Convex Quadrangulations of Bichromatic Point Sets

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Abstract

We consider quadrangulations of red and blue points in the plane where each face is convex and no edge connects two points of the same color. In particular, we show that the following problem is NP-hard: *Given a finite set S of points with each point either red or blue, does there exist a convex quadrangulation of S in such a way that the predefined colors give a valid vertex 2-coloring of the quadrangulation?* We consider this as a step towards solving the corresponding long-standing open problem on monochromatic point sets.

1 Introduction

A *quadrangulation* of a set S of n points in the Euclidean plane is a partition of the convex hull of S (denoted by $\mathsf{CH}(S)$) into quadrangles (i.e., 4-gons) such that the union of the vertices of the quadrangles is exactly the point set S and two quadrangles are disjoint or intersect either in a common vertex or a common edge. Hence, the quadrangulation is also a geometric (straight-line) planar graph with vertex set S. A quadrangulation is a *convex quadrangulation* if every quadrangle is convex. A point set admits a quadrangulation if and only if the number of points on the convex hull is even [3], but not every such set admits a convex quadrangulation, and deciding this in polynomial time is an open problem (posed by Joe Mitchell already in 1993 [15]).

A graph is *vertex k-colorable* (in brief *k-colorable*) if there exists a mapping of each vertex of the graph to exactly one of k colors such that no two vertices of the same color share an edge. A 2-colorable graph is a *bipartite graph.* It is known that every quadrangulation is bipartite. A *bichromatic point set* is a finite set S of points together with a mapping of each point to one of two colors. Throughout this paper, these colors will be *red* and *blue.*

Our main question is whether for a given bichromatic point set there is a convex quadrangulation s.t. the colors of the points define a valid 2-coloring of the quadrangulation. We call such a quadrangulation *valid.* Consider a 2-coloring of any quadrangulation. There are at least two vertices of each color, and it is easy to construct examples of quadrangulations with any valid number of vertices that have only two vertices of one color.

In Section 2, we prove that this bound differs for convex quadrangulations. Using observations of this section, we show that deciding whether a bichromatic point set has a valid convex quadrangulation is NP-complete. Mitchell's motivating question is left open.

Quadrangulations. Quadrangulations of point sets or polygons were discussed by many authors; see the survey by Toussaint [15]. Since not all polygons or point sets admit quadrangulations, even when the quadrangles are not required to be convex, the author surveys results on the characterization of those planar sets that always admit quadrangulations (convex and non-convex): quadrangulations of orthogonal polygons, simple polygons and point sets.

Lubiw [12] shows that determining whether a simple polygon with holes has a convex quadrangulation is NP-complete, even when quadrangles of any form are allowed; in contrast to that, there is a polynomial-time algorithm for a generalized variant of rectilinear polygons. Bose and Toussaint [3] show that a set S of n points admits a quadrangulation if and only if S has an even number of extreme points. They present an algorithm that computes a quadrangulation of S in $O(n \log n)$ time even in the presence of collinear points, adding an extra extreme point if necessary. Ramaswami, Ramos, and Toussaint [13] present efficient algorithms for converting triangulated domains to quadrangulations, while giving bounds on the number of Steiner points that might be required to obtain the quadrangulations. They show that a triangulated simple n-gon can be quadrangulated in linear time with the least number of outer Steiner points required for that triangulation, and that $\lfloor \frac{n}{3} \rfloor$ outer Steiner points are sufficient, and sometimes necessary, to quadrangulate a triangulated simple n-gon. They further show that $\lfloor \frac{n}{4} \rfloor$ inner Steiner points (and at most one outer Steiner point) are sufficient to quadrangulate a triangulated simple n-gon, and this can be done in linear time. The method can be used to quadrangulate arbitrary triangulated domains.

