

# ON THE ALEXANDROFF-BAKELMAN-PUCCI ESTIMATE FOR SOME ELLIPTIC AND PARABOLIC NONLINEAR OPERATORS

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ABSTRACT. In this work we prove the Alexandrov-Bakelman-Pucci estimate for (possibly degenerate) nonlinear elliptic and parabolic equations of the form

$$-\operatorname{div} (F (\nabla u(x))) = f(x) \quad \text{in } \Omega \subset \mathbb{R}^n$$

and

$$u_t(x, t) - \operatorname{div} (F (\nabla u(x, t))) = f(x, t) \quad \text{in } Q \subset \mathbb{R}^{n+1}$$

for  $F$  a  $\mathcal{C}^1$  monotone field under some suitable conditions. Examples of application such as the  $p$ -Laplacian or the Mean Curvature Flow are considered, as well as extensions of the general results to equations that are not in divergence form, such as the  $m$ -curvature flow.

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*Key words and phrases.* Non linear elliptic and parabolic equations, a priori estimates, maximum principle.

2000 *Mathematics Subject Classification:* 35J15, 35J60, 35J62, 35J92, 35J93, 35J96, 35K10, 35K59, 35K92, 35K93, 35K96.

Work partially supported by project MTM2007-65018, MICINN, Spain.

## 1. INTRODUCTION

In the decade of 1960, Aleksandrov [1], Bakel'man [2] and Pucci [38], independently established a Maximum Principle for linear elliptic equations in non-divergence form with bounded measurable coefficients, which asserts (see [26, Section 9.1]) that, whenever  $u \in \mathcal{C}(\overline{\Omega}) \cap W_{\text{loc}}^{2,n}(\Omega)$  satisfies

$$(1.1) \quad -\text{trace}(A(x)D^2u(x)) + \langle b(x), \nabla u(x) \rangle + c(x)u(x) \leq f(x) \quad \text{in } \Omega$$

for  $A(x)$  uniformly elliptic,  $c(x) \geq 0$  and  $f \in L^n(\Omega)$ , then,

$$(1.2) \quad \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \cdot \text{diam}(\Omega) \cdot \|f^+\|_{L^n(\Gamma^+(u))},$$

where  $\Omega \subset \mathbb{R}^n$  is a bounded domain,  $\Gamma^+(u) \subset \Omega$  is the set of points where  $u$  is concave and non-negative (called the upper contact set of  $u$ ), and  $C$  depends only on the dimension  $n$ , the ellipticity constants and the bounds on the coefficients. Analogous estimates hold for supersolutions.

The fact that the Aleksandroff-Bakel'man-Pucci estimate (in short ABP estimate) for linear operators mainly depends on the ellipticity constants of the matrix of coefficients and the geometry of  $\Omega$ , allowed the extension for uniformly elliptic fully nonlinear equations (see Caffarelli [7] and also Caffarelli and Cabré [6] and the references therein). The proof is based on the use of Pucci's extremal operators, that bound the class of uniformly elliptic equations, allowing to replace any equation in the class by two one-sided inequalities involving the Pucci operators. Then, the particular structure of Pucci's operators provides a natural link to linear equations in trace form, permitting to suitably adapt the classical ABP argument to the new framework.

The notion of solution which best suits elliptic equations in this generality is the notion of viscosity solution. However, this notion of solutions requires some continuity of the coefficients. In order to allow merely bounded measurable coefficients, Caffarelli, Crandall, Kocan and Świąch [8] introduce the notion of  $L^p$ -viscosity solution and prove the ABP estimate in this new setting via approximation.

All these results have a parabolic counterpart. Krylov [31] and then Tso [45] proved an ABP estimate for linear uniformly parabolic equations in non-divergence form with bounded measurable coefficients. To be more precise, given  $u \in \mathcal{C}(\overline{Q}) \cap W_{n+1}^{2,1}(Q)$  such that

$$\begin{aligned} u_t(x, t) - \text{trace}(A(x, t)D^2u(x, t)) \\ + \langle b(x, t), \nabla u(x, t) \rangle + c(x, t)u(x, t) \leq f(x, t) \quad \text{in } Q = \Omega \times (0, T), \end{aligned}$$

with  $\Omega \subset \mathbb{R}^n$  a bounded domain, the coefficients bounded and measurable,  $A(x, t)$  uniformly elliptic,  $c(x, t) \geq 0$  and  $f \in L^{n+1}(Q)$ , then we have,

$$\sup_Q u \leq \sup_{\partial_p Q} u^+ + C \cdot \text{diam}(\Omega)^{\frac{n}{n+1}} \cdot \|f^+\|_{L^{n+1}(\Gamma_p^+(u))},$$

where  $\partial_p Q$  denotes the parabolic boundary of  $Q$ ,  $\Gamma_p^+(u) \subset Q$  is the set of points where  $u(x, t)$  is concave in  $x$ , non-decreasing in  $t$  and non-negative (the parabolic upper contact set of  $u$ ), and  $C$  depends on  $n, d$ , the ellipticity constants and the bounds on the coefficients. One has analogous estimates for supersolutions.

Then, the theory for fully nonlinear uniformly parabolic equations was developed in a series of papers by Wang [46] making use of the notion of viscosity solution. As in the elliptic case, this requires continuity from the coefficients. The  $L^p$ -viscosity theory for parabolic equations with bounded measurable coefficients has been developed by Crandall, Fok, Kocan and Świąch in [14, 15].

It is worth mentioning that Cabré [5] proved an improvement of the ABP estimate, both in the elliptic and parabolic cases, replacing the quantity  $\text{diam}(\Omega)$  present in the equation by a more precise geometric quantity, allowing domains not necessarily bounded.

Other different improvements of the ABP inequality for the linear problem, regarding the integrability of the right-hand side and the optimality of the constants involved, have been proved by Fabes and Stroock [24], Escobar [21, 22] and Kuo and Trudinger [34].

We would also like to point out that Dávila, Felmer and Quaas [17, 18] and Imbert [28] have recently obtained ABP estimates for operators of  $p$ -Laplacian type in the elliptic case. Our method yields a different proof that can be easily extended to the parabolic case.

