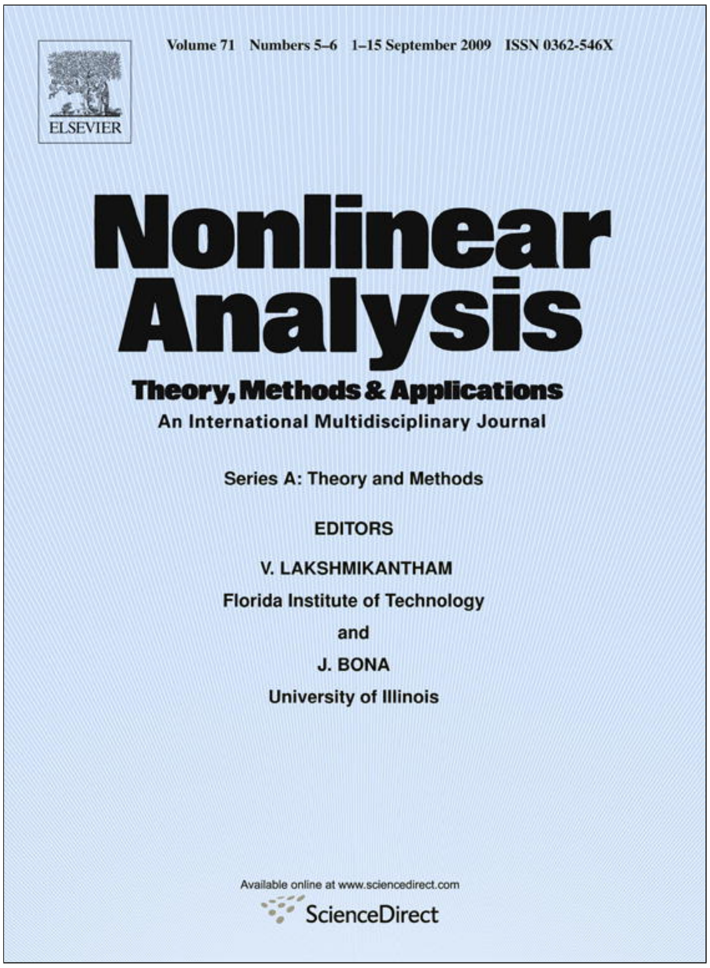
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The number of limit cycles in planar systems and generalized Abel equations with monotonous hyperbolicity

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ABSTRACT

We extend some previous results on the maximum number of isolated periodic solutions of generalized Abel equation and rigid systems. The key hypothesis is a monotonicity assumption on any stability operator (for instance, the divergence) along the solutions of a suitable transversal system. In such a case, at most two isolated periodic solutions exist. Under a simple additional assumption, we also prove a uniqueness result for limit cycles of rigid systems. Our results are easily applicable to special classes of equations, since the hypotheses hold when a suitable convexity property is satisfied.

1. Introduction

This paper is concerned with estimating the number of limit cycles (i.e. isolated cycles) of planar differential systems

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y),$$

with P, Q of class C^2 in a some planar region Ω. In the case of polynomial systems such a question is the object of the celebrated XVI Hilbert's problem (see for instance [1]). Hilbert's problem is still unanswered, even for quadratic systems. Recently, some progress has been done for special classes of systems, in particular for the so-called rigid systems

$$\dot{x} = -y + xF(x, y), \qquad \dot{y} = x + yF(x, y).$$

In this paper we look at rigid systems as embedded in the more general class of systems that can be "rigidized" through a Lie symmetry. That is, starting from a Lie symmetry among two transversal vector fields V and W, $[W, V] = \mu W$, we can transform any vector field X transversal to W to an equivalent one (in terms of stability) $\tilde{X} = V + FW$, for a scalar function F. Generalizing the idea of polar coordinates ($\tilde{r} = V \wedge W$, and $\tilde{\theta}$ the phase defined by the W-orbits), $\tilde{X}$ can be seen as a rigid system. This approach will be revisited both in the results and some examples. For the motivation purposes, we come back to the classical rigid systems, where $V = (-y, x)$, $W = (x, y)$ and $\mu \equiv 0$.

Rigid systems have a very simple form in polar coordinates,

$$\dot{r} = \sum_{j=1}^{n} a_j(\theta) r^j, \qquad \dot{\theta} = 1, \tag{1}$$

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where $a_j(\theta)$ is a 2π-periodic function, for $j = 1, \ldots, n$. If the rigid system is polynomial, then every $a_j(\theta)$ is a trigonometric polynomial of degree $j-1$. In a natural way, this leads to consider them as special cases of those considered in a problem posed by Pugh [2], who asked for upper bounds to the number of isolated periodic solutions of the equation

$$\dot{r} = \sum_{j=0}^{n} a_j(t) r^j, \tag{2}$$

where $a_j(t)$ is a T-periodic continuous function, for $j = 0, \ldots, n$. Particular cases are Riccati's equation, for $n = 2$, and Abel equation, for $n = 3$. Both have been extensively studied. For what is concerned to periodic solutions, Neto ([2]) proved that Riccati's periodic equation has at most two isolated periodic solutions. In the same paper, it was proved that Abel's equation has at most three isolated periodic solutions, if the leading coefficient $a_3(t)$ does not change sign (see also [3,4]). Moreover, Neto proved that for every n there exists an equation of the type

$$\dot{r} = a_3(\theta) r^3 + a_2(\theta) r^2,$$

having more than n isolated periodic solutions.

