 12$
11	$z \bar{z} + z \bar{z}^2$	$\alpha_{12} = 0$
12	$z^2 + z \bar{z} + z \bar{z}^2$	$\alpha_{10} + \alpha_{12} + \alpha_{13} = 6$
13	$z^2 + z \bar{z} - z^3$	$-\alpha_{11} - \alpha_{14} = 18$
14	$z^2 + \bar{z}^2 - z^3$	$-\alpha_{15} = 4$
15	$z^2 + \bar{z}^2 + z \bar{z}^2$	$\alpha_{10} + \alpha_{16} = 4$
16	$z \bar{z} + \bar{z}^2 + z \bar{z}^2$	$\alpha_{12} + \alpha_{17} = 6$
17	$z \bar{z} + \bar{z}^2 + z^3$	$\alpha_{11} + \alpha_{18} = -2$
18	$z^2 + \bar{z}^2 + \bar{z}^3$	$\alpha_{19} = 4$
19	$z \bar{z} + \bar{z}^2 + \bar{z}^3$	$\alpha_{20} = 2$
20	$iz^2 + \bar{z}^2$	$-\alpha_{24} = 0$
21	$z \bar{z} + iz^2$	$-\alpha_{27} = 4$
22	$z^2 + iz \bar{z} + (1+i)z^2 \bar{z}$	$\alpha_{21} + \alpha_{22} + 2\alpha_9 = 6$
23	$z^2 + (1+i)z \bar{z} + (1+i)z^2 \bar{z}$	$2\alpha_9 + \alpha_{21} + 2\alpha_{22} + 2\alpha_{28} = 12$
24	$(1+i)z^2 + z \bar{z} + (1+i)z^2 \bar{z}$	$2\alpha_9 + 2\alpha_{21} + \alpha_{22} + 2\alpha_{28} = 12$
25	$z^2 + (1+i)z \bar{z} + z^2 \bar{z}$	$\alpha_{21} + 2\alpha_{22} + 2\alpha_{28} = 6$
26	$z^2 - iz \bar{z} + \bar{z}^2 - z^2 \bar{z}$	$-\alpha_{21} - \alpha_{22} - \alpha_{23} + \alpha_{25} + \alpha_{27} = 0$
27	$iz^2 + z \bar{z} + \bar{z}^2 + z^2 \bar{z}$	$\alpha_{21} + \alpha_{22} + \alpha_{23} - \alpha_{24} + \alpha_{26} = -2$
28	$iz^2 + iz \bar{z} + \bar{z}^2$	$-\alpha_{24} + \alpha_{25} - \alpha_{26} - \alpha_{27} = -6$

$\text{Re}(C\bar{C}E)$ , and so the monomials  $ABE$ ,  $AB\bar{E}$ ,  $A^2\bar{A}B$ ,  $AB^2\bar{B}$ , and  $ABCC\bar{C}$  can be eliminated. On the other hand, since the systems of type  $\dot{z} = iz + Az^2 + Dz^3$  are isochronous centers, we can also deduce that the monomial  $A^2\bar{D}$  does not appear. Taking all this into account, we end Step 1 and we have that

$$\begin{aligned}
v_5 = & \frac{\pi}{3} \left[ \alpha_1 \text{Re}O + \text{Im} \left( \alpha_2 DF + \alpha_3 A\bar{I} + \alpha_4 AJ + \alpha_5 BI + \alpha_6 B\bar{J} + \alpha_7 CH + \alpha_8 C\bar{K} + \alpha_9 E^2 \right) \right. \\
& + \text{Re} \left( \alpha_{10} A^2 F + \alpha_{11} B^2 D + \alpha_{12} B^2 \bar{F} + \alpha_{13} A\bar{B}F + \alpha_{14} A\bar{B}\bar{D} + \alpha_{15} ACD + \alpha_{16} AC\bar{F} \right. \\
& \quad \left. + \alpha_{17} B\bar{C}F + \alpha_{18} B\bar{C}\bar{D} + \alpha_{19} A\bar{C}G + \alpha_{20} BC\bar{G} + \alpha_{21} A\bar{A}E + \alpha_{22} B\bar{B}E + \alpha_{23} C\bar{C}E \right) \\
& \left. + \text{Im} \left( \alpha_{24} A^3 C + \alpha_{25} A^2 \bar{B}C + \alpha_{26} A\bar{B}^2 C + \alpha_{27} B^3 \bar{C} + \alpha_{28} A^2 B^2 \right) \right].
\end{aligned}$$

Now we must search for the undetermined coefficients  $\alpha_j$ , for  $j = 1, \dots, 28$ .