The ABP estimate has proved to be a valuable tool in the proof of the Krylov-Safonov Harnack inequality for elliptic and parabolic linear equations in non-divergence form with bounded measurable coefficients [32, 33], and as the starting point of the regularity theory for uniformly elliptic and parabolic fully nonlinear equations (see [6, 46] and the references therein). Moreover, in a very simple way, it implies a Maximum Principle in domains of small measure for uniformly elliptic fully nonlinear equations, a useful tool when proving symmetry properties (it is interesting to emphasize that the improvement in [5] implies the Maximum Principle in narrow domains, such as thin infinite strips).

Although originally applied to linear equations, the proof of the ABP estimate involves genuinely nonlinear arguments that, as we will show, can be successfully applied to a wide variety of nonlinear equations.

The idea in the original argument (elliptic case) is to estimate the maximum of  $u$ , a solution to (1.1), by the measure of the image of the gradient mapping  $\nabla u$ . In this estimate, the gradient of  $u$  is interpreted as the Gauss (normal) mapping associated to the graph of  $u$ , and the argument is purely geometrical and does not involve the equation at all. The equation is used to bound the measure of the image of  $\nabla u$ .

It is our aim in this work to extend these ideas and to prove ABP-type estimates for more general nonlinear equations, modeled on the following

divergence-type equations

$$(1.3) \quad -\operatorname{div}(F(\nabla u(x))) \leq f(x) \quad \text{in } \Omega,$$

in the elliptic case, and

$$(1.4) \quad u_t(x, t) - \operatorname{div}(F(\nabla u(x, t))) \leq f(x, t) \quad \text{in } Q,$$

in the parabolic case. Some examples of fields under our scope are the “ $p$ -Laplacian field”  $F(\xi) = |\xi|^{p-2}\xi$ , for  $p \in (1, \infty)$ , the fields  $F(\xi) = e^{|\xi|^2}\xi$  and  $F(\xi) = (e^{|\xi|^2} - 1)\xi$  (a degenerate version of the latter) and the “mean curvature field”  $F(\xi) = (1 + |\xi|^2)^{-1/2}\xi$ .

The main idea in this paper is to estimate the measure of the image of  $F \circ \nabla u$  instead of just  $\nabla u$ , taking into account the deformation induced in the gradient field by the mapping  $F$ . We will show that, under some structure conditions on  $F$ , the geometrical argument in the ABP method still leads to a bound of the maximum of  $u$ . Again, as the mapping  $F$  is intrinsic to the problem, the equation bounds the measure of the image of  $F(\nabla u)$ .

In the parabolic case, the proof of the ABP estimate for (1.4) is based on the extension of the spatial intrinsic mapping  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  to an application from  $\mathbb{R}^{n+1}$  to  $\mathbb{R}^{n+1}$  by adding the Legendre transform of  $u$  as the last coordinate. Since [45], it is well-known that the Legendre transform is related to the proof of the parabolic ABP in the linear and uniformly parabolic cases [14, 15, 46]. The fact that the Legendre transform appears also in our framework seems to reflect that the equation is linear in  $u_t$  and the nonlinearity of the equation is mainly spatial.

A usual approach used to handle some particular quasilinear elliptic problems and to include lower-order terms (see [3, 8, 17], and also [26, Sections 9.1 and 10.2]) consists in appropriately weighting the measure of the image of  $\nabla u$  in order to cancel the effect of the extra quasilinear or first-order terms.

The main advantage of our approach is that it allows to prove elliptic and parabolic estimates in an efficient and coherent way. Nevertheless, notice that it is still possible to use weights in combination with the nonlinear mapping  $F$ , that is, to replace  $|F(\nabla u(\Omega))|$  by

$$\int_{F(\nabla u(\Omega))} g(\xi) d\xi,$$

with  $g(\xi)$  a positive weight. An appropriate election of the weight  $g$  allows to include extra terms, such as lower order terms, as necessary. For instance, we make use of this idea in Section 5 to treat the Mean Curvature Flow equation for graphs,

$$u_t(x, t) - \sqrt{1 + |\nabla u(x, t)|^2} \operatorname{div} \left( \frac{\nabla u(x, t)}{\sqrt{1 + |\nabla u(x, t)|^2}} \right) \leq f(x, t),$$

where the normal velocity induces an extra term accompanying the divergence. Furthermore, this result makes explicit the modularity of our approach; every part of the equation can be handled separately by means of a suitable combination of different techniques.

Finally, it is important to mention that our method allows to study other kind of related equations even in non-divergence form, as exemplified in Sections 4 and 6. The idea is that, if the operator belongs to a class of equations bounded by some extremal operators which contains some divergence-form representative, then, the ABP argument can be carried out for the extremal equations using the nonlinear mapping intrinsic to the divergence-form representative.

The paper is organized as follows. In Section 2, we present the ideas proving a general result for general elliptic and parabolic equations of the form (1.3) and (1.4) with  $F$  a  $\mathcal{C}^1$  monotone field satisfying certain conditions. In Section 3 and 4 we prove the ABP estimate for the  $p$ -Laplacian and Fully nonlinear with a  $p$ -Laplacian homogeneity. Then, in Section 5 we prove ABP-type estimates for the Prescribed Mean Curvature problem and the Mean Curvature Flow. Finally, in Section 6 we extend this results to equations given by a general symmetric polynomial such as the problem of the prescribed  $m$ -curvatures and its parabolic counterparts such as the  $m$ -curvature flow.

## 2. THE ABP ESTIMATE FOR A GENERAL CLASS OF NONLINEAR ELLIPTIC AND PARABOLIC EQUATIONS IN DIVERGENCE FORM

In this section, we are interested in proving the ABP estimate for problems (1.3) and (1.4), with  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfying the following assumptions:

- (F1)  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a field of class  $\mathcal{C}^1$ , with  $\text{curl}(F) = 0$ .
- (F2)  $F$  is monotone (i.e.  $\langle F(\xi_1) - F(\xi_2), \xi_1 - \xi_2 \rangle \geq 0$  for every  $\xi_1, \xi_2 \in \mathbb{R}^n$ ).
- (F3)  $|F(B_r(0))| \geq C \cdot |B_1(0)| \cdot r^{\alpha n}$  for some constants  $\alpha, C > 0$  independent of  $r > 0$ .

