Lecture 8

Design for performance

This lecture follows Chapter 10 of Doyle-Francis-Tannenbaum, skipping the material about 2-norm minimization.
Design for performance

• The performance criterion (the ability to follow a set of signals with prescribed error) is

\[ ||W_1S||_\infty < 1. \]

• In this lecture we attack the problem of finding a proper \( C \) so that the criterion is met.

• This could be done using the loopshaping method with \( W_2 = 0 \), but we will present an analytical method that works even when \( P \) or \( P^{-1} \) are unstable.

• For \( P^{-1} \) stable the problem is very easy. For \( P^{-1} \) unstable (with zeros in the open right half-plane) nonclassical (model matching) methods will be required.
$P^{-1}$ stable (I)

- First we assume that $P^{-1}$ is stable, i.e. $P$ has no zeros in $\mathbb{R}(s) \geq 0$. The weighting function $W_1$ is assumed to be stable and strictly proper. The later condition is not restrictive at all since trying to follow signals of arbitrary high frequency makes no sense.

- Let \( k = 1, 2, 3, \ldots \) and $\tau > 0$, and consider the transfer function

\[
J_{\tau}(s) = \frac{1}{(\tau s + 1)^k}.
\]

- The Bode plot of $J_{\tau}$ is as follows. The magnitude starts out at 0, is flat up to the corner frequency $1/\tau$, and then rolls out to $-\infty$ with slope $-k$. The phase starts at 0, is flat up to $0.1/\tau$, and afterwards rolls off to $-k\pi/2$. 
**$P^{-1}$ stable (II)**

- The behavior of $J_\tau$ that we have sketched shows that for low frequencies ($< 0.1/\tau$) it approximates the transfer function 1. This means that for $\tau \to 0$ it approximates 1 to all frequencies. This can be formulated in the following

**Lemma 1.** If $G$ is stable and strictly proper, then

$$\lim_{\tau \to 0} \|G(1 - J_\tau)\|_\infty = 0.$$  

- Now we will develop the design procedure, first with the additional assumption that $P$ is stable. From Lecture 4, the set of all internally stabilizing controllers is in this case

$$C = \frac{Q}{1 - PQ}, \quad Q \in Q.$$
$P^{-1}$ stable (III)

- Using this we get $W_1S = W_1(1 - PQ)$, and we are tempted to set $Q = P^{-1}$ to get $\|W_1S\|_\infty = 0 < 1$. This $Q$ is indeed, by assumption, stable, but it is not proper.

- Let us try

$$Q = P^{-1}J_\tau$$

with $k$ large enough to make $Q$ proper ($k$ must be equal to, or greater than, the relative degree of $P$).

- Then

$$W_1S = W_1(1 - J)$$

whose $\infty$-norm is less than 1 for sufficiently small $\tau$, by Lemma 1. This solves the problem.
If $P$ is unstable, the Youla-Kucera parametrization yields all the internally stabilizing controllers:

$$C = \frac{X + MQ}{Y - NQ}, \quad S = M(Y - NQ).$$

First do a coprime factorization of $P$ in $Q$:

$$P = \frac{N}{M}, \quad NX + MY = 1.$$ 

Set $k \geq$ the relative degree of $P$ and choose $\tau$ so small that

$$||W_1 MY (1 - J_\tau)||_\infty < 1.$$ 

This can be done, by Lemma 1, because $W_1 MY$ is stable and strictly proper ($W_1$ is strictly proper).
$P^{-1}$ stable (V)

- Set

$$Q = N^{-1}Y J_\tau.$$ 

Then

$$W_1S = W_1M(Y - NN^{-1}Y J_\tau) = W_1MY(1 - J_\tau)$$

and $||W_1S||_\infty < 1$.

- Finally, substitute $Q$ in the Youla-Kucera parametrization to get the controller.

- In general, $\tau$ must be found numerically, going down from an initial guess.
$P^{-1}$ stable (VI)

**Example.** Let

$$P(s) = \frac{1}{(s - 2)^2}, \quad W_1(s) = \frac{100}{s + 1}. \quad (1)$$

This weight has bandwidth 1 rad/s and should give less than 1% tracking error over that range.

- The coprime factorization of $P$ over $Q$ yields

$$N(s) = \frac{1}{(s + 1)^2}, \quad M(s) = \frac{(s - 2)^2}{(s + 1)^3}, \quad X(s) = 27\frac{s - 1}{s + 1}, \quad Y(s) = \frac{s + 7}{s + 1}. \quad (2)$$

- We set $k = 2$ and choose $\tau$ so that

$$W_1 MY(1 - J_\tau) = \frac{100(s - 2)^2(s + 7)}{(s + 1)^4} \left(1 - \frac{1}{(\tau s + 1)^2}\right)$$

has $\infty$-norm less than 1. One gets

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$10^{-1}$</th>
<th>$10^{-2}$</th>
<th>$10^{-3}$</th>
<th>$10^{-4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$-norm</td>
<td>199.0</td>
<td>19.97</td>
<td>1.997</td>
<td>0.1997</td>
</tr>
</tbody>
</table>

so we choose $\tau = 10^{-4}$. 