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Convex quadrangulations. Most of the work on convex quadrangulations is concerned with Steiner points. For example, Bremner et al. [4] prove that if the convex hull of S has an even number of points, then by adding at most $\frac{3n}{2}$ Steiner points in the interior of its convex hull, we can always obtain a point set that admits a convex quadrangulation. The authors also show that $\frac{n}{4}$ Steiner points are sometimes necessary. Heredia and Urrutia [8] improve these upper and lower bounds to $\frac{4n}{5}+2$ and $\frac{n}{3}$, respectively. Deciding in polynomial time whether a given (monochromatic) point set admits a convex quadrangulation without adding Steiner points seems to be a long-standing open problem. Only fixed-parameter-tractable algorithms and heuristics are known. Fevens, Meijer, and Rappaport [7] present a polynomial-time algorithm to determine whether a point set S admits a convex quadrangulation if S is constrained to lie on a constant number of nested convex polygons. Schiffer, Aurenhammer, and Demuth [14] propose a simple heuristic for computing large subsets of convex quadrangulations on a given set of points in the plane.

Quadrangulations of colored point sets. Cortés et al. [6] discuss aspects of quadrangulations of bichromatic point sets. They study bichromatic point sets that admit a quadrangulation, and whether, given two quadrangulations of the same bichromatic point set, it is possible to transform one into the other using certain local operations. They show that any bichromatic point set with convex layers having an even number of points with alternate colors has a valid quadrangulation, and any two such quadrangulations can be transformed into each other.

Alvarez, Sakai, and Urrutia [2] prove that a bichromatic set of n points can be quadrangulated by adding at most $\lfloor\frac{n-1}{3}\rfloor+\lfloor\frac{n}{2}\rfloor+1$ Steiner points and that $\frac{m}{3}$ Steiner points are occasionally necessary, where m is the number of quadrilaterals of the quadrangulation. They also show that there are 3-colored point sets with an even number of extreme points that do not admit a quadrangulation, even after adding Steiner points inside the set's convex hull.

Kato, Mori, and Nakamoto [9] define the *winding number* $\omega(S)$ for a 3-colored point set S, and prove that a 3-colored set S of n points in general position with a finite set P of Steiner points added is quadrangulatable if and only if $\omega(S)=0$. When $S\cup P$ is quadrangulatable, then $|P|\leq\frac{7n+34m-48}{18}$, where the number of extreme points is $2m$. This line of research is continued by Alvarez and Nakamoto [1], who study k-colored quadrangulation of k-colored sets of points, where $k\geq 2$. They show that if $\omega(S)=0$ or $k\geq 4$, then a k-colored quadrangulation of S can always be constructed using less than $\frac{(16k-2)n+7k-2}{39k-6}$ Steiner points. (We note that $\omega(S)=0$ for any bichromatic S where red and blue points on $\mathsf{CH}(S)$ alternate.)

2 The red and the blue graph of a convex quadrangulation

Let Q be a convex quadrangulation with a valid red-blue coloring of its n vertices. For every quadrangle, one diagonal connects the two red vertices of the quadrangle, and the other connects the two blue ones. We call them the *red diagonal* and the *blue diagonal*, respectively. Let G_{R} be the graph whose vertices are the red vertices of Q and whose edges are the red diagonals of all quadrangles of Q. Let G_{B} be defined analogously; since the colors are interchangeable, all the following statements hold equally for both graphs. Since every red edge has its own quadrangle and the faces (quadrangles) are convex, we obtain the following results.

Observation 1 *G_{R} is a plane simple graph.*

The following lemma is proven in the full version.

Lemma 2 *G_{R} is connected.*

Lemma 3 *Every minimal cycle of G_{R} contains exactly one blue point in its interior, and every inner blue point is contained in a minimal cycle of G_{R}. Blue points on the convex hull boundary are separated from the remaining set by a path in G_{R}.*

Proof. Consider the quadrangles that are adjacent to an inner blue point. The red diagonals of the quadrangles form a cycle that contain the blue point. Further, consider any minimal cycle of G_{R} and any edge therein. This edge corresponds to a quadrangle and there is one blue point of the quadrangle on each side of the edge. Observe that, in the same way, every blue point on the convex hull boundary is separated by a red path from the other blue vertices. ◻

Theorem 4 *Let n_{R} and n_{B} be the number of red and blue vertices, respectively, of a 2-colored convex quadrangulation. Then $n_{\mathrm{B}}\leq 2n_{\mathrm{R}}-2$.*

Proof. Observe that G_{R} and G_{B} have the same number e of edges. By Euler's Polyhedral Formula we have $n_{\mathrm{B}}-e+f_{\mathrm{B}}=2$, where f_{B} is the number of faces in the blue graph (including the outer face). Lemma 3 implies $n_{\mathrm{R}}=f_{\mathrm{B}}-1+\frac{h}{2}$, where h is the number of points in the convex hull. Hence, we get $n_{\mathrm{R}}+n_{\mathrm{R}}-\frac{h}{2}-1=e$. Since G_{R} is a plane geometric graph, we have $e\leq 3n_{\mathrm{R}}-3-\frac{h}{2}$. By plugging this into the previous equation we get the claimed inequality. ◻

Note that the inequality $e\leq n_{\mathrm{R}}-3-\frac{h}{2}$ is tight if and only if G_{R} is a triangulation. Figure 1 shows an example where the bound is tight.