From formula (2.3) we can deduce that the coefficients  $\alpha_j$  of the monomials of  $v_5$  have a common factor  $\pi$ . We also introduce a handling which permits a higher reliability on the numerical results, since we have observed that taking a system with integer coefficients and multiplying by  $\frac{3}{\pi}$  the numerical approximation of the constant obtained from Step 2, we get a result very close to an integer number. In fact, we obtain numbers  $v$  such that there exist  $n \in \mathbf{Z}$  satisfying  $|v - n| < 10^{-3}$ . Then, it is clear which is the integer result we must consider to be the correct one. For instance, if we take system 1 of Table 3 and apply to it Step 2 of our method, we have that  $\frac{3}{\pi} v_5 \approx 5.9999984$ . Therefore we consider that  $\frac{3}{\pi} v_5 = 6$ .

We choose 28 differential equations for which  $v_3 = 0$  and for which many of the monomials appearing in  $v_5$  vanish. We compute the value of the constant applying the numerical method explained above and we reach a linear system of 28 equations where each component of the independent vector is the result of one of the 28 numerical experiments, rounded off to the closer integer number. In Table 3, we show the differential equations chosen and the linear equations derived from them.

Solving the linear system, we obtain the coefficients of the expression of Theorem 3(ii). Of course, it coincides with the results obtained in the previous works, although this procedure cannot be strictly considered as a proof. A similar idea could be used to find the period constants.

In general, the problem of choosing which differential equations we have to consider to obtain the linear system of equations can be reduced to a problem of multivariate interpolation with multidimensional grid. Nevertheless, this approach gives rise to many redundant linear equations. In the present case, we have chosen the set of differential equations simply by looking at the nonzero monomials of  $v_5$ .

### 3.2. Computation of third Liapunov constant for a particular system.

In this example, we apply the numerical method to compute  $v_7$  for the particular quadratic system

$$\begin{cases} \dot{x} = -y + x^2 + 2xy, \\ \dot{y} = x + x^2 + 3xy - y^2, \end{cases}$$

for which it is known (see Remark 4) that  $v_3 = v_5 = 0$  and  $v_7 = \frac{25\pi}{32} \approx 2.45436926$ .

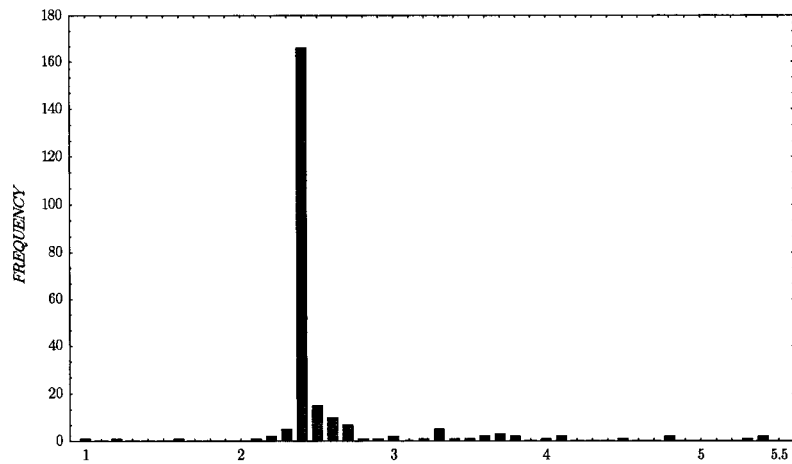
We are going to show how we obtained an approximation of this value from the numerical procedure explained at the beginning of this section without using that  $v_7$  is a rational multiple of  $\pi$ . The final value obtained from that procedure, after fixing  $x$ ,  $q$ , and  $m$ , will be called  $v_7(x, q, m)$ .