*Remark 2.1.* Notice that, as a consequence of (F1) and (F2), we have

$$\begin{aligned} \langle DF(\xi_2)(\xi_1 - \xi_2), (\xi_1 - \xi_2) \rangle &= \langle F(\xi_1) - F(\xi_2), (\xi_1 - \xi_2) \rangle + o(|\xi_1 - \xi_2|^2) \\ &\geq o(|\xi_1 - \xi_2|^2), \end{aligned}$$

and hence  $DF(\xi_2) \geq 0$ . Moreover, the condition  $\text{curl}(F) = 0$  implies that the matrix  $DF$  is symmetric.

An important situation when hypothesis (F3) is satisfied is when the mapping  $F$  is homogeneous of degree  $\alpha$ , as in the case of the  $p$ -Laplacian operator  $F(\xi) = |\xi|^{p-2}\xi$  which is homogeneous of degree  $p-1$ . Notice that the  $p$ -Laplacian only fulfills hypothesis (F1) when  $p \geq 2$  as for  $1 < p < 2$ ,  $F$  is not differentiable at 0 and the argument requires some adaptations, see Section 3.

Nevertheless, there are many examples of non-homogeneous mappings  $F$  for which (F3) holds, such as the field  $F(\xi) = e^{|\xi|^2} \xi$  which satisfies (F3) with  $C, \alpha = 1$ . A degenerate version of the latter is  $F(\xi) = (e^{|\xi|^2} - 1) \xi$ , for which (F3) holds with  $C = 1$  and  $\alpha = 3$ .

*Remark 2.2.* Hypothesis (F3) could be replaced in the sequel by the following more general hypothesis:

(F3')  $|F(B_r(0))| \geq \psi(r)$  for some  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  continuous and invertible.

Then, the arguments in this section yield ABP-type estimates of the form,

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + d \cdot \psi^{-1}(\|f^+\|_{L^n(\Gamma^+(u))}^n)$$

with analogous results in the parabolic case. For the sake of clarity, we have chosen to state the general results under hypothesis (F3), see Sections 5 and 6 for examples related to (F3)'.

**2.1. The ABP estimate for nonlinear elliptic equations in divergence form.** First, we address the elliptic equation

$$(2.1) \quad -\operatorname{div}(F(\nabla u)) \leq f(x) \quad \text{in } \Omega.$$

which we will also use in its trace form, namely,

$$(2.2) \quad -\operatorname{trace}(D_{\xi} F(\nabla u(x)) D^2 u(x)) \leq f(x).$$

Let us recall here the definition of viscosity solution, to be used in the sequel.

**Definition 2.3.** Let  $\mathcal{F} : \mathbb{R}^n \times S^n \rightarrow \mathbb{R}$ . A function  $u \in \mathcal{C}(\Omega)$  is a *viscosity subsolution* (resp. *supersolution*) of

$$(2.3) \quad \mathcal{F}(\nabla u, D^2 u) = f(x)$$

in  $\Omega$  if for all  $\hat{x} \in \Omega$  and  $\varphi \in \mathcal{C}^2(\Omega)$  such that  $u - \varphi$  attains a local maximum (minimum) at  $\hat{x}$ , one has

$$\mathcal{F}(\nabla \varphi(\hat{x}), D^2 \varphi(\hat{x})) \leq f(\hat{x}) \quad (\text{resp. } \geq).$$

We say that  $u \in \mathcal{C}(\Omega)$  is a *viscosity solution* of (3.1) in  $\Omega$  if it is both a viscosity subsolution and supersolution.

We also have to recall some necessary notions.

**Definition 2.4.** Let  $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ . The *upper contact set* of  $u$  is defined as,

$$\Gamma^+(u) := \left\{ y \in \Omega : \exists \xi \in \mathbb{R}^n \text{ such that } u(x) \leq u(y) + \langle \xi, x - y \rangle \ \forall x \in \Omega \right\}.$$

For convenience, we also define the following subset of  $\Gamma^+(u)$ ,

$$\Gamma_r^+(u) := \left\{ y \in \Omega : \exists \xi \in B_r(0) \text{ such that } u(x) \leq u(y) + \langle \xi, x - y \rangle \ \forall x \in \Omega \right\}.$$

In the following result we prove the ABP estimate for equation (2.1), one of the main results in this section.

**Theorem 2.5.** *Let  $\Omega \subset \mathbb{R}^n$  a bounded domain and  $f \in L^n(\Omega) \cap \mathcal{C}(\Omega)$ . Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfy assumptions (F1), (F2) and (F3). Consider  $u \in \mathcal{C}(\overline{\Omega})$  which satisfies*

$$-\operatorname{div}(F(\nabla u)) \leq f(x) \quad \text{in } \Omega,$$

*in the viscosity sense. Then, the ABP estimate holds, that is,*

$$(2.4) \quad \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + Cd \|f^+\|_{L^n(\Gamma^+(u))}^{\frac{1}{\alpha}}$$

*where  $d = \operatorname{diam}(\Omega)$  and  $C$  depends only on  $n$  and the constants  $C, \alpha$  in hypothesis (F3).*

*Analogously, whenever  $u \in \mathcal{C}(\overline{\Omega})$  is a viscosity solution of*

$$-\operatorname{div}(F(\nabla u)) \geq f(x) \quad \text{in } \Omega,$$

*we have the following estimate,*

$$(2.5) \quad \sup_{\Omega} u^- \leq \sup_{\partial\Omega} u^- + Cd \|f^-\|_{L^n(\Gamma^+(-u))}^{\frac{1}{\alpha}}$$

*where  $d, C$  are constants defined as before.*

*Proof.* We prove the first inequality, since the second one is similar. As usual, see [8], we start assuming  $u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$  and then we remove the assumption by regularization. Let  $x_0 \in \Omega$  such that  $\sup_{\Omega} u = u(x_0)$ , and assume  $u(x_0) > 0$ , since otherwise there is nothing to prove. Observe that

$$R_0 = \frac{\sup_{\Omega} u - \sup_{\partial\Omega} u}{d}$$

is the maximal slope of a plane that touches  $u$  at an interior point of  $\Omega$ . We can fix  $a > \sup_{\partial\Omega} u^+$ , such that,

$$r_0(u) = \frac{\sup_{\Omega} u - a}{d}$$

is positive, and then fix  $r < r_0(u)$ . We claim that then we can fix a compact set  $G \subset \Omega$ , such that,

$$(2.6) \quad \Gamma_r^+(u) \subset\subset G \subset\subset \Omega,$$

for  $\Gamma_r^+(u)$  defined as in Definition 2.4. In order to prove (2.6), notice that  $\hat{x} \in \Gamma_r^+(u)$  implies that there exists  $\xi \in B_r(0)$  such that  $u(x) \leq u(\hat{x}) + \langle \xi, x - \hat{x} \rangle$  for all  $x \in \Omega$ . Consequently,

$$\sup_{\Omega} u(x) - u(\hat{x}) \leq |\xi| d < rd.$$

Then,

$$u(\hat{x}) > a + (r_0(u) - r)d > \sup_{\partial\Omega} u^+ + (r_0(u) - r)d.$$

so (2.6) holds.