The interest in Abel's equation is also motivated by quadratic systems in the plane, whose study can be reduced (in different regions) to that of equations of the form

$$\dot{r} = a_3(\theta) r^3 + a_2(\theta) r^2 + \lambda r, \qquad r > 0, \quad \theta \in [0, 2\pi) \tag{3}$$

where $a_3(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degrees six and three, respectively, and λ is a fixed real number. Limit cycles of the quadratic system correspond to periodic solutions of (3). Gasull and Llibre (see [3]) showed that such equations have at most three isolated periodic solutions, provided either $a_3(\theta)$ or $a_2(\theta)$ do not change sign [3].

Recently, in [5], Gasull and Guillamon proved that every polynomial rigid system, with polar form

$$\dot{r} = a_m(\theta) r^m + a_k(\theta) r^k + \lambda r, \tag{4}$$

has at most two positive limit cycles. In such a case the functions a_m, a_k are trigonometric polynomials of degrees $m-1$, $k-1$, respectively, and one of them does not change sign.

We extend some of the above results by comparing the hyperbolicity properties of concentric limit cycles. Since we consider such systems as the polar forms of planar systems, in principle we should only be concerned with positive solutions of equations in the form (1). On the other hand, reducing to a suitable annulus in the plane, we may state some results even about equations in the form (2). The main idea consists in assuming that some "stability operator" increases (decreases) along the rays of a suitable system of polar-like coordinates. This implies the stability of external cycles to be different from the stability of internal cycles, giving a bound of two concentric limit cycles. Under some additional hypotheses, we also prove a limit cycles' uniqueness result.

There exist several operators that control the stability of limit cycles of a vector field X on the plane. The best known one is the divergence. More precisely, given a limit cycle $\Gamma = \{\gamma(t), t \in [0, T)\}$ of X, Γ is called hyperbolic if $\pi_\Gamma := \int_0^T \mathrm{div}(X(\gamma(t)))\mathrm{d}t \neq 0$. If $\pi_\Gamma < 0$ (resp., >0), then limit cycle is stable (resp., unstable). In general it is not easy to estimate the sign of the above integral, so that one usually tries to find a Dulac function, that is a function $B(x, y) > 0$ such that $\mathrm{div}(BX)$ has constant sign on the given region. The new vector field and the old one have the same orbits, so the orbital stability properties of BX are the same than those of X. A second stability operator can be obtained by means of the Lie brackets. In fact, for every vector field W transversal to X in a neighbourhood of Γ, it is always possible to write the Lie bracket of X and W as a linear combination, $[W, X] = \nu W + \beta X$. Then $\pi_\Gamma = \int_0^T \nu(\gamma(t))\mathrm{d}t$ (see for instance [6,7]). Here, again, there is a degree of freedom in the choice of a suitable ν, since the choice of W is arbitrary. There exist other operators that can be found in the literature (like orthogonal curvature or inverse of integrating factors); since they do not appear in the rest of this paper, we do not include them here for the sake of conciseness.

All of this paper is based on the existence of a suitable Lie symmetry that allows the comparison between concentric cycles even when we use the divergence as a stability operator. Although the method presented here seems to have potentially a wide range of applicability, our main applications will be restricted to rigid systems and to periodic scalar equations, not necessarily polynomial or with trigonometric coefficients. As an example of the type of results contained in this paper, we prove that if $f(r, \theta)$ is r-positively convex, then the equation

$$\dot{r} = f(r, \theta)$$

has at most two isolated positive periodic solutions. For what is concerned with generalized Abel equations, we prove that

$$\dot{r} = \sum_{j=1}^{m-1} a_j(t) r^j + a_m(t) r^m - \sum_{j=m+1}^{n} a_j(t) r^j, \quad 1 < m < n, \tag{5}$$

with $a_j(t) \geq 0$ for $j \neq m$, has at most two positive isolated periodic solutions.

The main results are developed in Section 2. We want to emphasize that, from a geometric point of view, the results that we obtain can be often interpreted as conditions of convexity (in a general metrics) that provide bounds for the number of limit cycles. In Section 3 we derive some corollaries of the main theorem which are very useful to visualize the convexity properties.

2. Radial-like derivatives of stability operators (main results)

Let $\Omega \subset \mathbb{R}^2$ be an annular planar region. For the sake of simplicity, from now on, all the functions and vector fields we consider are assumed to be of class C^2, even if in several cases such assumption could be weakened. Let X be a vector field on Ω. In the following, we denote by γ any orbit of the differential system

$$z' = X(z), \qquad z = (x, y) \in \Omega,$$

and by $\gamma^+ = \{\gamma(t) : t \geq 0\}$ its positive semi-orbit. For any α defined on Ω, we denote by $\partial_X \alpha$ the derivative of α along the solutions of $z' = X(z)$. Similarly for the derivative $\partial_X Y$ of a vector field Y. We say that a function α is an *S-operator* if, for every cycle $\gamma \in \Omega$, $\gamma(T) = \gamma(0)$, the condition

$$I_\alpha(\gamma) := \int_0^T \alpha(\gamma(t))\mathrm{d}t < 0, \tag{6}$$

implies the orbital asymptotic stability of γ. There exist at least three distinct S-operators. The best known one is the divergence div$X$, whose integral (6) gives the limit cycle's hyperbolicity. Recently, two additional S-operators have been found, considering the curvature of a suitable vector field, [8], or a suitable Lie symmetry, [6] or [7].