We consider  $x \in I_x = [0.01, 0.065]$ ,  $q \in I_q = [1.075, 1.425]$ , and  $m \in I_m = \{4, 5, 6, 7\}$ . Then, with the values of  $v_7(x, q, m)$  obtained for  $(x, q, m) \in I_x \times I_q \times I_m$ , we make a simple statistical estimation of the value of  $v_7$ .

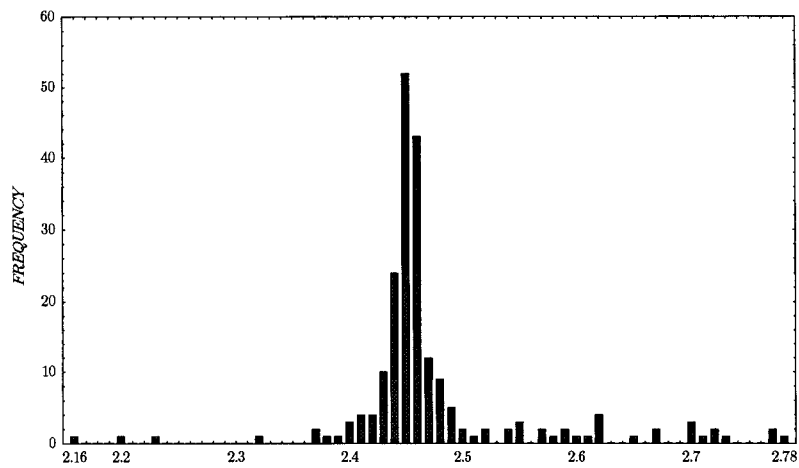
The estimator we consider is the sample mode. The main reason is that we believe that when the  $(x, q, m)$ -process works, it gives *good* approximations of  $v_7$ ; however, when it does not work (for instance, because of the numerical instability), the error can be big and often with the same sign. This is a definitive argument against the consideration of the mean and the median in our problem.

To display the accuracy of this example, we give some *graphics of the frequencies* (see Figure 1), considering from graphic to graphic one more digit of the values. In this way, it can be observed that (for this example) the method adjusts statistically up to the third digit. In the third graphic (corresponding to the fourth digit; see Figure 1(c)), we can appreciate a great dispersion of the values, which indicates the reliability of the method only up to the third digit, with a relative error of  $\delta \approx \frac{5}{2.45} 10^{-3} \approx 2 \cdot 10^{-3}$ .

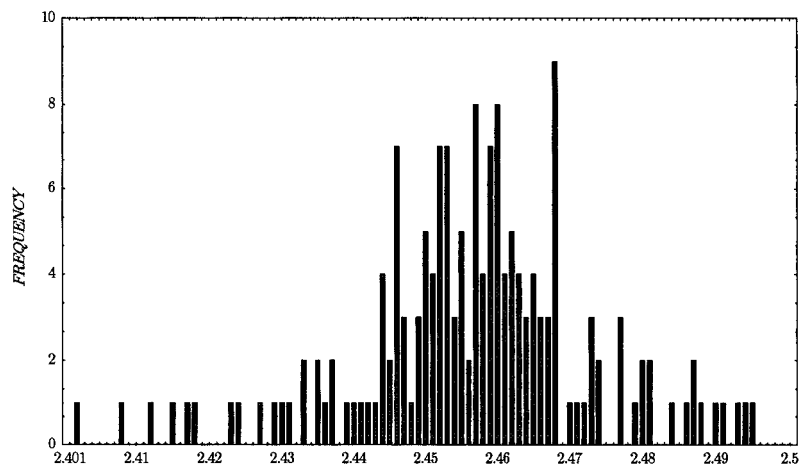
Obviously, this numerical conclusion is valid only for this example, but the procedure can be applied to any other system. To compute  $v_{2k+1}$ , notice that for the set of values given above,  $\pi(x) - x \approx v_{2k+1} 10^{-2(2k+1)}$ . On the other hand, remember that for the Runge–Kutta–Fehlberg 7–8 method,  $T/h \approx |w - \tilde{w}|$ , where  $w$  and  $\tilde{w}$  are the approximations given the RK7 and RK8 methods and  $T$  is the tolerance. Hence, if we want to compute  $v_{2k+1}$  with relative error  $10^{-t}$  (assuming step size in the Runge–Kutta–Fehlberg 7–8 method of order  $10^{-5}$ ), roughly speaking, it is necessary to work with a tolerance  $T \approx 10^{2-t-4k}$ .



