Geometry and Dynamics of singular symplectic manifolds

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Symplectic Geometry	Poisson Geometry
ω	Π
$\iota_{X_f}\omega = -df$	$X_f := \Pi(df, \cdot)$
one symplectic leaf	a symplectic foliation
Darboux theorem	Weinstein's splitting theorem
$\omega = \sum_{i=1}^n dx_i \wedge dy_i$	$\Pi = \sum_{i=1}^k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_i} + \sum_{kl} \phi_{kl}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l}$
$L_X \omega = 0$	$L_X \Pi = 0$
$H^1_{DR}(M) = \frac{\text{symplectic v.f}}{\text{Hamiltonian v.f}}$	$?= \frac{\text{Poisson v.f}}{\text{Hamiltonian v.f}}$
$H^k_{DR}(M)$ (cochains $\Omega^k(M)$)	$?:= H^k_{\Pi}(M)$ (cochains $\mathfrak{X}^m(M)$)

Plan for today

- Weinstein's splitting theorem and normal form theorems.

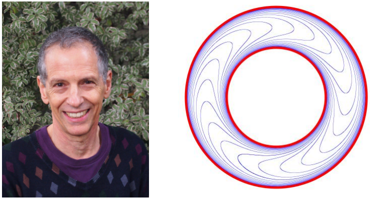

Figure: Alan Weinstein and Reeb foliation

- Poisson cohomology. Some computations.
- Compatible Poisson structures and commuting first integrals.

Schouten Bracket of vector fields in local coordinates

- Case of vector fields,
  $A = \sum_i a_i \frac{\partial}{\partial x_i}$ and $A = \sum_i b_i \frac{\partial}{\partial x_i}$. Then

$$[A, B] = \sum_i a_i (\sum_j \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j}) - \sum_i b_i (\sum_j \frac{\partial a_j}{\partial x_i} \frac{\partial}{\partial x_j})$$

- Re-denoting $\frac{\partial}{\partial x_i}$ as ζ_i **("odd coordinates ")**.
  Then $A = \sum_i a_i \zeta_i$ and $B = \sum_i b_i \zeta_i$ and $\zeta_i \zeta_j = -\zeta_j \zeta_i$ Now we can reinterpret the bracket as,

$$[A, B] = \sum_i \frac{\partial A}{\partial \zeta_i} \frac{\partial B}{\partial x_i} - \sum_i \frac{\partial B}{\partial \zeta_i} \frac{\partial A}{\partial x_i}$$

Schouten Bracket of multivector fields in local coordinates

We reproduce the same scheme for the case of multivector fields.

$$[A,B]=\sum_i \frac{\partial A}{\partial \zeta_i}\frac{\partial B}{\partial x_i}-(-1)^{(a-1)(b-1)}\sum_i \frac{\partial B}{\partial \zeta_i}\frac{\partial A}{\partial x_i}$$

is a $(a+b-1)$-vector field.
where

$$A=\sum_{i_1<\cdots<i_a} A_{i_1,\dots,i_a}\frac{\partial}{\partial x_{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_a}}=\sum_{i_1<\cdots<i_a} A_{i_1,\dots,i_a}\zeta_{i_1}\dots\zeta_{i_a}$$

and

$$B=\sum_{i_1<\cdots<i_b} B_{i_1,\dots,i_b}\frac{\partial}{\partial x_{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_b}}=\sum_{i_1<\cdots<i_b} B_{i_1,\dots,i_b}\zeta_{i_1}\dots\zeta_{i_b}$$

with $\frac{\partial(\zeta_{i_1}\dots\zeta_{i_p})}{\partial \zeta_{i_k}}:=(-1)^{(p-k)}\eta_{i_1}\dots\widehat{\eta}_{i_k}\eta_{i_{p-1}}$

Theorem (Schouten-Nijenhuis)

The bracket defined by this formula satisfies,

Graded anti-commutativity $[A,B]=-(-1)^{(a-1)(b-1)}[B,A]$.

