Balanced partitions of 3-colored geometric sets in the plane

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Abstract

In this work we study several problems regarding balanced partitions of 3-colored sets of points, and of 3-colored sets of lines. We say that a partition of a colored set is balanced if each of its two parts contains the same number of elements of each color. First we consider sets of lines and study the existence of segments that intersect the arrangement in a balanced set; we also study the dual problem, that is, we consider point sets in the plane and study partitions induced by double wedges. Then we consider point sets on a closed Jordan curve and study partitions induced by arcs. Last, we consider point sets in the plane lattice and study bipartitions induced by L-lines.

1 Introduction

Let \( S \) be a finite set of geometric objects distributed into classes or colors. A subset \( S_1 \subseteq S \) is said to be balanced if \( S_1 \) contains the same number of elements of \( S \) from each of the colors.

Naturally, if \( S \) is balanced, the complement of a balanced subset of \( S \) is also balanced, hence we talk of a balanced partition of \( S \). When the point set \( S \) is on the plane, and the balanced partition is defined by a geometric object \( \zeta \) splitting the plane into two regions, we say that \( \zeta \) is balanced.

In this paper we study problems on balanced bipartitions of 3-colored sets of points and lines in the plane. It is easy to construct balanced 3-colored point sets for which the only balanced partitioning line is the trivial one containing the whole pointset on one side. For example, consider an equilateral triangle \( p_1p_2p_3 \) and replace every vertex \( p_i \) by a very small disk \( D_i \) so that no line can intersect the three disks; place \( n \) red, \( n \) green, and \( n \) blue points inside the disks \( D_1, D_2 \) and \( D_3 \), respectively. It is obvious that for this configuration no balanced line exists. On the other hand, for every 3-colored set \( S \) of points there is a conic that simultaneously bisects the three colors: take the plane to be \( z = 0 \) in \( \mathbb{R}^3 \), lift the points vertically to the unit paraboloid \( P \), use the 3-dimensional ham-sandwich theorem for splitting evenly the lifted point set with a plane \( \Pi \), and use the projection of \( P \cap \Pi \) as halving conic in \( z = 0 \). Instead of changing the partitioning object, one may impose some constraints to the point set. For example, Bereg and Kano have recently proved [2] that if all vertices of the convex hull of \( S \) have the same color, then there exists a line that determines a halfplane containing exactly \( k \) points of each color, with \( 0 < k < n \). This result was recently extended to sets of points in higher dimensions, by Akopyan and Karasev [1].

In this work we present the following results: (a) In Section 2 we prove that for every 3-colored arrangement of lines there exists a segment that intersects exactly one line of each color. Moreover, if there are \( 2m \) lines of each color, there is a segment intersecting \( m \) lines of each color. (b) In Section 3 we show that given \( n \) red, \( n \) green and \( n \) blue points on any closed Jordan curve \( \gamma \), for every integer \( 0 \leq k \leq n \) there is a pair of disjoint intervals on \( \gamma \) whose union contains exactly \( k \) points of each color. (c) Given a set \( S \) of \( n \) red, \( n \) green and \( n \) blue points in the integer lattice, whose orthogonal convex hull is monochromatic, there exist one vertical and one horizontal ray with common apex, whose union splits the plane into two balanced regions; this is presented in Section 4. Due to lack of space, we omit most proofs from this extended abstract.

2 Balancing line arrangements

2.1 Cells in colored arrangements

Let \( L \) be a set of lines in the plane, partitioned into three sets \( R, G, \) and \( B \). We refer to the elements of \( R, G, \) and \( B \) as red, green, and blue, respectively. Let \( A(L) \) be the arrangement induced by the set \( L \). We assume that \( A(L) \) is simple, that is, there are no parallel lines and no more than two lines intersect at one point. In this section we prove that there always exists a 3-chromatic face in \( A(L) \), that is, a face that has at least one side of each color. We also extend this result to higher dimensions.
Consider a face $f$ of the 2-dimensional arrangement $\mathcal{A}(L)$. Consider the dual graph of $f$; we obtain a dual face $\hat{f}$ that contains a vertex for every bounding line of $f$, and contains an edge between two vertices of $\hat{f}$ if the intersection of the corresponding lines is part of the boundary of $f$ (see Figure 1(a)). Let $C$ be a simple cycle of vertices where each vertex is colored either red, green, or blue. Let $n_r(C), n_g(C)$, and $n_b(C)$ be the number of red, green, and blue vertices of $C$, respectively. We simply write $n_r$, $n_g$, and $n_b$ if $C$ is clear from the context. The type of an edge of $C$ is the multiset of the colors of its vertices. Let $n_{rr}, n_{rg}, n_{rb}, n_{gb}, n_{bb}$ be the number of edges of the corresponding type. Note that, if $f$ is bounded, then $\hat{f}$ is a simple cycle, where each vertex is colored either red, green, or blue. We say a bounded face $f$ is complete if $n_{rg} \equiv n_{rb} \equiv n_{gb} \equiv 1 \pmod{2}$ holds for $f$.