(a) Frequencies taking into account up to the first decimal.



(b) Frequencies taking into account up to the second decimal.



(c) Frequencies taking into account up to the third decimal.

FIG. 1. Frequencies for the value of  $v_7$  increasing the number of digits of precision.

**Appendix. Finiteness of basic monomials.** In this appendix we prove the following result.

PROPOSITION B. *The number of basic monomials associated with system (1.1) with  $F(z, \bar{z})$  polynomial is finite.*

REMARK 4. *Denote by  $\mathcal{B}$  the finite set of basic monomials associated to a polynomial system (1.1). Observe that the above result implies that the Liapunov constants are real-valued polynomials in the ring  $\mathbf{C}[\mathcal{B}]$ . For instance, consider system  $\dot{z} = iz + f_{20}z^2 + f_{11}z\bar{z} + f_{02}\bar{z}^2$ . The basic monomials associated with it are  $B_1 = f_{20}\bar{f}_{20}$ ,  $B_2 = f_{11}\bar{f}_{11}$ ,  $B_3 = f_{02}\bar{f}_{02}$ ,  $B_4 = f_{20}f_{11}$ ,  $B_5 = f_{20}^3\bar{f}_{02}$ ,  $B_6 = \bar{f}_{11}^3f_{02}$ ,  $B_7 = f_{20}^2\bar{f}_{11}f_{02}$ ,  $B_8 = f_{20}\bar{f}_{11}^2f_{02}$ , and their conjugates. Furthermore, also applying Theorems 1 and A we get that*

$$\begin{aligned} v_3 &= \alpha_1 \operatorname{Im} B_4, \\ v_5 &= \operatorname{Im} F_0, \\ v_7 &= \sum_{i=1}^3 B_i \operatorname{Im} F_i + \operatorname{Im}(B_4 F_4 + \bar{B}_4 F_5), \end{aligned}$$

where  $F_i = F_i(B_5, B_6, B_7, B_8)$  are linear functions in all the variables.

Indeed, it is a known fact that for quadratic systems

$$\begin{aligned} v_3 &= -2\pi \operatorname{Im}(B_4), \\ v_5 &= -\frac{2\pi}{3} \operatorname{Im}(2B_7 - 3B_8 - 2B_6), \\ v_7 &= -\frac{5\pi}{4} (B_2 - B_3) \operatorname{Im}(2B_8 + B_6). \end{aligned}$$

We will use the next technical result to prove Proposition B.

LEMMA 5. *Consider  $\alpha_1, \alpha_2, \dots, \alpha_s \in \mathbf{Z}$ . Then, there exists a positive integer  $m$ ,  $0 \leq m \leq s$ , and a reordering  $\{\beta_1, \dots, \beta_s\}$  of the numbers  $\alpha_1, \dots, \alpha_s$  such that*

$$\sum_{i=1}^m \beta_i \equiv 0 \pmod{s}.$$

*Proof.* Consider  $S_k := \sum_{i=1}^k \alpha_i$  for all  $k \leq s$ . If  $S_k \equiv S_j \pmod{s}$  for some  $k \neq j$  (let us suppose  $k > j$ ), then  $S_k - S_j = \alpha_{j+1} + \dots + \alpha_k \equiv 0 \pmod{s}$ , and so the lemma would be fulfilled. Otherwise,  $S_1, S_2, \dots, S_s$  belong to different classes in  $\mathbf{Z}_s$  and therefore one of them must be  $\bar{0} \in \mathbf{Z}_s$ . Then, the lemma also follows.  $\square$

