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On the symplectic classification of singular Lagrangian foliations

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Abstract

This paper is concerned with the classification problem of the symplectic structures on M^4 making a given foliation with singularities into a Lagrangian one in a neighbourhood of a singular leaf. Particularly, we are interested in determining the conditions that a given foliation has to satisfy to ensure the symplectic equivalence between this foliation and the simpler "canonical" foliation with the Darboux symplectic form. In order to do that, we define the concept of symplectic linearizable foliation in M^4 in a neighbourhood of a singular leaf, and we prove that a foliation is symplectically linearizable if and only if it is given by a transversally non-degenerate orientable integral pair $(\chi, \mathcal{A})$.

Key words: symplectic manifold, singular Lagrangian foliation, symplectically completely integrable system, S^1-Hamiltonian action
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1 Introduction

Let (M^4, w) be a symplectic manifold and let $\mathcal{F}$ be a generalized foliation on M^4 whose regular leaves are 2-dimensional. Assume further that $\mathcal{F}$ comes from a symplectically completely integrable system on M^4 and let $L = S^1$ be a singular leaf of $\mathcal{F}$. The goal of this paper is to classify the symplectic structures in a neighbourhood of L in M^4 making $\mathcal{F}$ into a foliation whose regular leaves are Lagrangian. In fact, we are going to characterize which foliations coming from a singular symplectically completely integrable foliations are symplectically linearizable and we will give an example of a singular symplectically completely integrable foliation for which the uniqueness of the germ cannot be obtained. The problem of classification will be given in terms of symplectic germs of singular Lagrangian foliation. First, let us review some well-known facts concerning symplectically completely integrable systems:

1.1 Symplectically completely integrable systems: some facts

Let (M^{2n}, w, H) be a Hamiltonian system and let $\{,\}$ be the Poisson bracket associated to it. Recall that the system is symplectically completely integrable if there exist $f_1, \ldots, f_n$, n first integrals of the system in involution (that is to say, $\{f_i, f_j\} = 0$, $\forall i, j$).

From now on, a integral pair $(\chi, \mathcal{A})$ will stand for the pair made up of χ, an algebra of Hamiltonian vector fields, and of $\mathcal{A}$, an abelian algebra of first-integrals.

It is known, thanks to Arnold–Liouville's theorem [1], that a regular symplectically completely integrable system with all leaves compact gives a foliation by tori on M^{2n}. Further, there is a foliation preserving symplectomorphism taking the former symplectic form into the "Darboux" one: $\sum_i dx_i \wedge dy_i$. Those x_i, y_i, allowing the explicit integration of the Hamiltonian system, have become known in the literature as action-angle coordinates.

The existence of singular integrals in $(\chi, \mathcal{A})$ is a quite common situation in many Hamiltonian systems bequeathed from classical mechanics (e.g the 3-body problem). Surprisingly, if the integrals become dependent in a "non-degenerate" way (realized in the condition on the Hamiltonian pair $(\chi, \mathcal{A})$ of being transversally elliptic in the sense of [6]) the existence of "singular" action-angle variables can be attained as well. That was proved firstly by Eliasson [7] and then by Dufour and Molino [6] using different tecniques.

1.2 Posing our problem

Notice that a symplectically completely integrable system gives a Lagrangian foliation in M^{2n}. And that every singularity of this Lagrangian foliation (corresponding to a singularity of a function in $\mathcal{A}$), gives an isotropic leaf of that foliation. At this point, we can think of finding the symplectic structures making a given foliation with singularities into a singular Lagrangian one in a neighbourhood of a singular compact leaf. So, in fact, we are looking for a symplectic neighboorhood classification of that isotropic leaf L_0 [9] done in a foliation-preserving way.

The organization of this paper is as follows. In section 2 we use Arnold–Liouville to solve the analogue of the classification problem in a neighbourhood of a regular leaf. In section 3 we deal with the singular case.