**Lemma 1** Consider a simple cycle $C$, where each vertex is colored either red, green, or blue. Then $n_{rg} \equiv n_{rb} \equiv n_{gb} \pmod{2}$.

**Proof.** The result follows from double counting. For $n_r$ we get the equation $2n_r = 2n_{rr} + n_{rg} + n_{rb}$. This directly implies that $n_{rg} \equiv n_{rb} \equiv n_{gb} \equiv 1 \pmod{2}$. We can do the same for $n_g$ and $n_b$ to obtain the claimed result. $\square$

**Theorem 2** Let $L$ be a set of lines in $\mathbb{R}^2$ colored with 3 colors so that each color appears at least once, and the arrangement $\mathcal{A}(L)$ induced by $L$ is simple. There always exists a complete face in $\mathcal{A}(L)$.

**Proof.** The result clearly holds if $|R| = |G| = |B| = 1$. For the general case, we start with one line of each color, and then incrementally add the remaining lines, maintaining a complete face $f$ at all times. Without loss of generality, assume that a red line $\ell$ is inserted into $\mathcal{A}(L)$. If $\ell$ does not cross $f$, we keep $f$. Otherwise, $f$ is split into two faces $f_1$ and $f_2$ (see Figure 1(b)). Similarly, $\hat{f}$ is split into $\hat{f}_1$ and $\hat{f}_2$ (with the addition of one red vertex, see Figure 1(c)). Because $\ell$ is red, the number $n_{gb}$ of green-blue edges does not change, that is, $n_{gb}(f_1) = n_{gb}(\hat{f}_1) + n_{gb}(\hat{f}_2)$. This implies that either $n_{gb}(f_1)$ or $n_{gb}(f_2)$ is odd. By Lemma 1 it follows that either $f_1$ or $f_2$ is complete. $\square$

Using the standard point-line duality in the plane, we obtain the following result:

**Corollary 1** Let $L$ be a set of lines in $\mathbb{R}^2$ colored with 3 colors so that each color appears at least once, and the arrangement $\mathcal{A}(L)$ induced by $L$ is simple. There always exists a segment intersecting exactly one line of each color of $\mathcal{A}(L)$.

We can extend the previous result to higher dimensions: every $(d+1)$-chromatic arrangement of hyperplanes in the $d$-dimensional space, contains a $(d+1)$-chromatic face. The result is sharp with respect to the number of colors. To extend the proof, we use similar techniques: first consider the dual of a cell in the arrangement as a triangulation of the $(d-1)$-dimensional sphere; then define types of faces of that triangulation based on the colors of their incident vertices, and prove an analogue of Lemma 1. The definition of complete cells is generalized accordingly, maintaining the property that a complete cell is $(d+1)$-chromatic. Based on that, we prove the following.

**Theorem 3** Let $H$ be a set of hyperplanes colored with $d+1$ colors so that each color appears at least once, and the arrangement $\mathcal{A}(H)$ induced by $H$ is simple. There always exists a complete cell in $\mathcal{A}(H)$.

### 2.2 3-colored point sets and balanced double wedges

A result equivalent to Corollary 1 is presented in the next theorem. This can be shown using standard duality transformation between points and lines.

**Theorem 4** Let $S$ be a set of points in general position in $\mathbb{R}^2$ colored with 3 colors so that each color appears at least once. Then, there exists a double wedge that contains exactly one point of $S$ of each color.

Next we turn our attention to balanced 3-colored point sets, and prove that a ham-sandwich-like theorem for double wedges exists.

**Theorem 5** Let $S$ be a balanced set of 3-colored $6n$ points in general position the plane. There exists a double wedge containing exactly $n$ points of $S$ of each color.

**Proof.** (Sketch) Without loss of generality we assume that the points of $S$ have distinct $x$-coordinates and distinct $y$-coordinates. For two distinct points $a$ and $b$ in the plane, let $\ell(a, b)$ denote the line passing through them. Consider the arrangement $\mathcal{A}$ of all the lines passing through two points from $S$.