$P^{-1}$ stable (VII)

• Now

$$Q(s) = N^{-1}(s)Y(s)J_{10-4}(s) = \frac{(s + 1)(s + 7)}{(10^{-4}s + 1)^2}$$

and the controller is

$$C(s) = \frac{X(s) + M(s)Q(s)}{Y(s) - N(s)Q(s)} = 10^4 \frac{(s + 1)^3}{s(s + 7)(10^{-4}s + 2)}.$$

• The figure shows the tracking error for unit sinusoidal inputs with $\omega = 0.5 \text{ rad/s}$ and $\omega = 1000 \text{ rad/s}$. The performance degradation for the later is clearly seen.
\( P^{-1} \) unstable (I)

- The interpolation theory (NP problem) presented in the previous lecture will be used here to solve the performance design problem for \( P^{-1} \) unstable.

- To simplify, we will assume that
  - \( P \) has no poles or zeros on the imaginary axis, only distinct poles and zeros in the right half-plane, and at least one zero in the right half-plane.
  - \( W_1 \) is stable and strictly proper.

These assumptions could be relaxed, but the development would be messier.

- To motivate the presentation, remember from Lecture 2 that \( C \) is internally stabilizing iff \( PC \) has no right half-plane pole-zero cancellations and \( 1 + PC \) has no poles in \( \Re(s) \geq 0 \). Since

\[
S = \frac{1}{1 + PC}
\]

we must have

- \( S(z) = 1 \) for \( z \) a zero of \( P \) in \( \Re(s) > 0 \).
- \( S(p) = 0 \) for \( p \) a pole of \( P \) in \( \Re(s) > 0 \).
$P^{-1}$ unstable (II)

- In terms of the weighted sensitivity function $G = W_1 S$ this translates to
  - $G(z) = W_1(z)$ for $z$ a zero of $P$ in $\Re(s) > 0$.  
  - $G(p) = 0$ for $p$ a pole of $P$ in $\Re(s) > 0$.

- The performance test $||W_1 S||_\infty < 1$ is $||G||_\infty < 1$, and the stability of $S$ requires $G$ to be analytic in the right half-plane. Hence we have a NP problem.

- Now we could solve this NP problem, get $S$, and finally obtain $C$. However, this does not guarantee that the final $C$ is proper, and so we must bring again coprime factorization into the problem.

- Remember that, using the Youla-Kucera parametrization,

\[
W_1 S = W_1 M(Y - NQ).
\]

The parameter $Q$ must be both stable and proper. Our approach is to drop first the properness requirement and find a suitable (improper) $Q_{im}$, and then roll it off at high frequency. This works because $W_1$ is strictly proper, and hence there is no performance requirement at high frequency.
$P^{-1}$ unstable (III)

The algorithm goes as follows. Given $P, W_1$:

Step 1. Do a coprime factorization of $P$ in $Q$:

$$P = \frac{M}{N}, \quad NX + MY = 1.$$ 

Step 2. Find a stable function $Q_{\text{im}}$ such that

$$||W_1M(Y - NQ_{\text{im}})||_\infty < 1.$$ 

Step 3. Set

$$J_\tau(s) = \frac{1}{(\tau s + 1)^k},$$

where $k$ is large enough that $Q_{\text{im}}J_\tau$ is proper and $\tau$ is small enough that

$$||W_1M(Y - NQ_{\text{im}}J_\tau)||_\infty < 1.$$ 

Step 4. Set $Q = Q_{\text{im}}J_\tau$ and $C = (X + MQ)/(Y - NQ)$. 
$P^{-1}$ unstable (IV)

- Step 2 is the model-matching problem: find a stable function $Q_{im}$ to minimize

$$||T_1 - T_2 Q_{im}||_\infty$$

where $T_1 = W_1 MY$ and $T_2 = W_1 MN$. Step 2 is feasible iff the minimum model-matching error $\gamma_{opt}$ is less than 1.

- Step 3 is feasible because

$$W_1 M (Y - N Q_{im} J_\tau) = W_1 M (Y - N Q_{im}) J_\tau + W_1 M Y (1 - J_\tau).$$

The first term on the right-hand side has $\infty$-norm less than 1 from Step 2 and the fact that $||J_\tau||_\infty \leq 1$, while the $\infty$-norm of the second term goes to zero as $\tau \to 0$ because of Lemma 1.
Flexible beam (I)

- Here we apply the preceding theory to a real experimental setup. It describes a flexible beam, one end fixed to the shaft of a high torque dc motor, and the other free, where a sensor is situated.

- The full model is infinite dimensional due to the continuous nature of the beam. A truncation of the model, taking into account only the first mode, yields for the transfer function from motor torque to deflection of the sensed tip

\[
P(s) = \frac{-6.4750s^2 + 4.0302s + 175.7700}{s(5s^3 + 3.5682s^2 + 139.5021s + 0.0929)},
\]

where the sensor, the motor and the amplifier have been incorporated.

