Geometry and Dynamics of singular symplectic manifolds

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Schouten Bracket of vector fields in local coordinates

- Case of vector fields,
  $A = \sum_i a_i \frac{\partial}{\partial x_i}$ and $A = \sum_i b_i \frac{\partial}{\partial x_i}$. Then

$$[A, B] = \sum_i a_i (\sum_j \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j}) - \sum_i b_i (\sum_j \frac{\partial a_j}{\partial x_i} \frac{\partial}{\partial x_j})$$

- Re-denoting $\frac{\partial}{\partial x_i}$ as ζ_i **("odd coordinates ")**.
  Then $A = \sum_i a_i \zeta_i$ and $B = \sum_i b_i \zeta_i$ and $\zeta_i \zeta_j = -\zeta_j \zeta_i$ Now we can reinterpret the bracket as,

$$[A, B] = \sum_i \frac{\partial A}{\partial \zeta_i} \frac{\partial B}{\partial x_i} - \sum_i \frac{\partial B}{\partial \zeta_i} \frac{\partial A}{\partial x_i}$$

Schouten Bracket of multivector fields in local coordinates

We reproduce the same scheme for the case of multivector fields.

$$[A, B] = \sum_i \frac{\partial A}{\partial \zeta_i} \frac{\partial B}{\partial x_i} - (-1)^{(a-1)(b-1)} \sum_i \frac{\partial B}{\partial \zeta_i} \frac{\partial A}{\partial x_i}$$

is a $(a + b - 1)$-vector field.
where

$$A = \sum_{i_1 < \cdots < i_a} A_{i_1, \ldots, i_a} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_a}} = \sum_{i_1 < \cdots < i_a} A_{i_1, \ldots, i_a} \zeta_{i_1} \ldots \zeta_{i_a}$$

and

$$B = \sum_{i_1 < \cdots < i_b} B_{i_1, \ldots, i_b} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_b}} = \sum_{i_1 < \cdots < i_b} B_{i_1, \ldots, i_b} \zeta_{i_1} \ldots \zeta_{i_b}$$

with $\frac{\partial(\zeta_{i_1} \ldots \zeta_{i_p})}{\partial \zeta_{i_k}} := (-1)^{(p-k)} \eta_{i_1} \ldots \widehat{\eta}_{i_k} \eta_{i_{p-1}}$

Theorem (Schouten-Nijenhuis)

The bracket defined by this formula satisfies,

Graded anti-commutativity $[A, B] = -(-1)^{(a-1)(b-1)}[B, A]$.

Graded Leibniz rule

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(a-1)b} B \wedge [A, C]$$

Graded Jacobi identity

$$(-1)^{(a-1)(c-1)}[A, [B, C]] + (-1)^{(b-1)(a-1)}[B, [C, A]] + (-1)^{(c-1)(b-1)}[C, [A, B]] = 0$$

If X is a vector field then, $[X, B] = L_X B$.