We will focus on the Lie symmetry approach, but the technique can be applied for any other S-operator as well. Let us suppose that V is the infinitesimal generator of a Lie symmetry of an auxiliary vector field W on Ω. In other words,

$$[W(z), V(z)] = \mu(z)\, W(z),$$

where [,] represents the Lie bracket, defined as $[W, V] = \partial_W V - \partial_V W$. Throughout all of this paper we assume V and W to be transversal on Ω. Let us set $V \wedge W = \det(V, W) = v_1 w_2 - v_2 w_1$, where $V = (v_1, v_2)$ and $W = (w_1, w_2)$. Under the previous assumptions, every vector field X transversal to W can be written as follows

$$X = AV + BW,$$

where

$$A = \frac{X \wedge W}{V \wedge W}, \qquad B = \frac{X \wedge V}{W \wedge V},$$

are scalar functions. Assuming $A \neq 0$ on Ω, one can divide X by A and consider the new vector field

$$\tilde{X} = V + FW, \qquad F := \frac{X \wedge V}{X \wedge W},$$

and its differential system

$$z' = \tilde{X}(z), \tag{7}$$

which has the same orbits as the previous one. The stability analysis may be carried over such a vector field, obtaining equivalent results.

Observe that the vector field X presents a kind of "angular+radial" structure since, typically, we will choose V to have a well defined return map in some region whereas W (transversal to V) will behave as a radial vector field. Thus, the directions defined by the orbits of W will play an important role in the radial-like strategy to study the stability and number of limit cycles, as next definitions and results show. For this purpose, and for the sake of simplicity of notation, we say that the function α *is a W-function* if both the following conditions hold:

(i) *$\partial_W \alpha \geq 0$ (≤ 0) on Ω;*
(ii) *every couple of cycles can be connected by an arc δ of a W-orbit such that $\partial_W \alpha > 0$ (< 0) at least at a point $\bar{x}$ of δ.*

The W-function condition for an S-operator provides simple conditions to bound the number of limit cycles of planar systems. The following result is a generalization to non-rigid systems and to any S-operator of Theorem 3 in [5]. In the following we assume the W-orbits to meet the V-cycles crossing them from the interior towards the exterior.

Theorem 1. *Let α be an S-operator on an annular planar region Ω, and let V, W satisfy $[W(z), V(z)] = \mu(z)\, W(z)$, for $z \in \Omega$. Assume α to be a W-function on Ω and W to have no cycles. Then, if V and W are transversal on Ω, the differential system (7) has at most two limit cycles in Ω. More precisely:*

(i) *if it has two cycles, then they have opposite stability character;*
(ii) *if it has a semistable cycle, then it has no other cycles.*

The stability character of the above cycles is determined by the sign of $\partial_W \alpha$.

Proof. We first show that (7) has at most three limit cycles, then we refine our statement proving that there exist at most two limit cycles.

Without loss of generality, we may assume that $\partial_W \alpha \geq 0$. Let us consider a couple of $\tilde{X}$-cycles. By the absence of critical points, every couple of $\tilde{X}$-cycles γ_1, γ_2 have to be concentric. We claim that a W-orbit δ either meets both γ_1 and γ_2, or it meets none of them. In fact, assume γ_1 to be contained in the bounded region defined by γ_2. Let δ be a W-orbit crossing γ_1 at z_1. Assume δ^+ to enter at z_1 the annular region A_{12} bounded by γ_1 and γ_2. If, by absurd, δ^+ were entirely contained in A_{12}, then it would be positively compact, with a non-empty positive limit set, a W-cycle, since W has no critical points. This contradicts the hypothesis that W has no cycles.

Hence every W-orbit δ entering A_{12} at a point of $z_1 \in \gamma_1$ meets γ_2. By the transversality, δ^+ has to leave A_{12} at a point of $z_2 \in \gamma_2$. Let us denote by $\Phi : \gamma_1 \to \gamma_2$ the map that sends such a z_1 to such a z_2: $\Phi(z_1) = z_2$.

Then, observe that

$$[W, \tilde{X}] = [W, V + FW] = (\mu + \partial_W F)\, W,$$

hence $\tilde{X}$ is a normalizer of W. By Theorem 4 in [9], the orbits of W are the isochrons of any cycle of X. In particular, any neighbourhood of a cycle of X is foliated by orbits of W. Since A_{12} is contained in the intersection of the foliated neighbourhoods of γ_1 and γ_2, both cycles have the same period, say T.

From the fact that α is a W-function on Ω, for every $t \in [0, T]$ one has $\alpha(\gamma_1(t)) \leq \alpha(\Phi(\gamma_1(t))) = \alpha(\gamma_2(t))$, and there exists at least a $\bar{t} \in [0, T]$ such that $\alpha(\gamma_1(\bar{t})) < \alpha(\Phi(\gamma_1(\bar{t}))) = \alpha(\gamma_2(\bar{t}))$. Hence,

$$I_\alpha(\gamma_1) = \int_0^T \alpha(\gamma_1(t))\mathrm{d}t < \int_0^T \Phi(\alpha(\gamma_1(t)))\mathrm{d}t = \int_0^T \alpha(\gamma_2(t))\mathrm{d}t = I_\alpha(\gamma_2).$$

Hence, if $I_\alpha(\gamma_1) > 0$, then no other cycles can enclose it, since all of them should be repelling (recall that α is an S-operator).