We now claim that,  $B_r(0) \subset \nabla u(\Gamma_r^+(u))$ . Indeed, take  $\xi \in B_r(0)$ , and consider the hyperplane  $l_{\xi}(x) = h + \langle \xi, x \rangle$  with the Legendre transform of  $x$ , that is,  $h = \sup_{y \in \Omega} \{u(y) - \langle \xi, y \rangle\}$ . Then,  $u(x) \leq l_{\xi}(x)$  in  $\Omega$  and

$u(z) = l_\xi(z)$  for some  $z \in \overline{\Omega}$ . We aim to prove that  $z \in \Omega$  so, suppose to the contrary that  $z \in \partial\Omega$ . We have that,

$$\begin{aligned} \sup_{\Omega} u &= u(x_0) \leq l_\xi(x_0) = l_\xi(z) + \langle \xi, x_0 - z \rangle \\ &= u(z) + \langle \xi, x_0 - z \rangle < a + rd < a + r_0(u) d = \sup_{\Omega} u, \end{aligned}$$

a contradiction, so the claim is proved.

Clearly,  $F(B_r(0)) \subset F(\nabla u(\Gamma_r^+(u)))$ ; hence, by hypothesis (F3), we get,

$$(2.7) \quad C|B_1(0)|r^{\alpha n} \leq |F(B_r(0))| \leq |F(\nabla u(\Gamma_r^+(u)))|.$$

We estimate the rightmost term in this inequality by means of the Area Formula. It states that if  $\Psi : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a Lipschitz map and  $h : A \rightarrow \mathbb{R}$  is a integrable function, then

$$\int_{\mathbb{R}^n} \left( \sum_{x \in A, z = \Psi(x)} h(x) \right) dz = \int_A |J\Psi(x)| h(x) dx,$$

where, in this case,  $J\Psi(x) = \det D\Psi$ . We now apply the formula with  $A = \Gamma_r^+(u)$ ,  $\Psi = F(\nabla u)$  and  $h = 1$ . We obtain,

$$|F(\nabla u(\Gamma_r^+(u)))| \leq \int_{\Gamma_r^+(u)} |\det DF(\nabla u(x))| dx$$

where,

$$|\det DF(\nabla u(x))| = \det D_\xi F(\nabla u(x)) \cdot \det(-D^2 u) = \det(-DF(\nabla u(x)))$$

since  $x \in \Gamma_r^+(u)$  and  $\det D_\xi F(\nabla u(x)) \geq 0$  by hypotheses (F1) and (F2), see also Remark 2.1. We observe that, according with (F1), the matrix  $D_\xi F(\nabla u(x))$  is symmetric.

We now recall (see [26, Section 9.1] and Lemma 6.4 below for a generalization) that if  $A$  and  $B$  are nonnegative symmetric matrices then,

$$(2.8) \quad \det(AB) \leq \left( \frac{\text{trace}(AB)}{n} \right)^n.$$

Taking  $A = D_\xi F(\nabla u(x))$  and  $B = -D^2 u$  we therefore have on  $\Gamma_r^+(u)$

$$\begin{aligned} |F(\nabla u(\Gamma_r^+(u)))| &\leq \int_{\Gamma_r^+(u)} \det(-DF(\nabla u(x))) dx \\ &\leq \int_{\Gamma_r^+(u)} \left[ \frac{-\text{div} F(\nabla u(x))}{n} \right]^n dx \leq \int_{\Gamma_r^+(u)} \left[ \frac{f^+(x)}{n} \right]^n dx. \end{aligned}$$

Then, combining (2.7) and the last expression, we get

$$C|B_1(0)|r^{\alpha n} \leq \int_{\Gamma_r^+(u)} \left[ \frac{f^+(x)}{n} \right]^n dx$$

and the result follows letting  $r \rightarrow r_0$  and then  $a \rightarrow \sup_{\partial\Omega} u$ .

This concludes the proof in the case  $u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ . In the general case, when  $u$  is merely continuous, we apply the regularization process in

[8, Appendix A], see also the proof of Theorem 2.8. Here  $F \in \mathcal{C}^1$ , (see hypothesis (F1)) is necessary, since it implies that the operator giving rise to our PDE is continuous when written in trace form (2.2), a necessary condition for the regularization process.  $\square$

**2.2. The ABP estimate for nonlinear parabolic equations in divergence form.** In this Section we present the parabolic counterpart of Theorem 2.5. First, we recall the definition of viscosity solution to be used in the sequel. We follow the sign convention in [16].

**Definition 2.6.** Let  $Q = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$  and  $\mathcal{F} : \mathbb{R}^n \times S^n \rightarrow \mathbb{R}$ . A function  $u \in \mathcal{C}(Q)$  is a *viscosity subsolution* (resp. *supersolution*) of

$$(2.9) \quad u_t + \mathcal{F}(\nabla u, D^2 u) = f(x, t)$$

in  $Q$  if for all  $(\hat{x}, \hat{t}) \in Q$  and  $\varphi \in \mathcal{C}^2(Q)$  such that  $u - \varphi$  attains a local maximum (minimum) at  $(\hat{x}, \hat{t})$  one has

$$\varphi_t(\hat{x}, \hat{t}) + \mathcal{F}(\nabla \varphi(\hat{x}, \hat{t}), D^2 \varphi(\hat{x}, \hat{t})) \leq f(\hat{x}, \hat{t}) \quad (\text{resp. } \geq).$$

We say that  $u \in \mathcal{C}(Q)$  is a *viscosity solution* of (3.1) in  $Q$  if it is both a viscosity subsolution and supersolution.