The structure of the red and the blue graph reveals a necessary condition of a bichromatic point set

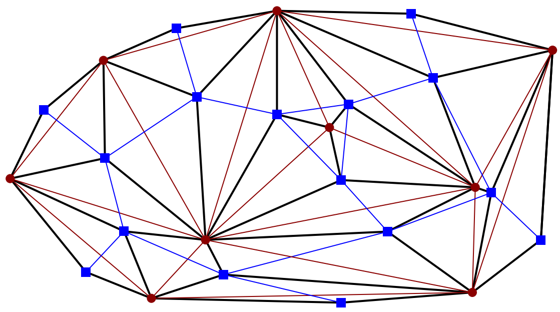

Figure 1: An example showing that the bound on the relation between the red (round) and blue (squared) points in a convex quadrangulation (indicated by thick black segments) is tight.

that allows a convex quadrangulation: Every segment between two red points must be intersected by a segment between two blue points. (Cortés et al. [5] give a quadratic-time algorithm to check for this property.)

3 NP-completeness

In this section we prove that the problem of deciding whether there exists a valid convex quadrangulation of a given bichromatic set of points is NP-hard.

Our reduction is from planar 3-SAT (cf. [11]). The construction is based in large parts on placing two red points sufficiently close to a crossing between two segments between blue points, s.t. exactly one of these blue segments is a diagonal of a quadrilateral in any convex quadrangulation (recall Lemma 3), and that the state of variables is propagated between the gadgets. Once there is a valid choice of these blue diagonals (corresponding to a satisfying variable assignment), we need to show that they are part of a valid convex quadrangulation. We later argue that the construction is possible with coordinates of polynomial size.

As common in this type of reductions, we transform an embedding of a planar 3-SAT instance to a bichromatic point set by replacing elements of the graph drawing by gadgets. For simplicity, we may consider the drawing to consist of edges that are represented by a sequence of orthogonal line segments (actually, one bend suffices, see [10]). An edge in this graph carries the truth value of a variable to the clause gadgets (possibly via a negation).

The main part of an edge gadget consists of a chain of *link gadgets*, each containing four blue points in convex position and two red points close to the crossing they define. Hence, one of the two blue edges must be a diagonal in any valid convex quadrangulation Q (if it exists). See Figure 2. If one of the segments is a diagonal of Q (say, the one from bottom-left to top-right), the edge gadget carries *true* (and the line segment is called the *T-diagonal* of the link gadget);

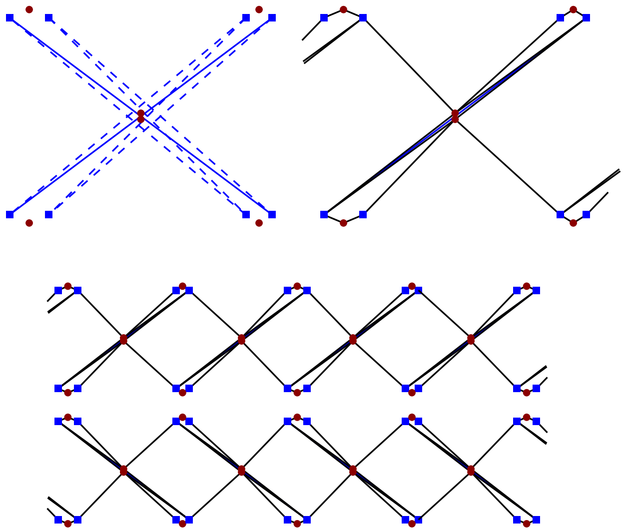

Figure 2: A link gadget to model edges in the graph. The middle red points are that close to the crossing of the solid blue segments s.t. there is no other segment passing between them (as indicated by the dashed lines). (Note that to this end, the three-point "caps" at the ends of the segments have to have slightly different width.) Exactly one of the blue segments has to be a diagonal of the quadrangulation, and combining these links propagates that decision. A possible quadrangulation is shown to the right. The link gadgets can be concatenated to form edges, as shown below.