2 The regular case

The aim of this section is to prove the uniqueness of the symplectic germ of a singular Lagrangian foliation in a neighbourhood of a regular compact leaf. First we need to recall:

Theorem 2.1 *(***Arnold–Liouville** *[1])*
Let (M^{2n}, w, H) be a Hamiltonian system and let $\{\ \}$ be the Poisson bracket associated to w. Let $f_1, f_2, \dots, f_n$ be n regular functions such that $\{f_i, f_j\} = 0$ $\forall i, j$, $\Gamma_c = \{p \in M^{2n} \mid f_1(p) = c_1, \dots, f_n(p) = c_n\}$. Assume that Γ_c is compact. Then there exist a neighbourhood U of Γ_c in M^{2n} and a coordinate system $(q_1, \dots q_n, \theta_1 \dots, \theta_n)$ such that:

1. *The new coordinates give a diffeomorphism $\rho : U \to W \times \mathbb{T}^n$ where W is an open set in $\mathbb{R}^n$ such that $\rho(\Gamma_c) = \{0\} \times \mathbb{T}^n$.*
2. *$w = \rho^* w_{can}$, where $w_{can} = \sum_1^n dq_j \wedge d\theta_j$.*
3. *$H \circ \rho^{-1}$ depends on $q_1, \dots, q_n$.*

Now let $\mathcal{F}$ be the foliation defined by the first-integrals f_i. Let w be a symplectic form such that $\mathcal{F}$ is Lagrangian. The Lagrangian condition implies that the w-Hamiltonian vector fields of the first integrals are always tangent to $\mathcal{F}$. So $\mathcal{F}$ is generated by w-Hamiltonian vector fields and then $\mathcal{F}$ is symplectically completely integrable for w. Now we can apply Arnold–Liouville to obtain a foliation preserving symplectomorphism ρ taking w to w_{can}. So far, we have proved the following:

Proposition 2.1 *Let $\mathcal{F}$ be a foliation given by a symplectic completely integrable system on (M^{2n}, w_0), let L be a compact regular leaf of $\mathcal{F}$ and let w be another symplectic structure on M^{2n} such that $\mathcal{F}$ is Lagrangian for w. Then there exist a neighbourhood $U(L)$ and a diffeomorphism $\phi : U(L) \longrightarrow \phi(U(L))$ such that:*

1. *$\rho^*(w) = w_0$.*
2. *ρ preserves $\mathcal{F}$ and leaves L fixed.*

3 The singular case

Let $\mathcal{F}$ be a foliation with singularities given by a symplectically completely integrable system such that $L = S^1$ is a singular leaf. Assume that w_1 and w_2

are symplectic forms on M^4 such that $\mathcal{F}$ becomes generically Lagrangian for both. Inspired by the regular case, we can pose the following question:

Does there exist a foliation preserving symplectomorphism taking w_1 to w_2?

The answer to that question is negative in general. For instance: Take $M^4 = S^1 \times \mathbb{R}^3$, $L_0 = \{(\theta, 0, 0, 0)\}$, $\mathcal{F} =< \frac{\partial}{\partial \theta}, x\frac{\partial}{\partial y} >$ This foliation is Lagrangian for both $\omega_1 = dx \wedge dy + dp \wedge d\theta$ and $\omega_2 = dx \wedge d\theta + dy \wedge dp$. But there cannot exist a foliation preserving symplectomorphism taking w_1 to w_2. Notice that the w_2-Hamiltonian vector fields corresponding to the first integrals p, x^2 are not tangent to the foliation at the leaf L_0.

With this example at hand one can foresee that in the singular case the non-degeneracy of the first integrals plays a central role to ensure the uniqueness. Furthermore, the Morse-Bott theory applied to the first integrals gives us some hints about the hypotheses that have to be set to guarantee the differentiable equivalence with the linear model. Before stating our results at the singular level we need some terminology:

3.1 Some definitions

Definition 3.1 *A* **symplectic germ of singular Lagrangian foliation** *is a triple $(M, w, \mathcal{F})_L$ where w is an L-symplectic germ defined in a neighbourhood of L in M making $\mathcal{F}$ into a generalized Lagrangian foliation and such that L is a singular leaf of this foliation.*

Definition 3.2 *Two triples $(M_1, w_1, \mathcal{F}_1)_L$ and $(M_2, w_2, \mathcal{F}_2)_L$ are equivalent if there exist neighbourhoods $U_1(L)$ and $U_2(L)$ of L in M_1 and M_2 respectively and a diffeomorphism $\phi: U_1(L) \to U_2(L)$ such that:*