We will also need the parabolic version of Definition 2.4.

**Definition 2.7.** Let  $Q = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$ . Given  $u : Q \rightarrow \mathbb{R}$ , the *parabolic upper contact set* of  $u$  is defined as

$$\Gamma_p^+(u) := \left\{ (x, t) \in Q : u(x, t) \geq \sup_{\partial_p Q} u^+, \text{ and } \exists \xi \in \mathbb{R}^n \text{ such that} \right. \\ \left. u(y, s) \leq u(x, t) + \langle \xi, y - x \rangle, \forall y \in \Omega, \text{ and } s \leq t \right\}.$$

For convenience, we also define the following subset of  $\Gamma_p^+(u)$ ,

$$\Gamma_{p,r}^+(u) := \left\{ (x, t) \in Q : u(x, t) \geq \sup_{\partial_p Q} u^+, \text{ and } \exists \xi \in B_r(0) \text{ such that} \right. \\ \left. u(y, s) \leq u(x, t) + \langle \xi, y - x \rangle, \forall y \in \Omega, \text{ and } s \leq t \right\}.$$

The following is the main result in this section, and the parabolic counterpart of Theorem 2.5.

**Theorem 2.8.** Let  $Q = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$  a bounded domain and  $f \in L^{n+1}(Q) \cap \mathcal{C}(Q)$ . Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfy assumptions (F1), (F2) and (F3). Consider  $u \in \mathcal{C}(\bar{Q})$  which satisfies

$$u_t - \operatorname{div}(F(\nabla u)) \leq f(x, t) \quad \text{in } Q$$

in the viscosity sense. Then, the ABP estimate holds, that is,

$$(2.10) \quad \sup_Q u \leq \sup_{\partial_p Q} u^+ + \tilde{C} d^{\frac{\alpha n}{\alpha n + 1}} \|f^+\|_{L^{n+1}(\Gamma_p^+(u))}^{\frac{n+1}{\alpha n + 1}}$$

where  $d = \text{diam}(\Omega)$ ,  $\partial_p Q$  denotes the parabolic boundary of  $Q$ , and

$$\tilde{C} = \left( \frac{\alpha n + 1}{C \cdot |B_1(0)| \cdot (n+1)^{n+1}} \right)^{\frac{1}{\alpha n + 1}}$$

with  $C, \alpha$  the constants in hypothesis (F3).

Analogously, whenever  $u \in \mathcal{C}(\overline{Q})$  is a viscosity solution of

$$u_t - \text{div}(F(\nabla u)) \geq f(x, t) \quad \text{in } Q,$$

we have the following estimate,

$$(2.11) \quad \sup_Q u^- \leq \sup_{\partial_p Q} u^- + \tilde{C} d^{\frac{\alpha n}{\alpha n + 1}} \|f^-\|_{L^{n+1}(\Gamma_p^+(-u))}^{\frac{n+1}{\alpha n + 1}}$$

where  $d, \tilde{C}, \alpha$  are constants defined as before.

We collect in the following lemma some technical results needed in the proof of Theorem 2.8 concerning regularization by sup-convolution, introduced by Jensen, Lions and Souganidis [29, 30]. It will play the role of [8, Lemma A.2] in the elliptic case. For the proof, see for instance [23, Lemma 3.1] (see also [11, 35] and the references therein).

**Lemma 2.9.** *Let  $Q \subset \mathbb{R}^{n+1}$  and  $u \in \mathcal{C}(\overline{Q})$  and define for every  $\epsilon > 0$ , its sup-convolution in space and time  $u^\epsilon(x, t)$  as*

$$(2.12) \quad u^\epsilon(x, t) = \sup_{(y, s) \in Q} \left\{ u(y, s) - \frac{|x - y|^2 + (t - s)^2}{2\epsilon} \right\} \quad \text{in } \mathbb{R}^{n+1}.$$

Then,

- (i)  $u^\epsilon$  is Lipschitz continuous on  $Q$ .
- (ii)  $u^\epsilon \rightarrow u$  as  $\epsilon \rightarrow 0^+$  uniformly on compact subsets of  $Q$ .
- (iii) The first and second derivatives of  $u^\epsilon$  exist for almost every  $(x, t) \in Q$  in the sense that,

$$\begin{aligned} u^\epsilon(w) &= u^\epsilon(z) + \langle D_w u^\epsilon(z), (w - z) \rangle \\ &\quad + \frac{1}{2} \langle D_w^2 u^\epsilon(z)(w - z), (w - z) \rangle + o(|w - z|^2) \end{aligned}$$

a.e.  $z = (x, t) \in Q$ .

- (iv)  $D_x^2 u^\epsilon(x, t) \geq -\frac{1}{\epsilon} I$  a.e.  $(x, t) \in Q$ .
- (v) If  $u_\eta^\epsilon(x, t)$  is a standard mollification of  $u^\epsilon(x, t)$ , then  $D_x^2 u_\eta^\epsilon(x, t) \geq -\frac{1}{\epsilon} I$  and

$$D^2 u_\eta^\epsilon(x, t) \rightarrow D^2 u^\epsilon(x, t) \quad \text{a.e. } (x, t) \in Q \quad \text{as } \eta \rightarrow 0.$$

Now, we are ready to proceed with the proof of Theorem 2.8. The proof is based on a geometrical argument similar to [45] using a nonlinear mapping intrinsic to the problem.