Consider a horizontal line $\ell$ that does not contain any point from $S$. We walk on $\ell$ from left to right.
in a continuous form. For any point \( p \in \ell \) we define an ordering \( \sigma_p \) of \( S \) as follows: consider the lines \( \ell(p, q), q \in S \) and sort them by slope. Let \( (p_1, \ldots, p_{n_0}) \) be the obtained sorting. By construction, any interval \( \{p_i, p_{i+1}, \ldots, p_j\} \) of an ordering of \( p \) corresponds to a set of points that can be covered by a double wedge whose apex is \( p \) (and vice versa).

Given a sorting \( \sigma_p = (p_1, \ldots, p_{n_0}) \) of \( S \) we associate a polygonal curve as follows: for every \( k \in \{1, 2, \ldots, 3n\} \) let \( b_k, \) and \( g_k \) be the number of blue and green points in the set \( S(p, k) = \{p_k, p_{k+1}, \ldots, p_{k+3n-1}\} \) of \( 3n \) points, respectively. We define the corresponding lattice point \( q_k := (b_k - n, g_k - n) \), and the path \( \phi(\sigma) = (q_1, \ldots, q_{3n}, -q_1, \ldots, -q_{3n}, q_1) \).

Observe that if \( q_k = (0, 0) \) for some \( p \in \ell \) and some \( k \leq 3n \), then there exists a balanced double wedge.

Assume, for the sake of contradiction, that \( q_k \neq (0, 0) \) for all orderings \( \sigma_p \) and all \( k \leq 3n \). We observe several important properties of \( \phi(\sigma) \):

- \( \phi(\sigma) \) is centrally symmetric (w.r.t. the origin).
- \( \phi(\sigma) \) is a closed path. Moreover, the interior of any edge of \( \phi(\sigma) \) cannot contain the origin.
- If the orderings of two points \( p \) and \( q \) are equal, then their paths \( \phi(\sigma_p) \) and \( \phi(\sigma_q) \) are equal.
- Path \( \phi(\sigma) \) has nonzero winding (with respect to the origin).

With these properties it can be seen that, when we walk from one cell to an adjacent one, not many changes can happen to the path \( \phi(\sigma) \). In particular, the winding must have the same sign. However, after we have moved \( p \) from \( p_x = -\infty \) to \( p_x = +\infty \), the corresponding orderings are reverse. In particular, their paths must have windings of different sign with respect to the origin. This contradicts the property that the motion of \( p \) does not affect the winding number of \( \phi(\sigma) \).

**Corollary 2** Let \( L \) be a balanced set of \( 6n \) lines inducing a simple arrangement. Then, there exists a segment intersecting exactly \( n \) lines of each color.

## 3 Balanced partitions on closed Jordan curves

In this section we consider balanced 3-colored point sets on closed Jordan curves. It is easy to see that, by homeomorphism, it suffices to give proofs for the unit circle. We show that there is always a bipartition of the set which is balanced and that can be realized by at most two disjoint arcs on the circle. To prove the claim we use the following arithmetic lemma:

**Lemma 6** For every integer \( n \geq 2 \), any integer \( k \in \{1, 2, \ldots, n\} \) can be obtained from \( n \) applying the functions \( f(x) = \lfloor x/2 \rfloor \) and \( g(x) = n - x \) to \( n \) at most \( 2 \log n + O(1) \) times.

Let \( S^1 \) be the unit circle in \( \mathbb{R}^2 \) with the usual parametrization \( f(t) = (\cos(t), \sin(t)), t \in [0, 2\pi) \). Let \( P \) be a balanced set of \( 3n \) points in \( S^1 \). The following theorem is a discrete version of the main result in [4]. There the methods are topological, while our approach is combinatorial, based on Lemma 6. The result can also be seen as a version of the *Necklace Theorem* [3] for closed curves. We say that a set \( Q \subseteq S^1 \) is a 2-arc set if it is the union of at most two disjoint arcs of \( S^1 \).

**Theorem 7** Let \( P \) be a balanced set of 3-colored \( 3n \) points in \( S^1 \). For each \( k \leq n \) there exists a 2-arc set \( P_k \subseteq S^1 \) containing exactly \( k \) points of each color.

