Geometry and Dynamics of singular symplectic manifolds

Session 9: Some applications of the path method in b-symplectic geometry

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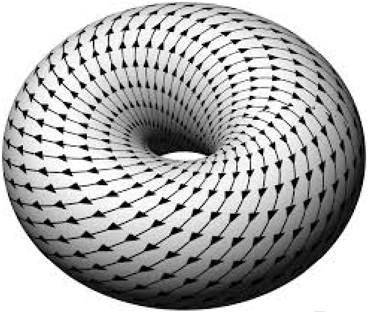
Outline

1. Reminder of last lecture
2. Classification of toric actions on b-symplectic manifolds
3. Integrable Systems on b-symplectic manifolds

A dual approach...

- b-Poisson structures can be seen as symplectic structures modeled over a Lie algebroid (the b-cotangent bundle).
- A vector field v is a **b-vector field** if $v_p \in T_pZ$ for all $p \in Z$. The **b-tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$



b-forms

- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are b-**forms**, ${}^b\Omega^p(M)$.The standard differential extends to

$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

- A **b-symplectic form** is a closed, nondegenerate, b-form of degree 2.
- This dual point of view, allows to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path method because we can play the same tricks as in the symplectic case.

What else?

Relative Moser theorem

Theorem

Let ω_0 and ω_1 be two b-symplectic forms on (M,Z). If $\omega_0|_Z = \omega_1|_Z$, then there exist neighborhoods U_0, U_1 of Z in M and a diffeomorphism $\gamma : U_0 \to U_1$ such that $\gamma|_Z = id_Z$ and $\gamma^*\omega_1 = \omega_0$.

Theorem

Let ω_0 and ω_1 be two b-symplectic forms on (M,Z). If they induce on Z the same restriction of the Poisson structure and their modular vector fields differ on Z by a Hamiltonian vector field, then there exist neighborhoods U_0, U_1 of Z in M and a diffeomorphism $\gamma : U_0 \to U_1$ such that $\gamma|_Z = id_Z$ and $\gamma^*\omega_1 = \omega_0$.

Global Moser theorem

Theorem

Suppose that M is compact and let ω_0 and ω_1 be two b-symplectic forms on (M, Z). Suppose that ω_t, for $0 \leq t \leq 1$, is a smooth family of b-symplectic forms on (M, Z) joining ω_0 and ω_1 and such that the b-cohomology class $[\omega_t]$ does not depend on t. Then, there exists a family of diffeomorphisms $\gamma_t : M \to M$, for $0 \leq t \leq 1$ such that γ_t leaves Z invariant and $\gamma_t^* \omega_t = \omega_0$.

Radko's theorem revisited

Theorem (Mazzeo-Melrose)

The b-cohomology groups of a compact M are computable by

$$^{b}H^{*}(M) \cong H^{*}(M) \oplus H^{*-1}(Z).$$

Corollary (Classification of b-symplectic surfaces à la Moser)

Two b-symplectic forms ω_0 and ω_1 on an orientable compact surface are b-symplectomorphic if and only if $[\omega_0] = [\omega_1]$.

Indeed

Theorem (Guillemin-M.-Pires)

$$^{b}H^{*}(M) \cong H^{*}_{\Pi}(M)$$

Moment maps in Symplectic Geometry

Definition (Symplectic case)

Let G be a compact Lie group acting symplectically on (M, ω).
The action is **Hamiltonian** if there exists an equivariant map $\mu : M \to \mathfrak{g}^*$ such that for each element $X \in \mathfrak{g}$,

$$-d\mu^X = \iota_{X^\#}\omega, \tag{1}$$

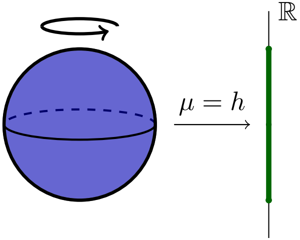
with $\mu^X = <\mu, X>$.
The map μ is called the **moment map**.