If $I_\alpha(\gamma_1) = 0$ and γ_2 encloses γ_1, then $I_\alpha(\gamma_2) > 0$. This both shows that γ_1 is not contained in a period annulus, and that at most one cycle can enclose γ_1.

Now we claim that if $I_\alpha(\gamma_1) < 0$, and γ_2 encloses γ_1, then $I_\alpha(\gamma_2) \geq 0$. In fact, if $I_\alpha(\gamma_2) < 0$, every cycle γ contained in the annular region A_{12} would satisfy $I_\alpha(\gamma) < 0$, so that γ_1's region of attraction would have a repelling external boundary, that is not compatible with its repulsiveness.

Concluding, (7) has at most three limit cycles: an attracting one, a semistable one and a repelling one.

We prove now that, if (7) has two hyperbolic cycles, then it has no semistable cycles. Assume by absurd that a semistable cycle γ exists, together with an attracting γ_1 and a repelling γ_2, with γ_1 enclosed in γ, and γ enclosed in γ_2. Consider the rotated family of vector fields $\tilde{X}_\theta := \cos(\theta)\tilde{X} + \sin(\theta)W$, $\theta \in [0, 2\pi)$. By the semi-stability of γ, the theory of rotated vector fields (see [10]) shows that for small positive (negative) values of θ the vector field $\tilde{X}_\theta$ has two limit cycles bifurcating from γ. Hence, for such values of θ, $\tilde{X}_\theta$ has at least four limit cycles. On the other hand, for small values of θ the vector field $\tilde{X}_\theta$ satisfies the theorem's hypotheses, since

$$[W, \tilde{X}_\theta] = [W, \cos(\theta)\tilde{X} + \sin(\theta)W] = \cos(\theta)[W, \tilde{X}] = \cos(\theta)\,(\mu + \partial_W F)\, W,$$

and on a compact set the condition $\partial_W \alpha > 0$ is preserved, hence $\tilde{X}_\theta$ has no more than 3 limit cycles, contradiction.

Now let us assume that two limit cycles exist, a semistable one and a hyperbolic one (two semistable ones is not compatible with $\partial_W \alpha > 0$). Without loss of generality, we may consider only the case of an inner semistable cycle γ and an outer repelling cycle γ_2. In this case γ is externally attracting. Let us apply again the bifurcation argument, getting two limit cycles γ_i and γ_e bifurcating from γ. The internal bifurcating cycle, γ_i, is repelling, while the external bifurcating cycle, γ_e, is attracting. Hence $\tilde{X}_\theta$ should have 3 limit cycles, repelling/attracting/repelling, which contradicts the condition $\partial_W \alpha > 0$. □

Remark 1. If $\partial_W \alpha \geq 0$ and on every W-orbit there exists a point where $\partial_W \alpha > 0$ then the absence of W-cycles can be proved by observing that α is a single-valued function.

Corollary 1. *Under the hypotheses of Theorem 1, assume Ω to be (negatively) asymptotically stable. Then Ω contains at most one limit cycle.*

Proof. Without loss of generality, we may assume that $\partial_W \alpha \geq 0$. By absurd, assume Ω to contain two limit cycles, γ_1 and γ_2, γ_1 surrounded by γ_2. As observed at the end of the proof of Theorem 1, γ_1 has to be an attractor and γ_2 a repellor. Hence, by the attractivity of Ω, a third limit cycle should exist, surrounding γ_2, and thus contradicting the bound on the number of limit cycles. □

3. Applications of the main result: The role of convexity

Theorem 1 is a general result that can be applied to any kind of systems and S-operators. In order to grasp the applicability of the result, we will now restrict ourselves to special classes of systems and operators.

It is known from [6,7] that ν is an S-operator when W and X are transversal and $[W, X] = \nu W + \beta X$. Using the notation of (7), $\nu = \mu + \partial_W F$ and we have:

Corollary 2. *Let V, W be transversal (except possibly at a unique singular point) vector fields with $[W(z), V(z)] = \mu(z)\, W(z)$. If $\partial_W\, (\mu + \partial_W F) \geq 0$ on an annular region Ω, then the thesis of Theorem 1 holds for the system (7).*

Most of the examples throughout the paper will be based on this operator though we will compare it to the divergence in some examples.

In the next corollary, we obtain a sufficient condition for the upper bound to the number of limit cycles in triangular systems in polar coordinates (r, θ), which reduces to a suitable convexity property.

We denote by $L_{\overline{\theta}}$ a ray; that is, the half-line $\{(r, \theta) : r > 0, \theta = \overline{\theta}\}$. We say that the annular region Ω^* is a *star-shaped annular region* if the intersection of Ω^* with every ray is a line segment (possibly different for each ray). Let V^*, defined on a star-shaped annular region Ω^*, be the vector field associated to the system

$$r' = f(r, \theta), \qquad \theta' = g(\theta). \tag{8}$$

Corollary 3. *Suppose that V^* has the form (8), with $g(\theta) > 0$ for all $\theta \in [0, 2\pi)$. Consider a function $h(r) > 0$ for all $r > 0$. If there exists $r_0 \geq 0$ such that*

$$h^2 f_{rr} - hh'' f - hh' f_r + h'^2 f \geq 0 \quad (\leq 0) \qquad \forall r > r_0, \quad \forall \theta \in [0, 2\pi),$$

and there exists a ray $L_{\overline{\theta}}$ where the above inequality holds strictly, then the thesis of Theorem 1 holds for the system (8).