1. *$\phi^* w_2 = w_1$.*
2. *ϕ sends $\mathcal{F}_1$ to $\mathcal{F}_2$ and $\phi(L) = L$.*

Definition 3.3 *A symplectic germ of singular Lagrangian foliation $(M^4, w, \mathcal{F})_L$ is symplectically linearizable if it is equivalent to the symplectic germ of singular Lagrangian foliation $(M^4, w_0, \mathcal{F}_0)_{L_0}$, where $w_0 = dx \wedge dy + dp \wedge d\theta$, $\mathcal{F}_0$ is given by $Y_1 = \frac{\partial}{\partial \theta}, Y_2 = y\frac{\partial}{\partial x} - \epsilon x \frac{\partial}{\partial y}$ and $L_0 = \{(\theta, 0, 0, 0)\}$, $\epsilon \in \{-1, 1\}$.*

Fix $p \in M^4$. From now on, let us assume the following on the integral pair at the point p: $\chi =< X_1, X_2 >$, whith $[X_1, X_2] = 0$, $X_1(p) \neq 0$, $X_2(p) = 0$; and $\mathcal{A} =< f_1, f_2 >$, with $d_p f_1 \neq 0$, $d_p f_2 = 0$. We denote by $X^T_{2_p}$ the transversal linearization of X at p.

Definition 3.4 *The integral pair $(\chi, \mathcal{A})$ is transversally non-degenerate at $p \in M$ if $\dim < X_{2_p}^T >= 1$ and $\dim < f_{2_p}^T >= 1$ and there exist coordinates (x, y) on $N_p^T = \mathrm{Ker} d_p f_1 / X_1(p)$ such that: $X_{2_p}^T = y\frac{\partial}{\partial x} - \epsilon x \frac{\partial}{\partial y}$, $f_{2_p}^T = x^2 + \epsilon y^2$, $\epsilon \in \{-1, 1\}$ ($\epsilon = 1$ in the elliptic case and $\epsilon = -1$ in the hyperbolic case).*

Definition 3.5 *A transversally non-degenerate integral pair $(\chi, \mathcal{A})$ is orientable either if $\epsilon = 1$ or if $\epsilon = -1$ and the foliation has no non-orientable separatrix diagrams in the sense of [2].*

3.2 The results

Theorem 3.1 Classification Theorem I [4]
Let $M^4 = S^1 \times \mathbb{R} \times \mathbb{R}^2$ and let $\mathcal{F}$ be the generalized foliation given by $\left\langle \frac{\partial}{\partial \theta}, y\frac{\partial}{\partial x} - \epsilon x \frac{\partial}{\partial y} \right\rangle$, $\epsilon \in \{-1, 1\}$. Let ω be a symplectic form in M^4 such that the regular leaves of $\mathcal{F}$ are Lagrangian submanifolds. Then there exist a neighbourhood U of the singular leaf $L_0 = \{(\theta, 0, 0, 0)\}$ and a foliation preserving diffeomorphism $f : U \to f(U)$ such that $f^*\omega = \overline{\omega}$, being $\overline{\omega} = dx \wedge dy + dp \wedge d\theta$.

Theorem 3.2 Classification Theorem II [5]
Let $\mathcal{F}$ be the generalized foliation in M^4 given by an orientable transversally non-degenerate integral pair $(\chi, \mathcal{A})$ and let $L = S^1$ be a singular leaf of $\mathcal{F}$ then any symplectic germ of singular Lagrangian foliation $(M^4, w, \mathcal{F})_L$ is symplectically linearizable.

The details of the proof of the above-mentioned theorems can be found in [4], [5]. For the sake of completeness, a sketch of the proof is provided here.
Sketch of the proof

Step 1. Proving Classification theorem I: We “deform” the symplectic structure preserving the foliation. The deformation is achieved defining a path of symplectic forms w_t “à la *Moser*”. A key point in the proof is the search of a foliation-preserving Hamiltonian action of an S^1. In the hyperbolic case the tecniques of the Morse Isochore Lemma [3] are used.

Step 2. Proving Classification Theorem II: We perform a *differential linearization* of the foliation in a neighbourhood of a singular leaf. This is done by extending the tricks of *“Morse theory”* to a tubular neighbourhood of L. Now we can apply the first theorem and we are done.

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