*Proof of Proposition B.* Denote by  $B = C_1^{\alpha_1} \dots C_n^{\alpha_n}$  a basic monomial of (1.1), where  $C_1, \dots, C_n$  are the coefficients of the system, and suppose that  $|w(C_i)| = i$  for all  $i$ . In fact,  $C_i$  is a generic label representing any coefficient of (1.1) of weight  $\pm i$ , while  $\alpha_i$  is the number of coefficients taking into account the repetitions (in a basic monomial of degree greater than two, there cannot coexist coefficients of weight  $i$  and  $-i$ ). Then, Proposition B follows if we are able to prove that

$$(A.1) \quad \alpha_1 + 2\alpha_2 + \dots + n\alpha_n = 0$$

has finitely many *minimal* solutions, where we say that a solution  $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}^n$  is a minimal solution of (A.1) if there does not exist any other solution  $(\beta_1, \dots, \beta_n)$  such that  $|\beta_i| \leq |\alpha_i|$  and  $\operatorname{sign}(\beta_i) = \operatorname{sign}(\alpha_i)$  for all  $i = 1, \dots, n$ .

In particular, the finiteness of minimal solutions will be proved if we prove that any minimal solution  $\alpha$  of (A.1) satisfies that  $|\alpha_i| \leq n$  for all  $i = 1, \dots, n$ .

Suppose that  $\alpha_k > n$  for some  $k \in \{1, \dots, n\}$ . First, we will see that this implies that  $\alpha_j \geq -n$  for any  $j$ . Otherwise, if for some  $j$  we have  $\alpha_j < -n$ , we can write  $\alpha_k = j + N_k$  and  $\alpha_j = -k - N_j$ , where  $N_k$  and  $N_j$  are positive numbers. Then,

$$\begin{aligned}\alpha &= (\alpha_1, \dots, \alpha_j, \dots, \alpha_k, \dots, \alpha_n) \\ &= (\alpha_1, \dots, -N_j, \dots, N_k, \dots, \alpha_n) + (0, \dots, 0, -k, \dots, j, 0, \dots, 0),\end{aligned}$$

and this situation breaks the property of minimality of  $\alpha$ .

Now, we keep the assumption that  $\alpha_k > n$  for some  $k \in \{1, \dots, n\}$ . Since  $\alpha$  must be a solution of (A.1), we deduce that there must exist  $\alpha_{j_1}, \dots, \alpha_{j_r}$ , all of them negative, such that  $\sigma_k := \sum_{i=1}^r j_i \alpha_{j_i} < -kn$ . We can write

$$\sigma_k = \begin{matrix} (-\alpha_{j_1}) \\ -(j_1 + \dots + j_1 \end{matrix} + \dots + \begin{matrix} (-\alpha_{j_r}) \\ j_r + \dots + j_r \end{matrix}.$$

Since  $j_i \leq n$  for all  $i$  and  $\sigma_k < -kn$ , the number of terms involved in the last expression of  $\sigma_k$  is greater than or equal to  $k$ . Then, we choose any string of  $k$  terms in  $\sigma_k$  and we call it  $\sigma'_k$ . It is obvious that

$$\sigma'_k = \sum_{i=1}^r j_i \alpha'_{j_i} \quad \text{with} \quad 0 \geq \alpha'_{j_i} \geq \alpha_{j_i} \quad \text{for all } i.$$

Then, since  $\sigma'_k$  is a string of length  $k$ , we can apply Lemma 5 to it and deduce that it contains a partial sum,  $\sigma''_k$ , such that  $\sigma''_k \equiv 0 \pmod{k}$ . Then, if we write  $\sigma''_k = \sum_{i=1}^r j_i \alpha''_{j_i} = -k\mu$ , with  $0 < \mu \leq n$ , we can construct the following solution  $\beta := (\beta_1, \dots, \beta_n)$  of (A.1):

$$\beta_i = \begin{cases} \alpha''_{j_i} & \text{if } i \in \{j_1, \dots, j_r\}, \\ \mu & \text{if } i = k, \\ 0 & \text{if } i \notin \{k, j_1, \dots, j_r\}. \end{cases}$$

From their definitions, it is clear that  $0 \geq \alpha''_{j_i} \geq \alpha_{j_i}$  for all  $i \in \{1, \dots, r\}$ . On the other hand,  $0 < \mu \leq n < \alpha_k$ . Then,  $\beta$  is a solution which breaks again the minimality of  $\alpha$  and gives the contradiction we were searching for. Therefore, the proposition is proved.  $\square$

**REMARK 6.** Observe that the proof of Proposition B provides the bound  $(2n+1)^n$  for the maximum number of monomials of the form  $C_1^{\alpha_1} \dots C_n^{\alpha_n}$ , because  $|\alpha_i| \leq n$  for all  $i = 1, \dots, n$ . Furthermore, note that if the degree of  $F$  in system (1.1) is  $d$ , then  $n = d+1$ . As can be seen in Remark 4, the bound  $(2d+3)^{d+1}$  is not a sharp one.