Graded Leibniz rule

$$[A, B\wedge C]=[A,B]\wedge C+(-1)^{(a-1)b}B\wedge[A,C]$$

Graded Jacobi identity

$$(-1)^{(a-1)(c-1)}[A,[B,C]]+(-1)^{(b-1)(a-1)}[B,[C,A]]+(-1)^{(c-1)(b-1)}[C,[A,B]]=0$$

If X is a vector field then, $[X,B]=L_XB$.

Poisson cohomology computation kit

- Space of cochains $\mathfrak{X}^m(M)$.
- Differential $d_\Pi(A) := [\Pi, A]$.
- Poisson cohomology

$$H^k_\Pi(M) := \frac{\ker d_\Pi : \mathfrak{X}^k(M) \longrightarrow \mathfrak{X}^{k+1}(M)}{\mathrm{Im} d_\Pi : \mathfrak{X}^{k-1}(M) \longrightarrow \mathfrak{X}^k(M)}$$

- Computation is difficult. It can be infinite-dimensional. Tools: Mayer-Vietoris, spectral sequences.
- Particular cases: (M, Π) symplectic $H^k_\Pi(M) \cong H^k_{DR}(M)$.
- (M, Π) b-Poisson, $H^k_\Pi(M) \cong H^k_{DR}(M) \oplus H^{k-1}_{DR}(Z)$.

Poisson cohomology computation kit

- Hamiltonian vector fields $X_f = -[\Pi, f]$ (1-coboundary).
- Poisson vector fields $[\Pi, X] = -L_X \Pi = 0$ (1-cocycle).
- Poisson structures $[\Pi, \Pi] = 0$ (2-cocycle).
- Compatible Poisson structures $[\Pi_1, \Pi_2] = 0$ (2-cocycle).
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$$H^1_\Pi = \frac{\text{Poisson vector fields}}{\text{Hamiltonian vector fields}}.$$

Example 5: Cauchy-Riemann equations and Hamilton's equations

- Take a holomorphic function on $F : \mathbb{C}^2 \longrightarrow \mathbb{C}$ decompose it as $F = G + iH$ with $G, H : \mathbb{R}^4 \longrightarrow \mathbb{R}$.

  Cauchy-Riemann equations for F in coordinates $z_j = x_j + iy_j$, $j = 1, 2$

$$\frac{\partial G}{\partial x_i} = \frac{\partial H}{\partial y_i}, \quad \frac{\partial G}{\partial y_i} = -\frac{\partial H}{\partial x_i}$$

- Reinterpret these equations as the equality

$$\{G, \cdot\}_0 = \{H, \cdot\}_1 \quad \{H, \cdot\}_0 = -\{G, \cdot\}_1$$

  with $\{\cdot, \cdot\}_j$ the Poisson brackets associated to the real and imaginary part of the symplectic form $\omega = dz_1 \wedge dz_2$ $(\omega = \omega_0 + i\omega_1)$.
- Check $\{G, H\}_0 = 0$ and $\{H, G\}_1 = 0$ (integrable system).

Example 2: Determinants in $\mathbb{R}^3$

- Dynamics: Given two functions $H, K \in \mathcal{C}^\infty(\mathbb{R}^3)$. Consider the system of differential equations:

$$(\dot{x}, \dot{y}, \dot{z}) = dH \wedge dK \qquad (1)$$

  H and K are constants of motion (the flow lies on $H = cte.$ and $K = cte.$)

- Geometry: Consider the brackets,

$$\{f, g\}_H := \det(df, dg, dH) \quad \{f, g\}_K := \det(df, dg, dK)$$

  They are antisymmetric and satisfy Jacobi,
  $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$
  The flow of the vector field

$$\{K, \cdot\}_H := \det(dK, \cdot, dH)$$

  and $\{-H, \cdot\}_K$ is given by the differential equation (1) and

$$\{H, K\}_H = 0, \quad \{H, K\}_K = 0$$