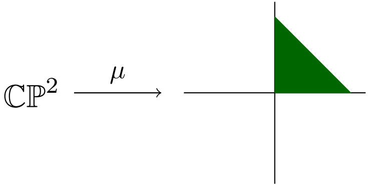
Delzant theorem for symplectic manifolds

Theorem (Delzant)

Toric manifolds are classified by Delzant's polytopes. More specifically, the bijective correspondence between these two sets is given by the image of the moment map:

$$\begin{array}{ccc} \{\textit{toric manifolds}\} & \longrightarrow & \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, F) & \longrightarrow & F(M) \end{array}$$





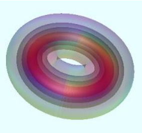
$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{it_1} z_1 : e^{it_2} z_2]$$

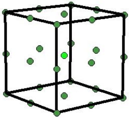
Why toric?

Given an integrable system $F = (f_1, \ldots, f_n)$ ($\{f_i, f_j\} = 0$) and a compact fiber,

Theorem (Arnold-Liouville)

There exist semilocal action-angle coordinates $(p_1, \theta_1, \ldots, p_n, \theta_n)$ such that $\omega = \sum_{i=0}^{n} dp_i \wedge d\theta_i$, $F = (p_1, \ldots, p_n)$ with linear Hamiltonian flow on the torus.





Liouville tori (left) and Bohr-Sommerfeld orbits read from the polytope (right)

Some applications:

- Singular fibrations: Symplectic Morse-Bott classification.
- Geometric Quantization: Guillemin-Sternberg, Sniaticky.

Surfaces and circle actions

The only orientable compact surfaces admitting an effective action by circles are the two sphere $\mathbb{S}^2$ and the 2-torus $\mathbb{T}^2$ and the action is equivalent to the standard action by rotations.

In the symplectic case the standard rotation on $\mathbb{T}^2$ is not Hamiltonian (only symplectic).

$$d\theta_1 \wedge d\theta_2(\frac{\partial}{\partial \theta_1}, \cdot) = d\theta_2.$$

In the b-symplectic case, the toric surfaces are either the sphere or the torus.

The b-line

The b-line is constructed by gluing copies of the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ together in a zig-zag pattern and $\mathbb{R}_{>0}$-valued labels ("weights") on the points at infinity to prescribe a smooth structure.

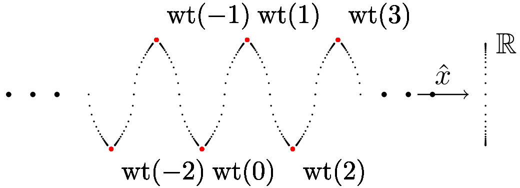
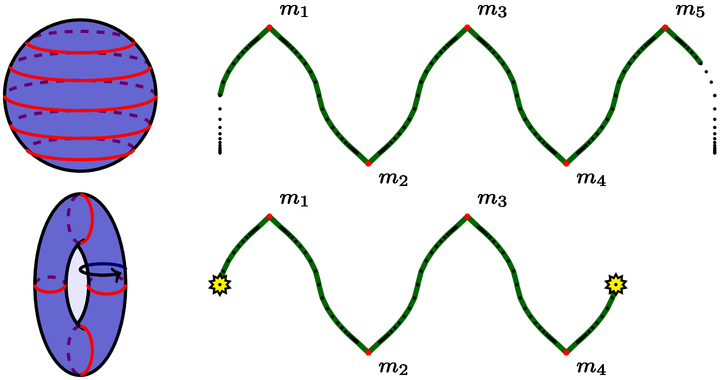


Figure: A weighted b-line with $I = \mathbb{Z}$

The weights are given by the modular periods associated to each connected component of Z.

b-surfaces and their moment map

A toric b-surface is defined by a smooth map $f : S \longrightarrow {}^b\mathbb{R}$ or $f : S \longrightarrow {}^b\mathbb{S}^1$ (a posteriori the moment map).



Classification of toric b-surfaces

Theorem (Guillemin, M., Pires, Scott)

A toric b-symplectic surface is equivariantly b-symplectomorphic to either $(\mathbb{S}^2, Z)$ or $(\mathbb{T}^2, Z)$, where Z is a collection of latitude circles.

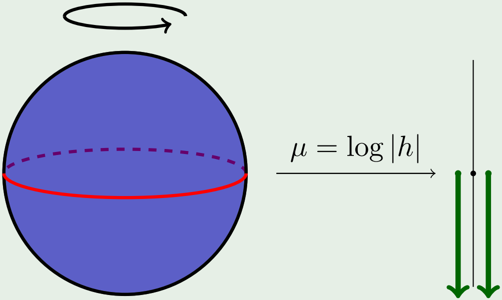
The action is the standard rotation, and the b-symplectic form is determined by the modular periods of the critical curves and the regularized Liouville volume.