*Proof of Theorem 2.8.* Replacing  $u$  with  $u - \sup_{\partial_p Q} u^+$ , we can suppose that  $\sup_{\partial_p Q} u^+ = 0$  during the proof. First, we will assume  $u \in \mathcal{C}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  and then we will remove this requirement by regularization. Let  $(x_0, t_0) \in Q$  be such that,

$$M(u) = \sup_Q u = u(x_0, t_0),$$

and fix a positive number  $r < M$ . Following [45], we define the map,

$$\Phi(x, t) = (\nabla u(x, t), u(x, t) - (x - x_0) \cdot \nabla u(x, t)).$$

We claim that

$$D = \{(\xi, h) : \xi \in B_{r/d}(0), d|\xi| < h < r\} \subset \Phi(\Gamma_{p, \frac{r}{d}}^+(u)).$$

To see this, take a pair  $(\xi, h) \in D$  and consider the hyperplane,

$$l_\xi(x) = \xi \cdot (x - x_0) + h.$$

From the condition  $h > d|\xi|$ , we obtain that  $l_\xi > u$  on  $\partial_p Q$ . Moreover,  $l_\xi(x_0) = h < M = u(x_0, t_0)$  implies that if we translate the hyperplane  $l_\xi$  along the  $t$  direction, it will touch the graph of  $u$  at some (possibly non unique) point  $(\hat{x}, \hat{t})$  with  $\hat{t} \leq t_0$ . At those points  $(\hat{x}, \hat{t})$  where  $l_\xi$  and  $u$  contact for the first time, we have

$$u(\hat{x}, \hat{t}) = l_\xi(\hat{x}) \quad \text{and} \quad u(x, t) \leq l_\xi(x) \quad \forall t \leq \hat{t}.$$

From this contact condition one readily recovers

$$\xi = \nabla u(\hat{x}, \hat{t}) \quad \text{and} \quad h = u(\hat{x}, \hat{t}) - (\hat{x} - x_0) \cdot \nabla u(\hat{x}, \hat{t}),$$

that is,  $(\xi, h) = \Phi(\hat{x}, \hat{t})$ , so the claim is proved.

Now, we introduce the spatially-nonlinear map

$$\begin{aligned} \mathcal{F} : \mathbb{R}^{n+1} &\rightarrow \mathbb{R}^{n+1} \\ (\xi, h) &\mapsto (F(\xi), h). \end{aligned}$$

Clearly,  $\mathcal{F}(D) \subset \mathcal{F}(\Phi(\Gamma_{p, \frac{r}{d}}^+(u)))$  and consequently,

$$|\mathcal{F}(D)| \leq |\mathcal{F}(\Phi(\Gamma_{p, \frac{r}{d}}^+(u)))|.$$

Then, we have to estimate the left-hand side of this inequality. Using hypothesis (F3), we get,

$$\begin{aligned} |\mathcal{F}(D)| &= \iint_{\mathcal{F}(D)} d\xi dh = \int_0^r |F(B_{h/d}(0))| dh \\ &\geq \frac{C|B_1(0)|}{d^{\alpha n}} \int_0^r h^{\alpha n} dh = \frac{C|B_1(0)|}{d^{\alpha n}(\alpha n + 1)} r^{\alpha n + 1}. \end{aligned}$$

Then, using the Area Formula as in the elliptic case, we deduce,

$$\begin{aligned}
 r^{\alpha n+1} &\leq \frac{(\alpha n+1)}{C|B_1(0)|} d^{\alpha n} \left| \mathcal{F}\left(\Phi\left(\Gamma_{p,\frac{r}{d}}^+(u)\right)\right) \right| \\
 (2.13) \quad &= \frac{(\alpha n+1)}{C|B_1(0)|} d^{\alpha n} \iint_{\mathcal{F}\left(\Phi\left(\Gamma_{p,\frac{r}{d}}^+(u)\right)\right)} d\xi dh \\
 &\leq \frac{(\alpha n+1)}{C|B_1(0)|} d^{\alpha n} \iint_{\Gamma_{p,\frac{r}{d}}^+(u)} |\det D(\mathcal{F}(\Phi(x,t)))| dx dt.
 \end{aligned}$$

Next, we estimate the right-hand side of (2.13). Since  $D_\xi F(\nabla u(x,t)) \geq 0$  by hypothesis (F2) (see Remark 2.1), we have that,

$$|\det D(\mathcal{F}(\Phi(x,t)))| = \det \begin{bmatrix} D_\xi F(\nabla u(x,t)) & 0 \\ 0 & 1 \end{bmatrix} \cdot |\det D\Phi(x,t)|.$$

We observe that an elementary row operation yields,

$$\begin{aligned}
 \det D\Phi(x,t) &= \det \begin{bmatrix} D^2 u(x,t) & \nabla u_t(x,t) \\ -(x-x_0) \cdot D^2 u(x,t) & u_t(x,t) - (x-x_0) \cdot \nabla u_t(x,t) \end{bmatrix} \\
 &= \det \begin{bmatrix} D^2 u(x,t) & \nabla u_t(x,t) \\ 0 & u_t(x,t) \end{bmatrix} = \det \begin{bmatrix} D^2 u(x,t) & 0 \\ 0 & u_t(x,t) \end{bmatrix}
 \end{aligned}$$

Hence, for any  $(x,t) \in \Gamma_{p,\frac{r}{d}}^+(u)$ ,

$$\begin{aligned}
 |\det D(\mathcal{F}(\Phi(x,t)))| &= \det \left( \begin{bmatrix} D_\xi F(\nabla u(x,t)) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -D^2 u(x,t) & 0 \\ 0 & u_t(x,t) \end{bmatrix} \right) \\
 &= \det(AB).
 \end{aligned}$$

Since, the matrices  $A, B$  in this expression are symmetric and positive semi-definite, we can apply (2.8) and get,

$$\begin{aligned}
 (2.14) \quad &\iint_{\Gamma_{p,\frac{r}{d}}^+(u)} |\det D(\mathcal{F}(\Phi(x,t)))| dx dt \\
 &\leq \iint_{\Gamma_{p,\frac{r}{d}}^+(u)} \left( \frac{u_t(x,t) - \operatorname{div}(F(\nabla u(x,t)))}{n+1} \right)^{n+1} dx dt \\
 &\leq \iint_{\Gamma_p^+(u)} \left( \frac{|f^+(x,t)|}{n+1} \right)^{n+1} dx dt.
 \end{aligned}$$

Putting together (2.13) and (2.14) we get the result in the case  $u \in \mathcal{C}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$ .

We still have to remove the regularity hypothesis  $u \in \mathcal{C}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$ . We do so by regularization, in a similar fashion to [8].

First, replacing  $u$  with  $u - \sup_{\partial_p Q} u - k$  for some  $k > 0$  we can assume without loss of generality that  $\sup_{\partial_p Q} u = -k$  during the proof upon letting  $k \rightarrow 0$  in the end.