if the other segment (being called the *F-diagonal*) is a diagonal of Q, the edge gadget carries *false*. Two of these links are joined such that the T-diagonal of the previous link crosses the F-diagonal of the next link, and vice versa, and thus Q cannot have a T-diagonal and an F-diagonal in the same edge gadget.

A variable gadget works by connecting three edge gadgets in a way that they all have either the T-edge or the F-edge as a diagonal; an arbitrary number of edges from the same variable vertex can be connected in that way. The variable gadget is shown in Figure 3. Further, we need bends in the edge gadgets to connect horizontal and vertical parts, as well as negation gadgets. All of these are mere appropriate combinations of link gadgets, figures and exact descriptions of these gadgets are provided in the full version.

For the clause gadgets, we have a pair of red points that span a segment intersected by exactly three potential blue diagonals. There is only one state of the three incident edge gadgets that prevents all three blue diagonals. By appropriately adding negation gadgets, we make this configuration appear exactly when all three edge gadgets carry *false*. See Figure 4.

It remains to argue that the parts of the convex hull not covered by gadgets can be quadrangulated. As these parts are enclosed by a polygonal chain in which red and blue vertices alternate, we can add

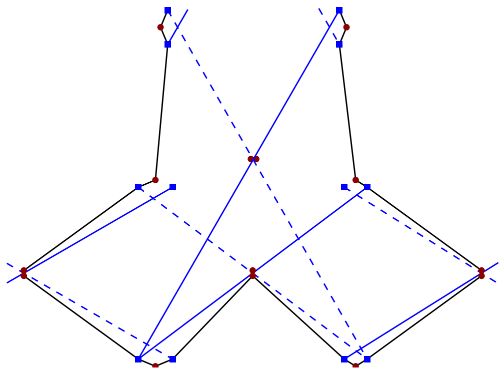

Figure 3: A variable gadget, showing the possible set of blue diagonals. It "splits" an edge, propagating the (negated) truth value it carries.

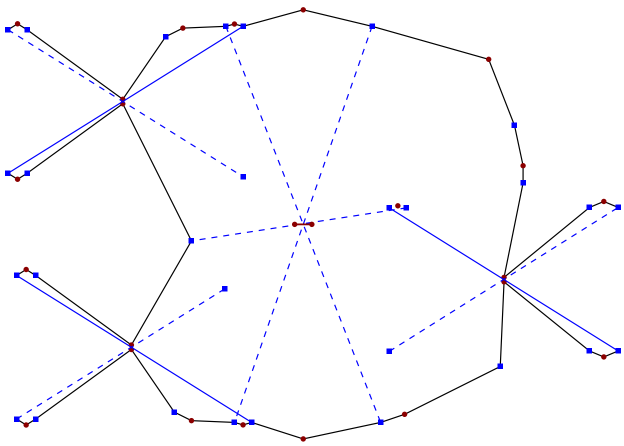

Figure 4: The clause gadget. The two red points in the middle connected by a red segment are closer than drawn. The only impossible combination of blue diagonals for the link gadgets is the one including all three solid segments. Negating the top-left edge makes this the configuration with all literals set to *false*.

Steiner points to find a quadrangulation (see the full version for details). Thus, after adding these Steiner points to our construction, there is a bichromatic convex quadrangulation of our point set if and only if the corresponding planar 3-SAT instance is satisfiable.

Finally, let us remark that the points can be placed in general position using coordinates of polynomial size. Before placing two close red points, we find out which distance allows placing the points sufficiently close to each other (possibly after a perturbation).

Theorem 5 *Given a set of red and blue points in the plane, it is NP-complete to decide whether there is a valid convex quadrangulation of that point set.*

Note that actually, the bichromatic setting is a way to forbid certain edges in the quadrangulation. For our reduction, it is sufficient to forbid those between the close red points in the gadgets. However, we do not know how to achieve this in an unconstrained setting.

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