The weights $w(a)$ of the codomain of the moment map are given by de modular periods of the connected components of the critical hypersurface.

The S^1-b-sphere

Example

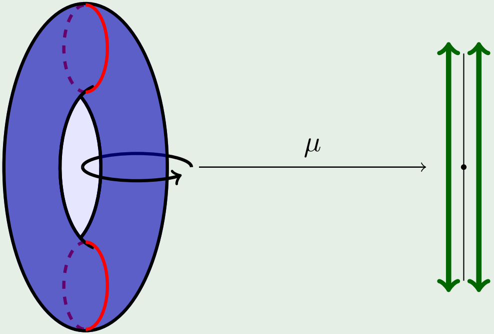
$(\mathbb{S}^2, \omega = \frac{dh}{h} \wedge d\theta)$, with coordinates $h \in [-1,1]$ and $\theta \in [0, 2\pi]$. The critical hypersurface Z is the equator, given by $h = 0$. For the $\mathbb{S}^1$-action by rotations, the moment map is $\mu(h,\theta) = \log|h|$.



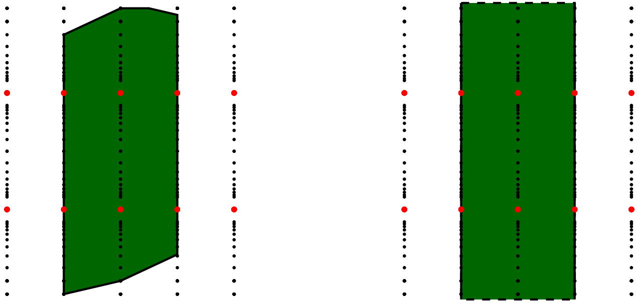
The S^1-b-torus

Example

On $(\mathbb{T}^2, \omega = \frac{d\theta_1}{\sin\theta_1} \wedge d\theta_2)$, with coordinates: $\theta_1, \theta_2 \in [0, 2\pi]$. The critical hypersurface Z is the union of two disjoint circles, given by $\theta_1 = 0$ and $\theta_1 = \pi$. Consider rotations in θ_2 the moment map is $\mu : \mathbb{T}^2 \longrightarrow \mathbb{R}^2$ is given by $\mu(\theta_1, \theta_2) = \log\left|\frac{1+\cos(\theta_1)}{\sin(\theta_1)}\right|$.



The polytopes



This information can be recovered by doing reduction by stages: Hamiltonian reduction of an action of $\mathbb{T}_Z^{n-1}$ and the classification of toric b-surfaces.

The semilocal model

Fix ${}^b\mathfrak{t}^*$ with $wt(1) = c$.
For any Delzant polytope $\Delta \subseteq \mathfrak{t}_Z^*$ with corresponding symplectic toric manifold $(X_\Delta, \omega_\Delta, \mu_\Delta)$, the **semilocal model** of the b-symplectic manifold is

$$M_{\mathrm{lm}} = X_\Delta \times \mathbb{S}^1 \times \mathbb{R} \qquad \omega_{\mathrm{lm}} = \omega_\Delta + c\frac{dt}{t} \wedge d\theta$$

where θ and t are the coordinates on $\mathbb{S}^1$ and $\mathbb{R}$ respectively. The $\mathbb{S}^1 \times \mathbb{T}_Z$ action on M_{lm} given by $(\rho, g) \cdot (x, \theta, t) = (g \cdot x, \theta + \rho, t)$ has moment map $\mu_{\mathrm{lm}}(x, \theta, t) = (y_0 = t, \mu_\Delta(x))$.

A b-Delzant theorem

Theorem (Guillemin,M.,Pires, Scott)

The maps that send a b-symplectic toric manifold to the image of its moment map

$$\{(M, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*)\} \to \{b\text{-Delzant polytopes in } {}^b\mathfrak{t}^*\} \tag{2}$$

and

$$\{(M, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*/\langle N\rangle)\} \to \{b\text{-Delzant polytopes in } {}^b\mathfrak{t}^*/\langle N\rangle\} \tag{3}$$

are bijections.