- This \( P(s) \) has a pole on the imaginary axis, so we perturb it to

\[
P(s) = \frac{-6.4750s^2 + 4.0302s + 175.7700}{5s^4 + 3.5682s^3 + 139.5021s^2 + 0.0929s + 10^{-6}}.
\]
Flexible beam (II)

- For a system like this, it does not make much sense to make it follow sinusoidal signals. Rather, we would like it to follow step inputs with specified transient, for instance

settling time = 8 s, \quad \text{overshoot} \leq 10\%.

- We will accomplish this by shaping $T(s)$, the transfer function from $r$ to $y$, so that it approximates a standard second order model

$$T_{id}(s) = \frac{\omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2}.$$  

- A settling time of 8 seconds requires

$$\frac{4.6}{\xi \omega_n} = 8$$

and the overshoot of ten percent is

$$\exp \left( \frac{-\xi \pi}{\sqrt{1 - \xi^2}} \right) = 0.1.$$
Flexible beam (III)

- The solution is \( \xi = 0.5912 \) and \( \omega_n = 0.9583 \), so we take \( \xi = 0.6, \omega_n = 1 \) and

\[
T_{id}(s) = \frac{1}{s^2 + 1.2s + 1}.
\]

The ideal sensitivity function is then

\[
S_{id}(s) = 1 - T_{id}(s) = \frac{s(s + 1.2)}{s^2 + 1.2s + 1}.
\]

- Now we will argue that the sensible weighting function for this setup is

\[
W_1(s) = S_{id}^{-1}(s) = \frac{s^2 + 1.2s + 1}{s(s + 1.2)}.
\]
Flexible beam (IV)

- The argument goes as follows. From Step 2 of the algorithm, one gets

\[ F = W_1 M (Y - NQ_{im}) \]

which is a constant times an all-pass function. The procedure then rolls off \( Q_{im} \) at high frequency to result in the weighted sensitivity function

\[ W_1 S = W_1 M (Y - NQ_{im} J_\tau). \]

This means that, except at high frequencies, \( W_1 S \sim F \). With the form of \( W_1 \) that we propose, this is

\[ S \sim FS_{id}. \]

- Now \( F \), being an all-pass function, behaves like a time delay, except at high frequencies, so we get

\[ S \sim (\text{time delay}) S_{id}. \]

Hence, the actual step response equals the ideal step response plus a time delay, which does not change the overshoot and, hopefully, is small and does not change the settling time too much.
Flexible beam (V)

• One further adjustment is required: $W_1(s)$ must be stable and strictly proper, so we change it to

$$W_1(s) = \frac{s^2 + 1.2s + 1}{(s + 0.001)(s + 1.2)(0.001s + 1)}.$$ 

• First, since $P \in Q$, no coprime factorization is required and we take $N = P$, $M = 1$, $X = 0$ and $Y = 1$, and the controller parametrization is then $C = \frac{Q}{1 - PQ}$. 

• The model matching problem is to minimize

$$||W_1M(Y - NQ_{im})||_\infty = ||W_1(1 - PQ_{im})||_\infty.$$ 

Since $T_2 = W_1P$ has only one right half-plane zero, that of $P$, at $s = 5.5308$, the model-matching problem is next to trivial and

$$\min ||W_1(1 - PQ_{im})||_\infty = |W_1(5.5308)| = 1.0210.$$ 

This means that $||W_1S||_\infty < 1$ is not achievable with this $W_1$, so we scale it down by

$$W_1 \rightarrow \frac{0.9}{1.0210}W_1 \quad \text{and now} \quad |W_1(5.5308)| = 0.9.$$
Flexible beam (VI)

• The optimal $Q_{im}$ is then

$$Q_{im} = \frac{T_1 - T_1(5.5308)}{T_2} = \frac{W_1 - 0.9}{W_1 P},$$

that is

$$Q_{im}(s) = \frac{0.0008s^6 + 0.0221s^5 + 0.1768s^4 + 0.7007s^3 + 3.8910s^2 + 0.0026s}{s^3 + 6.1081s^2 + 6.8897s + 4.9801}.$$  

• To get a proper $Q$, we take

$$J_\tau(s) = \frac{1}{(\tau s + 1)^3}$$

and compute $||W_1(1 - PQ_{im}J_\tau)||_\infty$ for decreasing values of $\tau$ until we go below 1:

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>0.1</th>
<th>0.05</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$-norm</td>
<td>1.12</td>
<td>1.01</td>
<td>0.988</td>
</tr>
</tbody>
</table>

We take $\tau = 0.04$. 
Finally, \( Q = Q_{\text{im}} J_{0.04} \) and we compute the controller. The controller \( C \) so obtained is of very high order (quotient of two order 10 polynomials). The common way to alleviate this is to approximate \( Q \) by a lower-order transfer function (it cannot be done directly with \( C \) due to the internal stability constraint, while any proper and stable \( Q \) resembling the computed one will do).

The output to a step input for the controller just computed is

![Graph](image)

and it clearly satisfies the specifications, in spite of all the approximations and *ad hoc* changes performed.