Proof. Observe that V^* normalizes the system W_h^* defined as

$$r' = h(r), \qquad \theta' = 0. \tag{9}$$

More precisely, $[W_h^*, V^*] = \mu^* W_h^*$, with

$$\mu^* = \frac{hf_r - h'f}{h}.$$

In addition, W_h^*-orbits are rays and every ray meets every V^*-cycle. By the main result in [6], μ^* is an S-operator, hence we just have to compute $\partial_{W_h^*}\mu^*$, whose numerator gives the function in the statement. Finally, one can apply the Theorem 1 to get the desired result. □

It is to be noticed the absence of g from the formula in the above corollary, due to the independence of θ' from r. We consider now some special cases.

Corollary 4. *Let V^* have the form (8) in polar coordinates, with $g(\theta) > 0$ for all $\theta \in [0, 2\pi)$. If one of the following holds:*

(1) *there exist $r_0 \geq 0$ and $\xi \in \mathbb{R}$, such that $r^2 f_{rr} - \xi^2 r f_r + \xi f \geq 0\, (\leq 0)$ for all $\theta \in [0, 2\pi)$ and for all $r > r_0$,*
(2) *there exist $r_0 \geq 0$ such that $f_{rr} \geq f_r\, (\leq f_r)$ for all $\theta \in [0, 2\pi)$ and for all $r > r_0$,*

and there exists a ray $L_{\overline{\theta}}$ where such an inequality holds strictly, then the thesis of Theorem 1 holds for the system (8).

Proof. (1) Choosing $h(r) = r^{\xi}$, leads to the inequality in the statement.

(2) Let us choose $h(r) = \mathrm{e}^r$. Then the formula in Corollary 3 becomes $f_{rr} \geq f_r$ for $\theta \in [0, 2\pi)$ and for all $r > 0$. □

Remark 2. In the above corollary, if one takes $\xi = 0$, that is $h(r) \equiv 1$, one gets the condition $f_{rr} \geq 0\, (\leq 0)$ for $r > 0$. In the following, functions satisfying $f_{rr} \geq 0$ will be called *radially convex* functions. For instance, the equation $\dot{r} = r^2 - 3r + 2$, which has two isolated periodic solutions at $r = 1$ and $r = 2$, satisfies the condition $f_{rr} > 0$.

A two-variable convex function is radially convex, but the vice versa is not true, as the function $K(r\cos\theta, r\sin\theta) = (r-1)^2 + \cos\theta$ shows. Such a function is radially convex, but it has a saddle point at $(r, \theta) = (1, 0)$.

Moving back to Corollary 2, we recall that, essentially, the existence of a Lie symmetry $[W(z), V(z)]$=$\mu(z)\, W(z)$ plus the condition $\partial_W\, (\mu + \partial_W F) \geq 0$ imply the bound of two limit cycles for system (7). In particular, when V and W commute, that is when $[V(z), W(z)] \equiv 0$, this upper bound is controlled by the "second derivative" $\partial_W\, (\partial_W F)$. This amounts to impose the convexity of F on the W-orbits. Such a property is strongly dependent on the parameterization of such orbits, but since if W is a commutator of V, then every IW, where I is a first integral of V, is itself a commutator, one can choose among infinitely many distinct commutators. More generally, if W is normalized by V, then, for every function ψ, ψW is normalized by V, so that one has an infinite family of normalized vector fields to choose among.

A simple way to get a class of such systems consists in choosing the couple of commuting vector fields $V(x, y) = (-y, x)$ and $W(x, y) = \eta(x^2 + y^2)\, (x, y)$, where $\eta(x^2 + y^2)$ is any scalar function. In such a case, the derivative of a function α in the direction of W is simply related to its radial derivative.

$$\partial_W \alpha = \eta(x^2 + y^2)(x\alpha_x + y\alpha_y) = \eta(r^2) r \alpha_r.$$

The simplest case occurs when one chooses $\eta(x^2 + y^2) \equiv 1$. As observed previously, every vector field can be written as

$$\begin{cases} x' = -y\, A(x, y) + x\, B(x, y), \\ y' = x\, A(x, y) + y\, B(x, y), \end{cases} \tag{10}$$

critical points being characterized by the simultaneous vanishing of A and B. The functions A and B are simply related to the angular velocity $\dot{\theta}$ and $\dot{r}$,

$$\dot{\theta} = A, \qquad \dot{r} = rB.$$

Such a system is more general than the one considered in (8), since here we do not assume $\dot\theta$ to depend only on θ. On the other hand, if the condition $\dot\theta = A \neq 0$ is satisfied in some region, then we may divide the vector field by A getting an equivalent vector field with $\dot\theta \equiv 1$,

$$\dot\theta = 1, \qquad \dot r = \frac{r\,B(r,\theta)}{A(r,\theta)}, \tag{11}$$

which is the polar form of

$$\begin{cases} x' = -y + x\,F(x,y), \\ y' = x + y\,F(x,y), \end{cases} \tag{12}$$