Toric b-manifolds can be of two types:

1. ${}^b\mathbb{T}^2 \times X$ (with X a toric symplectic manifold of dimension $(2n-2)$)
2. ${}^b\mathbb{S}^2 \times X$ and manifolds obtained via symplectic cutting (for instance, $m\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$, with $m, n \geq 1$).

b-integrable systems

Definition

b-integrable system A set of b-functions[a] $f_1, \dots, f_n$ on (M^{2n}, ω) such that

- $f_1, \dots, f_n$ Poisson commute.
- $df_1 \wedge \dots \wedge df_n \neq 0$ as a section of $\Lambda^n({}^bT^*(M))$ on a dense subset of M and on a dense subset of Z

[a] $c \log|x| + g$

Example

The symplectic form $\frac{1}{h} dh \wedge d\theta$ defined on the interior of the upper hemisphere H_+ of S^2 extends to a b-symplectic form ω on the double of H_+ which is S^2. The triple $(S^2, \omega, log|h|)$ is a b-integrable system.

Example

If $(f_1, \dots, f_n)$ is an integrable system on M, then $(\log|h|, f_1, \dots, f_n)$ on $H_+ \times M$ extends to a b-integrable on $S^2 \times M$.

Action-angle coordinates for b-integrable systems

The compact regular level sets of a b-integrable system are (Liouville) tori.

Theorem (Kiesenhofer-M.-Scott)

Around a Liouville torus there exist coordinates $(p_1, \dots, p_n, \theta_1, \dots, \theta_n) : U \to B^n \times \mathbf{T}^n$ such that

$$\omega|_U = \frac{c}{p_1} dp_1 \wedge d\theta_1 + \sum_{i=2}^{n} dp_i \wedge d\theta_i, \tag{4}$$

and the level sets of the coordinates $p_1, \dots, p_n$ correspond to the Liouville tori of the system.

Reformulation of the result

Integrable systems semilocally $\rightsquigarrow$ twisted cotangent lift[a] of a $\mathbb{T}^n$ action by translations on itself to $(T^*\mathbb{T}^n)$.

[a] We replace the Liouville form by $c \log |p_1| d\theta_1 + \sum_{i=2}^{n} p_i d\theta_i$.

Proof

1. Topology of the foliation. In a neighbourhood of a compact connected fiber the b-integrable system F is diffeomorphic to the b-integrable system on $W := \mathbf{T}^n \times B^n$ given by the projections $p_1, \dots, p_{n-1}$ and $\log |p_n|$.
2. Uniformization of periods: We want to define integrals whose (b-)Hamiltonian vector fields induce a $\mathbf{T}^n$ action. Start with $\mathbf{R}^n$-action:

$$\begin{array}{rcccl} \Phi & : & \mathbf{R}^n \times (\mathbf{T}^n \times B^n) & \to & \mathbf{T}^n \times B^n \\ & & ((t_1, \dots, t_n), m) & \mapsto & \Phi^{(1)}_{t_1} \circ \cdots \circ \Phi^{(n)}_{t_n}(m). \end{array}$$

   Uniformize to get a $\mathbf{T}^n$ action with fundamental vector fields Y_i.
3. The vector fields Y_i are Poisson vector fields (check $\mathcal{L}_{Y_i}\mathcal{L}_{Y_i}\omega = 0$).
4. The vector fields Y_i are b-Hamiltonian with primitives $\sigma_1, \dots, \sigma_n \in {}^bC^\infty(W)$. In this step the properties of b-cohomology are essential.Use this action to drag a local normal form (Darboux-Carathéodory) in a whole neighbourhood.

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A picture...

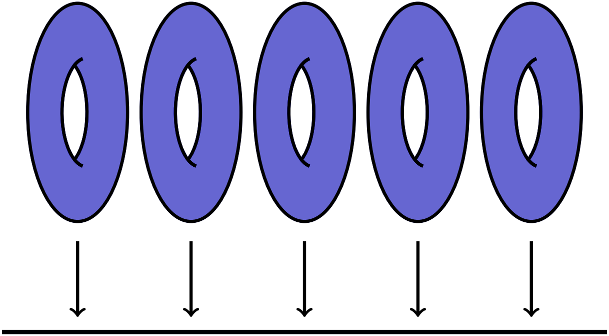

Figure: Fibration by Liouville tori

Applications to KAM theory (surviving torus under perturbations) on b-symplectic manifolds (Kiesenhofer-M.-Scott).