Session 9: Some applications of the path method in $b$-symplectic geometry

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1. Reminder of last lecture

2. Classification of toric actions on $b$-symplectic manifolds

3. Integrable Systems on $b$-symplectic manifolds
- b-Poisson structures can be seen as symplectic structures modeled over a Lie algebroid (the b-cotangent bundle).

- A vector field $v$ is a $b$-vector field if $v_p \in T_pZ$ for all $p \in Z$. The $b$-tangent bundle $bTM$ is defined by

$$
\Gamma(U, bTM) = \left\{ \text{b-vector fields on } (U, U \cap Z) \right\}
$$
The $b$-cotangent bundle $bT^*M$ is $(bTM)^*$. Sections of $\Lambda^p(bT^*M)$ are $b$-forms, $b\Omega^p(M)$. The standard differential extends to

$$d : b\Omega^p(M) \to b\Omega^{p+1}(M)$$

A $b$-symplectic form is a closed, nondegenerate, $b$-form of degree 2.

This dual point of view, allows to prove a $b$-Darboux theorem and semilocal forms via an adaptation of Moser’s path method because we can play the same tricks as in the symplectic case.

What else?
Relative Moser theorem

Theorem

Let $\omega_0$ and $\omega_1$ be two $b$-symplectic forms on $(M, Z)$. If $\omega_0|_Z = \omega_1|_Z$, then there exist neighborhoods $U_0, U_1$ of $Z$ in $M$ and a diffeomorphism $\gamma : U_0 \to U_1$ such that $\gamma|_Z = \text{id}_Z$ and $\gamma^*\omega_1 = \omega_0$.

Theorem

Let $\omega_0$ and $\omega_1$ be two $b$-symplectic forms on $(M, Z)$. If they induce on $Z$ the same restriction of the Poisson structure and their modular vector fields differ on $Z$ by a Hamiltonian vector field, then there exist neighborhoods $U_0, U_1$ of $Z$ in $M$ and a diffeomorphism $\gamma : U_0 \to U_1$ such that $\gamma|_Z = \text{id}_Z$ and $\gamma^*\omega_1 = \omega_0$. 
Global Moser theorem

Theorem

Suppose that $M$ is compact and let $\omega_0$ and $\omega_1$ be two $b$-symplectic forms on $(M, Z)$. Suppose that $\omega_t$, for $0 \leq t \leq 1$, is a smooth family of $b$-symplectic forms on $(M, Z)$ joining $\omega_0$ and $\omega_1$ and such that the $b$-cohomology class $[\omega_t]$ does not depend on $t$. Then, there exists a family of diffeomorphisms $\gamma_t : M \to M$, for $0 \leq t \leq 1$ such that $\gamma_t$ leaves $Z$ invariant and $\gamma_t^* \omega_t = \omega_0$. 

Eva Miranda (UPC)
Radko’s theorem revisited

**Theorem (Mazzeo-Melrose)**

The $b$-cohomology groups of a compact $M$ are computable by

\[ b^*H^*(M) \cong H^*(M) \oplus H^{*-1}(\mathbb{Z}). \]

**Corollary (Classification of $b$-symplectic surfaces à la Moser)**

Two $b$-symplectic forms $\omega_0$ and $\omega_1$ on an orientable compact surface are $b$-symplectomorphic if and only if $[\omega_0] = [\omega_1]$.

Indeed

**Theorem (Guillemin-M.-Pires)**

\[ b^*H^*(M) \cong H^*_\Pi(M) \]
Definition (Symplectic case)

Let $G$ be a compact Lie group acting symplectically on $(M, \omega)$. The action is **Hamiltonian** if there exists an equivariant map $\mu : M \rightarrow g^*$ such that for each element $X \in g$,

$$-d\mu^X = \iota_X \# \omega,$$

(1)

with $\mu^X = \langle \mu, X \rangle$.

The map $\mu$ is called the **moment map**.
Theorem (Delzant)

Toric manifolds are classified by Delzant’s polytopes. More specifically, the bijective correspondence between these two sets is given by the image of the moment map:

\[
\begin{align*}
\{ \text{toric manifolds} \} & \quad \longrightarrow \quad \{ \text{Delzant polytopes} \} \\
(M^{2n}, \omega, \mathbb{T}^n, F) & \quad \longrightarrow \quad F(M)
\end{align*}
\]

\[
(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{it_1} z_1 : e^{it_2} z_2]
\]
Why toric?