where $F = \frac{B}{A}$. The system (11) can be obtained from system (8) setting $f(r,\theta) = rF = \frac{rB}{A}$. Actually, system (11) is only apparently a special case of system (8). In fact, since $g(\theta) > 0$, one can always divide the vector field equivalent to the system (8) by $g(\theta)$, getting a new system with the same orbits for which $\dot\theta \equiv 1$. Vice versa, one can obtain system (8) from (11) multiplying by $g(\theta)$. Summing up, whenever the angular speed is positive, one can reduce to systems with constant angular speed. For such systems, geometrically characterized by the orbits' star-shapedness, the condition of Corollary 2 assumes the following simple form,

$$\partial_W\left(\partial_W F(x,y)\right) = \partial_W\left(\partial_W \frac{B(x,y)}{A(x,y)}\right) \neq 0. \tag{13}$$

Such a condition is evidently related to a convexity property of F along the W-orbits. In rectangular coordinates it appears as follows,

$$\partial_W\left(\partial_W F\right) = x\,F_x + y\,F_y + x^2 F_{xx} + 2xyF_{xy} + y^2 F_{yy} = rF_r + r^2 F_{rr} \geq 0, \tag{14}$$

which is equivalent to

$$F_r + rF_{rr} = (rF_r)_r \geq 0, \tag{15}$$

since $r > 0$.

Observe that another simple condition could be obtained using the condition $f_{rr} \geq 0$ with $f := rF$, which would give:

$$f_{rr} = (rF)_{rr} = 2F_r + rF_{rr} \geq 0. \tag{16}$$

On the other hand, since the divergence for a rigid system is

$$\mathrm{div}X = 2F + xF_x + yF_y = 2F + \partial_W F = 2F + rF_r,$$

we have that

$$\partial_W \mathrm{div} = 3(x\,F_x + y\,F_y) + x^2 F_{xx} + 2xyF_{xy} + y^2 F_{yy} = 3rF_r + r^2 F_{rr} \geq 0,$$

equivalent to

$$3F_r + rF_{rr} \geq 0. \tag{17}$$

Thus, any of the conditions (15)–(17) implies an upper bound of two limit cycles to the system. The similarity of such conditions is not a coincidence. In fact we have the following statement.

Lemma 1. *If α is an S-operator for (12) and $\kappa \in \mathbb{R}$, then $\kappa F + \alpha$ is an S-operator for (12), too.*

Proof. For every periodic solution $\gamma(t)$, one has

$$\begin{aligned}\int_0^{2\pi} F(\gamma(t))\mathrm{d}t &= \int_0^{2\pi} \frac{\dot r(\gamma(t))}{r(\gamma(t))}\mathrm{d}t = \int_0^{2\pi} (\ln r(\gamma(t)))^{\dot{}}\,\mathrm{d}t \\ &= \ln r(\gamma(2\pi)) - \ln r(\gamma(0)) = 0.\end{aligned}$$

Hence one has

$$\int_0^{2\pi} \kappa F(\gamma(t)) + \alpha(\gamma(t))\mathrm{d}t = \int_0^{2\pi} \alpha(\gamma(t))\mathrm{d}t. \quad \square$$

The above proposition shows that one can add any multiple of F to, say, the divergence $2F + rF_r$ and use its radial derivative in place of the conditions (15)–(17) in order to get the upper bound of two limit cycles. As a consequence we have the following corollary, that we state without proof.

Corollary 5. *If there exists $\kappa^* \in \mathbb{R}$ such that*

$$\kappa^* F_r + rF_{rr} \geq 0,$$

in a star-shaped annular region Ω^*, then the thesis of Theorem 1 holds on Ω^*.

This freedom (expressed by means of κ^*) to choose an operator has to be managed adequately in each case. Next we present a "canonical" example to illustrate how to choose a suitable κ^*.

Example 1. Consider the system given in polar coordinates by

$$\begin{cases} r' = r\,G(r^2), \\ \theta' = 1, \end{cases} \tag{18}$$

with $G(z) = (z - b_1)(z - b_2)$ and $b_2 > b_1 > 0$. Clearly, system (18) has two limit cycles, located at $\{r = b_j\}$, for $j = 1, 2$. Such cycles are hyperbolic, hence they resist to small perturbation. In general, one does not know how much the system can be perturbed, without destroying the limit cycles. We apply Theorem 1 to give an estimate of how much the above system can be perturbed, preserving the limit cycles. Let us replace the constants b_1 and b_2 with periodic functions: $G(z,\theta) = (z - b_1(\theta))(z - b_2(\theta))$, $b_2(\theta) > b_1(\theta) > 0$. Computing the divergence, one has div $= 2\,G(r^2,\theta) + 2\,r^2\,G_r(r^2,\theta)$, and we consider the class of operators

$$\alpha_\kappa = (\kappa + 2)G(r^2,\theta) + 2\,r^2\,G_r(r^2,\theta).$$

Direct computations give

$$\partial_r \alpha_\kappa = 2\,r\left((\kappa + 4)G_r(r^2,\theta) + 2\,r^2\,G_{rr}(r^2,\theta)\right),$$

which, in the case $G(z,\theta) = (z - b_1(\theta))(z - b_2(\theta))$ leads to

$$\partial_r \alpha_\kappa = -2\,r\left(-12\,r^2 - 2\,\kappa\,r^2 + 4\,b_2(\theta)^2 + \kappa\,b_2(\theta)^2 + \kappa\,b_1(\theta)^2 + 4\,b_1(\theta)^2\right).$$