For  $\epsilon > 0$  consider  $u^\epsilon(x, t)$ , the sup-convolution of  $u$  in space and time as defined in (2.12). It is known (see for instance [23, Lemma 3.1] or [35, Section 4]) that  $u^\epsilon$  satisfies,

$$(u^\epsilon)_t - \operatorname{div}(F(\nabla u^\epsilon)) \leq f_\epsilon(x, t) \quad \text{in } Q_{2(\epsilon\|u\|_{L^\infty(Q)})^{1/2}}$$

in the viscosity sense (here is used that  $F \in \mathcal{C}^1$ ), for

$$f_\epsilon(x, t) = \sup_{|x-y|^2 + |t-s|^2 \leq 4\epsilon\|u\|_{L^\infty(Q)}} f(y, s),$$

and,

$$Q_\delta = \{(x, t) \in Q : \operatorname{dist}((x, t), \partial Q) > \delta\},$$

where  $\partial Q$  and  $\operatorname{dist}(\cdot, \partial Q)$  denote, respectively, the boundary of  $Q$  and the distance to the boundary in  $\mathbb{R}^{n+1}$ .

Fix  $r < M(u)$  as before, and consider  $u_\eta^\epsilon$ , an standard mollification of  $u^\epsilon$ . By uniform convergence  $u_\eta^\epsilon \rightarrow u^\epsilon$  as  $\eta \rightarrow 0$  and  $u^\epsilon \rightarrow u$  as  $\epsilon \rightarrow 0$ , we have that  $r < M(u_\eta^\epsilon)$  for  $\epsilon, \eta$  small enough.

Furthermore, taking  $\epsilon$  small enough, we can find an open set  $G \subset \Omega$  such that

$$\Gamma_{p, \frac{r}{d}}^+(u) \subset\subset G \subset\subset Q_{2(\epsilon\|u\|_{L^\infty(Q)})^{1/2}}$$

where every set is a compact subset of the one containing it. Then, by uniform convergence we also have,

$$\Gamma_{p, \frac{r}{d}}^+(u_\eta^\epsilon) \subset\subset G$$

for  $\epsilon, \eta$  small enough.

Now, since  $u_\eta^\epsilon \in \mathcal{C}^2$ , we can proceed as in the regular case to get,

$$\begin{aligned} r^{\alpha n+1} &\leq \frac{(\alpha n+1)}{C|B_1(0)|} \frac{d^{\alpha n}}{(n+1)^{n+1}} \iint_{\Gamma_{p, \frac{r}{d}}^+(u_\eta^\epsilon)} \left( (u_\eta^\epsilon)_t(x, t) \right. \\ &\quad \left. - \operatorname{trace}(D_\xi F(\nabla u_\eta^\epsilon(x, t)) D^2 u_\eta^\epsilon(x, t)) \right)^{n+1} dx dt. \end{aligned}$$

Lemma 2.9 implies that  $-(1/\epsilon)I \leq D^2 u_\eta^\epsilon \leq 0$  on  $\Gamma_{p, \frac{r}{d}}^+(u_\eta^\epsilon)$  and by dominated convergence, we can pass to the limit in  $\eta$  in the above expression to get

$$\begin{aligned} r^{\alpha n+1} &\leq \frac{(\alpha n+1)}{C|B_1(0)|} \frac{d^{\alpha n}}{(n+1)^{n+1}} \iint_G \left( (u^\epsilon)_t(x, t) \right. \\ &\quad \left. - \operatorname{trace}(D_\xi F(\nabla u^\epsilon(x, t)) D^2 u^\epsilon(x, t)) \right)^{n+1} dx dt. \end{aligned}$$

As a consequence of Lemma 2.9 we have that  $(u^\epsilon)_t$ ,  $\nabla_x u^\epsilon$  and  $D_x^2 u^\epsilon$  exist almost everywhere and

$$(u^\epsilon)_t - \operatorname{trace}(D_\xi F(\nabla_x u^\epsilon) D_x^2 u^\epsilon) \leq f_\epsilon^+(x, t) \quad \text{a.e. in } Q_{2(\epsilon\|u\|_{L^\infty(Q)})^{1/2}}.$$

Hence, plugging this expression into our estimate, we can pass to the limit as  $\epsilon \rightarrow 0$  using the continuity of  $f$  to get

$$r^{\alpha n+1} \leq \frac{(\alpha n + 1)}{C |B_1(0)|} \frac{d^{\alpha n}}{(n+1)^{n+1}} \iint_G (f^+(x, t))^{n+1} dx dt.$$

Since the set  $G$  is arbitrary, we can shrink it to  $\Gamma_{p, \frac{r}{d}}^+(u)$  and finally let  $r \rightarrow M(u)$  to conclude.  $\square$

### 3. ABP ESTIMATES FOR OPERATORS WITH SINGULARITIES

In this Section we provide some examples of operators for which proving an ABP estimate is possible even if they do not fulfill hypothesis (F1). Namely, we consider operators with a singularity at  $\xi = 0$  as they are the divergence of fields  $F(\xi)$  that are merely continuous at  $\xi = 0$ .

The idea of the proofs consist in removing the singularity by considering perturbed fields  $F_\epsilon$  that are under the scope of our general results and then passing to the limit.

We consider, for instance, the  $p$ -Laplacian operator with  $1 < p < 2$ , given by

$$\begin{aligned} \Delta_p u &= \operatorname{div}(|\nabla u|^{p-2} \nabla u) \\ &= |\nabla u|^{p-2} \cdot \operatorname{trace} \left[ \left( I + (p-2) \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) D^2 u \right]. \end{aligned}$$

In this case, the operator can be readily written in the form (2.1) using the map  $F(\xi) = |\xi|^{p-2} \xi$ .

As the case  $p \geq 2$  is an straightforward consequence of the results in Section 2, we will focus on the range  $1 < p < 2$ . Due to the fact that the  $p$ -Laplacian operator is singular when  $1 < p < 2$ , we have to adapt the notion of viscosity solution in Definition 2.3. Notice that the singularity of the operator is not bounded, so it is not possible to use the lower and upper semicontinuous envelopes (relaxations) in defining the notion of viscosity solution (see [16, Section 9] and also [25, Chapter 2]). Instead, we adopt the definition proposed in a series of papers by Birindelli and Demengel, see [4] and the references therein. An alternative but equivalent definition (in the case  $f = 0$ ) can be found in [25, Section 2.1.3].