Given an integrable system $F = (f_1, \ldots, f_n)$ (\{f_i, f_j\} = 0) and a compact fiber,

**Theorem (Arnold-Liouville)**

There exist semilocal action-angle coordinates $(p_1, \theta_1, \ldots, p_n, \theta_n)$ such that

$$\omega = \sum_{i=0}^{n} dp_i \wedge d\theta_i, \quad F = (p_1, \ldots, p_n)$$

with linear Hamiltonian flow on the torus.

Liouville tori (left) and Bohr-Sommerfeld orbits read from the polytope (right)

**Some applications:**
- **Singular fibrations**: Symplectic Morse-Bott classification.
- **Geometric Quantization**: Guillemin-Sternberg, Sniaticky.
The only orientable compact surfaces admitting an effective action by circles are the two sphere $\mathbb{S}^2$ and the 2-torus $\mathbb{T}^2$ and the action is equivalent to the standard action by rotations.

In the symplectic case the standard rotation on $\mathbb{T}^2$ is not Hamiltonian (only symplectic).

$$d\theta_1 \wedge d\theta_2(\frac{\partial}{\partial \theta_1}, \cdot) = d\theta_2.$$ 

In the $b$-symplectic case, the toric surfaces are either the sphere or the torus.
The \textit{b-line} is constructed by gluing copies of the extended real line \( \mathbb{R} := \mathbb{R} \cup \{ \pm \infty \} \) together in a zig-zag pattern and \( \mathbb{R}_{>0} \)-valued labels ("weights") on the points at infinity to prescribe a smooth structure.

The weights are given by the \textbf{modular periods} associated to each connected component of \( \mathbb{Z} \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{b-line.png}
\caption{A weighted \textit{b}-line with \( I = \mathbb{Z} \)}
\end{figure}
A toric $b$-surface is defined by a smooth map $f : S \rightarrow \mathbb{R}$ or $f : S \rightarrow \mathbb{S}^1$ (a posteriori the moment map).
Classification of toric $b$-surfaces

Theorem (Guillemin, M., Pires, Scott)

A toric $b$-symplectic surface is equivariantly $b$-symplectomorphic to either $(S^2, Z)$ or $(T^2, Z)$, where $Z$ is a collection of latitude circles.

The action is the standard rotation, and the $b$-symplectic form is determined by the modular periods of the critical curves and the regularized Liouville volume.

The weights $w(a)$ of the codomain of the moment map are given by the modular periods of the connected components of the critical hypersurface.
Example

\((S^2, \omega = \frac{dh}{h} \wedge d\theta)\), with coordinates \(h \in [-1, 1]\) and \(\theta \in [0, 2\pi]\). The critical hypersurface \(Z\) is the equator, given by \(h = 0\). For the \(S^1\)-action by rotations, the moment map is \(\mu(h, \theta) = \log |h|\).
The $S^1$-$b$-torus

Example

On $(\mathbb{T}^2, \omega = \frac{d\theta_1}{\sin \theta_1} \wedge d\theta_2)$, with coordinates: $\theta_1, \theta_2 \in [0, 2\pi]$. The critical hypersurface $Z$ is the union of two disjoint circles, given by $\theta_1 = 0$ and $\theta_1 = \pi$. Consider rotations in $\theta_2$ the moment map is $\mu: \mathbb{T}^2 \rightarrow \mathbb{R}^2$ is given by $\mu(\theta_1, \theta_2) = \log \left| \frac{1+\cos(\theta_1)}{\sin(\theta_1)} \right|$. 
This information can be recovered by doing reduction by stages: Hamiltonian reduction of an action of $\mathbb{T}^{n-1}_\mathbb{Z}$ and the classification of toric $b$-surfaces.
Fix $bt^*$ with $wt(1) = c$.

For any Delzant polytope $\Delta \subseteq t^*_Z$ with corresponding symplectic toric manifold $(X_\Delta, \omega_\Delta, \mu_\Delta)$, the **semilocal model** of the $b$-symplectic manifold is

$$M_{lm} = X_\Delta \times S^1 \times \mathbb{R} \quad \omega_{lm} = \omega_\Delta + c\frac{dt}{t} \wedge d\theta$$

where $\theta$ and $t$ are the coordinates on $S^1$ and $\mathbb{R}$ respectively. The $S^1 \times T_Z$ action on $M_{lm}$ given by $(\rho, g) \cdot (x, \theta, t) = (g \cdot x, \theta + \rho, t)$ has moment map $\mu_{lm}(x, \theta, t) = (y_0 = t, \mu_\Delta(x))$. 
Theorem (Guillemin, M., Pires, Scott)

The maps that send a $b$-symplectic toric manifold to the image of its moment map

$$\{(M, Z, \omega, \mu : M \to b^t)\} \to \{b\text{-Delzant polytopes in } b^t\}$$  (2)

and

$$\{(M, Z, \omega, \mu : M \to b^t/\langle N \rangle)\} \to \{b\text{-Delzant polytopes in } b^t/\langle N \rangle\}$$  (3)

are bijections.