This expression vanishes for $r^2(\theta) = \frac{(\kappa+4)(b_1(\theta)^2+b_2(\theta)^2)}{2(\kappa+6)}$, giving rise to different ways to apply our results:

- Choosing $\kappa^* < \min\left\{\frac{8b_1(\theta)^2-4b_2(\theta)^2}{b_2(\theta)^2-b_1(\theta)^2} : \theta \in [0, 2\pi]\right\}$, we have that $\partial_r \alpha_{\kappa^*} > 0$ in an annular region containing the annulus defined by $max\{\, b_1(\theta) : \theta \in [0, 2\pi]\} \le r \le \min\{\, b_2(\theta) : \theta \in [0, 2\pi]\}$. Applying Corollary 5, then, we would predict the maximum number of limit cycles in such a region.
- Choosing $\kappa^* > \max\left\{\frac{8b_1(\theta)^2-4b_2(\theta)^2}{b_2(\theta)^2-b_1(\theta)^2} : \theta \in [0, 2\pi]\right\}$, we have that $\partial_r \alpha_{\kappa^*} < 0$ in an annular region containing the curve $\{r = b_1(\theta)\}$ and $\partial_r \alpha_{\kappa^*} > 0$ in an annular region containing the curve $\{r = b_2(\theta)\}$. Thus, the maximum bounds given by Corollary 5 would be too loose in this case, indicating that this is not a good choice for κ^*.

For systems in rectangular coordinates expressed as in (12), Corollary 5 assumes the following form:

Corollary 6. *If there exists $\kappa^* \in \mathbb{R}$ such that*

$$\Delta_{F,\kappa^*} := r\,(\kappa^*\,F_r + r\,F_{rr}) = \kappa^*(x\,F_x + y\,F_y) + x^2F_{xx} + 2xyF_{xy} + y^2F_{yy} \ge 0,$$

in a star-shaped annular region Ω^*, then the thesis of Theorem 1 holds for system (12) on Ω^*.

If we write $F(x,y) = \sum_{j=0}^{n} P_j(x,y)$, where the functions $P_j(x,y)$ are homogeneous polynomials of degree j, that is,

$$\begin{cases} x' = -y + x\left(\displaystyle\sum_{j=0}^{n} P_j(x,y)\right), \\ y' = x + y\left(\displaystyle\sum_{j=0}^{n} P_j(x,y)\right), \end{cases} \tag{19}$$

we have:

$$\begin{aligned} \Delta_{F,\kappa^*} &= \kappa^*(x\,F_x + y\,F_y) + x^2F_{xx} + 2xyF_{xy} + y^2F_{yy} \\ &= \sum_{j=0}^{n}\left(\kappa^* jP_j(x,y) + j(j-1)P_j(x,y)\right) = \sum_{j=0}^{n}\left(\kappa^* + j - 1\right) j\,P_j(x,y). \end{aligned}$$

The coefficient κ^* can be used to get rid of a single P_{j^*} when the remainder $\Delta_{F,\kappa^*} - (\kappa^* + j^* - 1)j^*\,P_{j^*}$ does not change sign. In such a case one chooses $\kappa^* = 1 - j^*$.

Corollary 7. *If there exists m, $1 < m < n$, such that $P_j(x,y) \ge 0$ for $1 < j < m$, $P_j(x,y) \le 0$ for $j > m$ then system (19) has at most two limit cycles. If, additionally, there exists $1 \le \underline{j} < m$, such that $P_{\underline{j}} > 0$, and $m < \bar{j} \le n$, such that $P_{\bar{j}} > 0$, then the system has a unique limit cycle.*

Proof. Applying Corollary 6, choosing $\kappa^* = 1 - m$ proves that there exist at most two limit cycles. If there exist $\underline{j}$ and $\bar{j}$ as in the hypotheses, then using $H(x,y) = x^2 + y^2$ as a Lyapunov function, one can prove that the origin and the point at infinity are both repellors for system (19). Let Ω^* be the global attractor of $\mathbb{R}^2 \setminus \{O\}$. Since $\dot\theta \equiv 1$, Ω^* is an asymptotically stable annulus. Then we apply Corollary 1 to get the thesis. $\square$

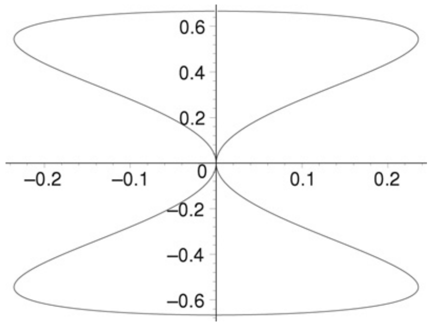

Fig. 1. The curve $\Delta_{F,-2}=0$ of Example 2. The region beyond $\Delta_{F,-2}=0$ cannot contain more than two limit cycles.

We consider now some specific examples.