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#### REFERENCES

- [1] A.A. ANDRONOV, E.A. LEONTOVICH, I.I. GORDON, AND A.G. MAIER, *Theory of Bifurcations of Dynamic Systems on a Plane*, John Wiley, New York, 1973.
- [2] A. CIMA, A. GASULL, V. MAÑOSA, AND F. MAÑOSAS, *Algebraic properties of the Liapunov and period constants*, Rocky Mountain J. Math., 27 (1997), pp. 471–501.

- [3] C. CHICONE AND M. JACOBS, *Bifurcation of critical periods for plane vector fields*, Trans. Amer. Math. Soc., 312 (1989), pp. 433–486.
- [4] W.W. FARR, C. LI, I.S. LABOURIAU, AND W.F. LANGFORD, *Degenerate Hopf bifurcation formulas and Hilbert's 16th problem*, SIAM J. Math. Anal., 20 (1989), pp. 13–30.
- [5] A. GASULL, A. GUILLAMON, C. LI, AND Z. ZHANG, *Study of perturbed Lotka-Volterra systems via Abelian integrals*, J. Math. Anal. Appl., 198 (1996), pp. 703–728.
- [6] A. GASULL, A. GUILLAMON, AND V. MAÑOSA, *An explicit expression of the first Liapunov and period constants with applications*, J. Math. Anal. Appl., 211 (1997), pp. 190–212.
- [7] A. GASULL AND R. PROHENS, *Effective computation of the first Lyapunov quantities for a planar differential equation*, Appl. Math. (Warsaw), 24 (1997), pp. 243–250.
- [8] I.S. GRADSHTEYN AND I.M. RYZHIK, *Table of Integrals, Series and Products*, Academic Press, New York, 1980, p. 14.
- [9] F. GÖBBER AND K.D. WILLAMOWSKI, *Liapunov approach to multiple Hopf bifurcation*, J. Math. Anal. Appl., 71 (1979), pp. 333–350.
- [10] B. HASSARD AND Y.H. WAN, *Bifurcation formulae derived from Center Manifold Theory*, J. Math. Anal. Appl., 63 (1978), pp. 297–312.
- [11] D. KINCAID AND W. CHENEY, *Numerical Analysis*, Brooks/Cole, Pacific Grove, CA, 1991, pp. 436–441.
- [12] Y. LIN AND J.-B. LI, *Normal form of planar autonomous system and periodic critical point*, Acta Math. Sinica, 34 (1991), pp. 490–501 (in Chinese).
- [13] Y.-R. LIU AND J.-B. LI, *Theory of values of singular point in complex autonomous differential systems*, Sci. China Ser. A, 33 (1990), pp. 10–23.
- [14] N.G. LLOYD AND J.M. PEARSON, *REDUCE and the bifurcation of limit cycles*, J. Symbolic Comput., 9 (1990), pp. 215–224.
- [15] E.J. PONCE-NÚÑEZ AND E. GAMERO, *Generating Hopf bifurcation formulae with MAPLE*, Internat. Ser. Numer. Math., 97 (1991), pp. 295–298.
- [16] V.G. ROMANOVSKII, *Calculation of Lyapunov numbers in the case of two pure imaginary roots*, Differential Equations, 29 (1993), pp. 782–784.
- [17] K.S. SIBIRSKII, *Algebraic Invariants for Differential Equations and Matrices*, Kshinev, Shtiintsa, 1976 (in Russian).
- [18] H. ŻOŁĄDEK, *Quadratic systems with center and their perturbations*, J. Differential Equations, 109 (1994), pp. 223–273.