Toric $b$-manifolds can be of two types:

1. $bT^2 \times X$ (with $X$ a toric symplectic manifold of dimension $(2n - 2)$)
2. $bS^2 \times X$ and manifolds obtained via symplectic cutting (for instance, $m\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$, with $m, n \geq 1$).
**b-integrable systems**

**Definition**

*b-integrable system* A set of *b*-functions $f_1, \ldots, f_n$ on $(M^{2n}, \omega)$ such that

1. $f_1, \ldots, f_n$ Poisson commute.
2. $df_1 \wedge \cdots \wedge df_n \neq 0$ as a section of $\Lambda^n(bT^*(M))$ on a dense subset of $M$ and on a dense subset of $\mathbb{Z}$

\[ a \log |x| + g \]

**Example**

The symplectic form $\frac{1}{\hbar} dh \wedge d\theta$ defined on the interior of the upper hemisphere $H_+$ of $S^2$ extends to a *b*-symplectic form $\omega$ on the double of $H_+$ which is $S^2$. The triple $(S^2, \omega, \log|\hbar|)$ is a *b*-integrable system.

**Example**

If $(f_1, \ldots, f_n)$ is an integrable system on $M$, then $(\log|\hbar|, f_1, \ldots, f_n)$ on $H_+ \times M$ extends to a *b*-integrable on $S^2 \times M$. 
Action-angle coordinates for $b$-integrable systems

The compact regular level sets of a $b$-integrable system are (Liouville) tori.

**Theorem (Kiesenhofer-M.-Scott)**

Around a Liouville torus there exist coordinates\((p_1,\ldots,p_n,\theta_1,\ldots,\theta_n) : U \to B^n \times T^n\) such that

\[
\omega|_U = \frac{c}{p_1} dp_1 \wedge d\theta_1 + \sum_{i=2}^{n} dp_i \wedge d\theta_i, \tag{4}
\]

and the level sets of the coordinates $p_1,\ldots,p_n$ correspond to the Liouville tori of the system.

**Reformulation of the result**

Integrable systems semilocally $\leftrightarrow$ twisted cotangent lift\(^{a}\) of a $T^n$ action by translations on itself to $(T^*T^n)$.

\(^{a}\)We replace the Liouville form by $c \log |p_1|d\theta_1 + \sum_{i=2}^{n} p_id\theta_i$. 
Proof

1. **Topology of the foliation.** In a neighbourhood of a compact connected fiber the $b$-integrable system $F$ is diffeomorphic to the $b$-integrable system on $W := \mathbb{T}^n \times B^n$ given by the projections $p_1, \ldots, p_{n-1}$ and $\log |p_n|$.

2. **Uniformization of periods:** We want to define integrals whose $(b)$-Hamiltonian vector fields induce a $\mathbb{T}^n$ action. Start with $\mathbb{R}^n$-action:

$$\Phi : \mathbb{R}^n \times (\mathbb{T}^n \times B^n) \to \mathbb{T}^n \times B^n$$

$$(t_1, \ldots, t_n, m) \mapsto \Phi_{t_1}^{(1)} \circ \cdots \circ \Phi_{t_n}^{(n)}(m).$$

Uniformize to get a $\mathbb{T}^n$ action with fundamental vector fields $Y_i$.

3. The vector fields $Y_i$ are Poisson vector fields (check $\mathcal{L}_{Y_i} \mathcal{L}_{Y_i} \omega = 0$).

4. The vector fields $Y_i$ are $b$-Hamiltonian with primitives $\sigma_1, \ldots, \sigma_n \in ^b\mathcal{C}^\infty(W)$. In this step the properties of $b$-cohomology are essential. Use this action to drag a local normal form (Darboux-Carathéodory) in a whole neighbourhood.
Proof

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   \[ \Phi : \mathbb{R}^n \times (T^n \times B^n) \rightarrow T^n \times B^n \]

   \[ ((t_1, \ldots, t_n), m) \mapsto \Phi^{(1)}_{t_1} \circ \cdots \circ \Phi^{(n)}_{t_n}(m). \]

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Applications to KAM theory (surviving torus under perturbations) on $b$-symplectic manifolds (Kiesenhofer-M.-Scott).

Figure: Fibration by Liouville tori