Lyapunov exponents for the buck converter

Some numerical and analytical results

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The buck converter

In adimensional variables \((v \sim x, i \sim y)\)

\[
\begin{align*}
\dot{x} &= -x + y \\
\dot{y} &= -\gamma x + \gamma V \theta(r(t) - x)
\end{align*}
\]
The QR decomposition (I)

- System in $\mathbb{R}^n$ and its tangent map with respect to a trajectory $x_0(t), z = x - x_0(t),$

$$\frac{dx}{dt} = F(x, t), \quad \frac{dz}{dt} = \frac{\partial F}{\partial x} \bigg|_{x_0(t)} z \equiv DF(t) \cdot z$$

- Solution: $z(t) = M(t)z(0)$ where $M(t) = T e^{\int_{t_0}^{t} DF(s)ds}$

- SVD decomposition: $M(t) = U(t) \cdot A(t) \cdot V^T(t)$, with $U(t), V(t)$ orthogonal, $A(t)$ diagonal.

- Infinite-time Lyapunov exponents: eigenvalues $\lambda_i$ of

$$\Lambda = \lim_{t \to +\infty} \left( M^T(t)M(t) \right)^{\frac{1}{2t}}$$
The \( QR \) decomposition (II)

- \( QR \) decomposition: \( M(t) = Q(t) \cdot R(t) \), where
  - \( Q(t) \) is orthogonal
  - \( R(t) \) is upper-triangular and with positive diagonal elements \( \Delta_i(t) \)
- It can be shown that
  \[
  \lambda_i = \lim_{t \to \infty} \frac{1}{t} \log \Delta_i(t)
  \]

- The \( QR \) formulation is optimal: it uses the minimum number of variables.
The QR decomposition (III)

- From $\dot{M} = DF \cdot M$ it follows
  
  $$Q^T \dot{Q} + \dot{R}R^{-1} = Q^T \cdot DF \cdot Q \equiv S.$$ 

- $Q$ can be parameterized with $n(n-1)/2$ angles $\theta_{ij}$.

- One gets the separable equations
  
  $$\dot{\Delta}_i = S_{ii}(\theta) \Delta_i, \quad i = 1, \ldots, n$$

  plus the nonlinear equations

  $$\dot{\theta}_{ij} = g_{ij}(\theta), \quad i > j$$
**QR for the buck converter (I)**

- If \((x_0(t), y_0(t))\) is a reference trajectory

\[
DF = \begin{pmatrix}
1 & 0 \\
-\gamma(1 + V\delta(r(t) - x_0(t))) & 0 \\
r & 0 \\
\end{pmatrix}
\]

- Since \(n = 2\), there’s only one angle \(\theta_{12}(t) \equiv \alpha(t)\) and

\[
Q^T \dot{Q} = \begin{pmatrix}
0 & \dot{\alpha} \\
-\dot{\alpha} & 0 \\
\end{pmatrix}
\]

\[
R(t) = \begin{pmatrix}
\Delta_1(t) & r_{12}(t) \\
0 & \Delta_2(t) \\
\end{pmatrix}
\]
QR for the buck converter (II)

The equations to be solved are

\[
\begin{align*}
\frac{\dot{\Delta}_1}{\Delta_1} &= -\cos^2 \alpha + (\gamma - 1) \sin \alpha \cos \alpha + \gamma V \delta(r(t) - x_0(t)) \sin \alpha \cos \alpha \\
\frac{\dot{\Delta}_2}{\Delta_2} &= -\sin^2 \alpha + (1 - \gamma) \cos \alpha \sin \alpha - \gamma V \delta(r(t) - x_0(t)) \cos \alpha \sin \alpha \\
\dot{\alpha} &= -\sin^2 \alpha - \cos \alpha \sin \alpha - \gamma (1 + V \delta(r(t) - x_0(t))) \cos^2 \alpha
\end{align*}
\]

With \(\Delta_{1,2}(t) = \exp \lambda_{1,2}(t)\), one has

\[
\frac{\lambda_1(t) + \lambda_2(t)}{t} = -1
\]

If \(\lambda_1(t) \equiv \lambda(t)\), \(\lambda(t)/t\) will asymptotically give the largest Lyapunov exponent (LLE).
LLE for the buck converter (I)

- Numerical integration, step function approximated by an \( \arctan \) with high coefficient

- Gives the LLE of the dominant attracting set.
LLE for the buck converter (II)

• For a $T$-periodic orbit switching at $t = t_c$

$$\delta(r(t) - x_0(t)) = \frac{1}{|\dot{r}(t_c) - \dot{x}_0(t_c)|} \delta(t - t_c)$$

• If $\nu = \gamma V/|\dot{r}(t_c) - \dot{x}_0(t_c)|$ one gets

$$\dot{\alpha} = \sin^2 \alpha + \sin \alpha \cos \alpha + (\gamma + \nu \delta(t - t_c)) \cos^2 \alpha$$
$$\dot{\lambda} = -\cos^2 \alpha + (\gamma + \nu \delta(t - t_c) - 1) \sin \alpha \cos \alpha,$$

• Exact integration + discontinuities at $t = t_0$:

$$\tan \alpha(t_c^+) - \tan \alpha(t_c^-) = \nu$$
$$\lambda(t_c^+) - \lambda(t_c^-) = \frac{1}{2} \log \frac{1 + \tan^2(\alpha(t_c^+))}{1 + \tan^2 \alpha((t_c^-))}$$
LLE for the buck converter (III)

- Recurrence relation for $\alpha_n = \alpha(nT)$ and $\lambda_n = \lambda(nT)$:

\[
\begin{align*}
    r_{n+1} &= \frac{A + B + r_n}{1 - AB - Ar_n} \\
    \lambda_{n+1} &= \frac{1}{2} \log \left( \frac{(\tan^2 \alpha_{n+1} + 1)(\tan^2 \alpha_n + \tan \alpha_n + \gamma)}{(\tan^2 \alpha_n + 1)(\tan^2 \alpha_{n+1} + \tan \alpha_{n+1} + \gamma)} \right) \\
    &\quad + \log \left( \frac{((M_n + \nu)^2 + M_n + \nu + \gamma)}{(M_n^2 + M_n + \gamma)} \right) - \frac{1}{2} T + \lambda_n
\end{align*}
\]

where $\mu = \sqrt{\gamma - \frac{1}{4}}$, $A = \tan \mu T$, $B = \nu / \mu$, $M_n = \mu r_n - 1/2$ and

\[r_n = \tan \left( \mu t_c + \arctan \frac{\tan \alpha_n + 1/2}{\mu} \right).\]
**LLE for the buck converter (IV)**

- Numerical iteration for \( \lambda_n \) (left) and asymptotic values of \( \alpha_n \) and \( \lambda_n \) (right) in terms of \( V \):

- \( \lambda(t) \): linear + bounded oscillation
- When \( (\alpha_n) \) does not converge, \( \lambda = -1/2 \).
Analytical results

- The above results have been proved analytically and extended to higher period orbits.
- Dominant attractor (numerical) + $T$-periodic (analytic) + $2T$-periodic (analytic)
Summary and things to do

- $QR$ equations written for the buck converter
- Solved numerically for any value of $V$ and analytically for periodic orbits.
- Analytical results reproduce numerical one when the attractor is periodic.
- Use results for periodic orbits to compute analytical approximations of LLE in chaotic regime.
- Extend to higher order (corrections to linear LLE)
References


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