Example 2. Let us consider the system

$$\begin{cases} x' = -y + x\left(x^4 - 5x^3 - 2x^2 + y^6 - 2y^4 + 3y^3\right), \\ y' = x + y\left(x^4 - 5x^3 - 2x^2 + y^6 - 2y^4 + 3y^3\right), \end{cases} \tag{20}$$

obtained choosing $F(x,y) = x^4 - 5x^3 - 2x^2 + y^6 - 2y^4 + 3y^3$. Choosing $\kappa^* = -2$ in order to eliminate the cubic terms from Δ_{F,κ^*}, one has $\Delta_{F,-2} = 4x^4 + 4x^2 + 18y^6 - 8y^4$. Solving the bi-quadratic equation in x, it turns out that the curve $\Delta_{F,-2} = 0$ is given implicitly by

$$x^2 = \frac{-1 + \sqrt{1 - 18y^6 + 8y^4}}{2},$$

defined only for $|y| < \frac{2}{3}$, see Fig. 1.

Example 3. Consider, for instance, the system

$$\begin{cases} x' = -y + x\left(x^4 - x^3 - x^2 + y^6 + y^4 + 3y^3 - 1\right), \\ y' = x + y\left(x^4 - x^3 - x^2 + y^6 + y^4 + 3y^3 - 1\right), \end{cases}$$

obtained from choosing $F(x,y) = x^4 - x^3 - x^2 + y^6 + y^4 + 3y^3 - 1$. Choosing again $\kappa^* = -2$, in order to eliminate the cubic terms from Δ_{F,κ^*}, one has $\Delta_{F,-2} = 4x^4 + 2x^2 + 18y^6 + 4y^4 \geq 0$ for all x and y and vanishes only at the origin. Hence Corollary 7 applies and the system has exactly one limit cycle.

3.1. Generalized Abel equations

Here we extend Corollary 7 to non-polynomial systems. Next corollary is a refinement of a result published in [2] and originally attributed to S. Smale. The thesis comes from a convexity property of a system obtained by means of a singular transformation.

Corollary 8. *If there exists m, $1 < m < n$, such that $a_j(\theta) \geq 0$ for $j < m$, $a_j(\theta) \leq 0$ for $m < j$, then Eq. (1) has at most two positive periodic solutions.*

Proof. Let us recall first that (1) is given by $\dot r = \sum_{j=1}^{n} a_j(\theta) r^j$, $\dot\theta = 1$. We apply the transformation

$$u = r^{1-m}, \qquad r = u^{\frac{1}{1-m}}, \qquad r > 0, \quad u > 0,$$

and denote the transformed equation by

$$\dot u = \phi(u,\theta), \quad u > 0. \tag{21}$$

Positive periodic solutions of (1) are taken into positive periodic solutions of (21). Assume by absurd the existence of three distinct periodic solutions $0 < u_1(\theta) < u_2(\theta) < u_3(\theta)$. Consider Eq. (21) only on the region $\{(u,\theta) : u_1(\theta) \leq u \leq u_3(\theta)\}$, corresponding to an annular region $\overline{\Omega}$ in the rectangular coordinates obtained from the polar coordinates (u,θ). Eq. (21) is the polar form of a planar system defined on $\overline{\Omega}$, hence we may apply Theorem 1. The term $a_j(\theta) r^j$ of Eq. (1) is transformed into the term $(1-m) a_j(\theta) u^{\frac{j-m}{1-m}}$. In order to apply point (1) of Corollary 4, we derive such a term twice with respect to u, getting

$$a_j(\theta) \frac{(j-m)(j-1)}{1-m} u^{\frac{j+m-2}{1-m}}.$$

Such a second derivative vanishes identically for $j = 1$ and for $j = m$, hence the linear and the m-degree terms have no effect on its sign. Under the chosen hypotheses, every term in the transformed equation has positive sign, hence we have

$$\phi_{uu} \geq 0,$$

which contradicts the existence of three distinct positive periodic solutions. □

In the above corollary the transformation takes the linear term into a linear term, while taking the mth degree term into a constant one (constant w. r. t. r). This allows to eliminate two terms by differentiating twice w. r. t. r, but no more than two. In the next corollary we consider two special cases.

Corollary 9. *The following equations with continuous coefficients have at most two positive periodic solutions.*

(1) $\dot{r} = a_n(\theta)r^n + a_k(\theta)r^k + a_1(\theta)r, \quad n > k > 1,\ a_n(\theta) \geq 0\ (a_n(\theta) \leq 0)$;
(2) $\dot{r} = a_n(\theta)r^n + a_k(\theta)r^k + a_1(\theta)r, \quad n > k > 1,\ a_k(\theta) \geq 0\ (a_k(\theta) \leq 0)$.

Proof. In both cases two terms can have arbitrary sign, while all the other ones – just one – have constant sign. Hence we may apply the previous corollary. □

Observe that the sign of $\int_0^{2\pi} a_1(\theta)\mathrm{d}\theta$ determines the stability of the solution $r = 0$ and, hence, it is a key point for the stability alternation between periodic orbits. For instance, in the case a_n (or a_m) positive, $\int_0^{2\pi} a_1(\theta)\mathrm{d}\theta < 0$ would guarantee the coexistence of $r = 0$ together with the two positive periodic orbits (see also Lemma 8 in [5]).

The above corollary (see also Theorem 3 in [5]) extends Theorem C (a) in [3]. We emphasize that the systems considered in the above corollaries are not necessarily the polar form of polynomial systems, since the functions $a_j(\theta)$ are not supposed to be trigonometric polynomials of degree $j - 1$.

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