**Proof.** (Sketch) Let \( I \) be the set of numbers \( k \) such that a subset \( P_k \) as in the theorem exists. We prove that \( I = \{1, \ldots, n\} \). By Lemma 6 it suffices to prove that: i) \( n \in I \), ii) If \( k \in I \) then \( n - k \in I \), and iii) If \( k \in I \) then \( k/2 \in I \).

Claims i) and ii) follow from the fact that \( S^1 \) and the complementary of any 2-arc set are 2-arc sets, respectively. To prove iii), we lift \( P \) to \( \mathbb{R}^3 \) using the moment curve. Abusing slightly the notation, we identify each point \( f(t) = (\cos(t), \sin(t)), t \in [0, 2\pi) \) on \( S^1 \) with its corresponding parameter \( t \). We assume that \( 0 \notin P_k \); otherwise we can change the parametrization of \( S^1 \) by rotating around the origin to assure this fact.

Then, for \( t \in S^1 \) we define \( \gamma(t) = \{t, t^2, t^3\} \); also, if \( S \subseteq S^1 \), we define \( \gamma(S) = \{\gamma(s) | s \in S\} \). As \( 0 \notin P_k \), any two disjoint arcs in \( S^1 \) become two disjoint intervals in \( \gamma(S^1) \). Now we apply the ham-sandwich theorem to \( \gamma(P_k) \) and obtain a plane that cuts each chromatic class in \( \gamma(P_k) \) in half. In order to finish the proof, we must show that this plane induces the desired partition of \( P \). Notice that any plane in \( \mathbb{R}^3 \) will intersect \( \gamma(S^1) \) in at most 3 points, hence the projection will be a 2-arc set.

The above proof can be generalized to \( c \) colors: if \( P \) contains \( n \) points of each color on \( S^1 \), then for each \( k \in \{1, \ldots, n\} \) there exists a \((c - 1)\)-arc set \( P_k \subseteq S^1 \) such that \( P_k \) contains exactly \( k \) points of each color (where the definition of a \((c - 1)\)-arc set is the natural extension of the 2-arc set). In the full version we also show that the bound in the number of intervals is tight.

## 4 L-lines in the plane lattice

We now consider a balanced partition problem for 3-colored point sets in the integer plane lattice. An *L-line with corner* \( q \) is the union of two different rays with common apex \( q \), each of them being either vertical or horizontal. Figure 2 shows a balanced L-line with apex \( q \).
L-lines in the lattice play a role comparable to that of ordinary lines in the real plane. For ordinary lines, Bereg and Kano [2] proved that if $S$ is a balanced 3-color point set whose convex hull is monochromatic, then there always exists a balanced L-line for $S$ (other than the trivial ones) may not exist. We extend the result in [2] to the plane lattice. As in the Euclidean plane, there exist problem instances for which the only balanced L-line is the trivial one, see for example Figure 3. To prevent this, we impose a condition on $S$: all vertices in the orthogonal convex hull of $S$ have the same color. Observe that this condition is the natural translation to the one used for ordinary lines by Bereg and Kano. Moreover, an equivalent condition is the fact that for every L-line that splits $S$ into two non-empty subsets, each subset contains at least one red point.

**Theorem 8** Let $S$ be a balanced set of 3-colored $3n$ points in the integer lattice. If the orthogonal hull is monochromatic, then there exists a nontrivial balanced L-line.

**Proof.** (Sketch) We use a technique similar to that described in the proof of Theorem 5. Imagine sweeping the lattice with a horizontal line (from top to bottom). At any point of the sweep, we sort the points above the line from top to bottom, and the remaining points from left to right. By doing so, we obtain some orderings of the point set. Once we have swept all the points, we do a second sweep, this time vertical. As in the proof of Theorem 5, we associate a curve to each instant of the sweep. Moreover, the winding cannot change before and after sweeping a point. Once we have rotated the sweep line twice (i.e., we have rotated the line $\pi$ radians), we obtain a reverse ordering, obtaining a similar contradiction. Details will be given in a full version of this paper.

**5 Concluding remarks**

Observe that our results on double wedges can be viewed as partial answers to the following interesting open problem: Find all $k$ such that, for any set of $n$ red, $r$ green and $b$ blue points in general position in the plane, there exists a double wedge containing exactly $k$ points of each color. We have given here an affirmative answer for $k = 1, n/2$ and $n - 1$ (Theorems 4 and 5). On the other hand, Theorem 7 can be viewed as an affirmative answer for all $k = 1, \ldots, n$ if the points belong to a circle (or in general, if they are in convex position).

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**References**