**Definition 3.1.** Let  $\mathcal{F} : \mathbb{R}^n \times S^n \rightarrow \mathbb{R}$ . A function  $u \in \mathcal{C}(\Omega)$  is a *viscosity subsolution* (resp. *supersolution*) of

$$(3.1) \quad \mathcal{F}(\nabla u, D^2 u) = f(x)$$

in  $\Omega$  if for all  $\hat{x} \in \Omega$  we have:

- (i) Either for all  $\varphi \in \mathcal{C}^2(\Omega)$  such that  $u - \varphi$  attains a local maximum (minimum) at  $\hat{x}$  with  $\nabla \varphi(\hat{x}) \neq 0$  one has

$$\mathcal{F}(\nabla \varphi(\hat{x}), D^2 \varphi(\hat{x})) \leq f(\hat{x}) \quad (\text{resp. } \geq).$$

- (ii) Or there exists an open ball  $B_\delta(\hat{x}) \subset \Omega$ ,  $\delta > 0$  such that  $u \equiv C$  in  $B_\delta(\hat{x}) \subset \Omega$  and  $f(x) \geq 0$  for all  $x \in B_\delta(\hat{x}) \subset \Omega$  (resp.  $f(x) \leq 0$ ).

We say that  $u \in \mathcal{C}(\Omega)$  is a *viscosity solution* of (3.1) in  $\Omega$  if it is both a viscosity subsolution and supersolution.

It is interesting that the notion of solution in Definition 3.1 is equivalent to the one in Definition 2.3 when  $\mathcal{F}$  is continuous, see [18].

Next, we present the ABP estimate for the  $p$ -Laplacian in the whole range  $1 < p < \infty$ .

**Theorem 3.2.** *Assume that  $\Omega \subset \mathbb{R}^n$  is a bounded domain and let  $1 < p < \infty$ . Let  $f \in L^n(\Omega) \cap \mathcal{C}(\Omega)$  and consider  $u \in \mathcal{C}(\overline{\Omega})$ , a viscosity solution of*

$$(3.2) \quad -\Delta_p u \leq f(x) \quad \text{in } \Omega,$$

*in the sense of Definition 3.1. Then, the ABP estimate holds, that is*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C d \|f^+\|_{L^n(\Gamma^+(u))}^{\frac{1}{(p-1)}}$$

*with  $d = \text{diam}(\Omega)$  and  $C = (n \cdot \min\{1, p-1\} \cdot |B_1(0)|^{1/n})^{-\frac{1}{p-1}}$ .*

*Analogously, whenever  $u \in \mathcal{C}(\overline{\Omega})$  is a viscosity solution of*

$$(3.3) \quad -\Delta_p u \geq f(x) \quad \text{in } \Omega,$$

*we have the following estimate*

$$\sup_{\Omega} u^- \leq \sup_{\partial\Omega} u^- + C d \|f^-\|_{L^n(\Gamma^+(-u))}^{\frac{1}{(p-1)}}$$

*where  $d, C$  are constants defined as before.*

As we have already mentioned, Theorem 3.2 follows directly from Theorem 2.5 when  $p \geq 2$ . Hypotheses (F1), (F2) and (F3) can be checked easily, namely,  $F$  is differentiable, it is well-known that for any  $p > 1$ ,

$$\langle |\xi_1|^{p-2} \xi_1 - |\xi_2|^{p-2} \xi_2, \xi_1 - \xi_2 \rangle > 0 \quad \forall \xi_1, \xi_2 \in \mathbb{R}^n, \xi_1 \neq \xi_2,$$

and, by homogeneity,

$$|F(B_r(0))| = |B_1(0)| \cdot r^{n(p-1)}.$$

In the case  $p \in (1, 2)$   $F$  has a discontinuity and the argument requires some adaptations.

We will need the Pucci extremal operators (see [6] for its properties), defined as follows,

$$(3.4) \quad \begin{aligned} \mathcal{M}_{\theta, \Theta}^+(X) &= \Theta \sum_{\lambda_i > 0} \lambda_i(X) + \theta \sum_{\lambda_i < 0} \lambda_i(X) \\ \mathcal{M}_{\theta, \Theta}^-(X) &= \theta \sum_{\lambda_i > 0} \lambda_i(X) + \Theta \sum_{\lambda_i < 0} \lambda_i(X), \end{aligned}$$

where  $\lambda_i(X)$  is the  $i$ th eigenvalue of the matrix  $X$ . We will use the following result similar to [17, Lemma 4].

**Lemma 3.3.** *Let  $1 < p < 2$  and  $f \geq 0$ . Consider  $u \in \mathcal{C}(\overline{\Omega})$ , a viscosity solution of*

$$-\Delta_p u(x) \leq f(x) \quad \text{in } \Omega$$

*in the sense of Definition 3.1. Then, for every  $\epsilon > 0$ ,  $u$  is also a viscosity solution of*

$$(3.5) \quad -(|\nabla u(x)|^2 + \epsilon)^{\frac{p-2}{2}} \cdot \mathcal{M}_{(p-1),1}^+(D^2 u(x)) \leq f(x) \quad \text{in } \Omega.$$

*Analogously, whenever  $f \leq 0$  and  $u \in \mathcal{C}(\overline{\Omega})$  is a viscosity solution of*

$$-\Delta_p u(x) \geq f(x) \quad \text{in } \Omega$$

*then, for every  $\epsilon > 0$ ,  $u$  is also a viscosity solution of*

$$(3.6) \quad -(|\nabla u(x)|^2 + \epsilon)^{\frac{p-2}{2}} \cdot \mathcal{M}_{(p-1),1}^-(D^2 u(x)) \geq f(x) \quad \text{in } \Omega.$$

We include the proof of this Lemma for the sake of completeness.

*Proof.* Let  $\varphi \in \mathcal{C}^2(\Omega)$  such that  $u - \varphi$  attains a local maximum (minimum) at  $\hat{x}$ . According to Definition 3.1, we can suppose  $\nabla \varphi(\hat{x}) \neq 0$ . Then, by definition, we have

$$-|\nabla \varphi(\hat{x})|^{p-2} \cdot \text{trace} \left[ \left( I + (p-2) \frac{\nabla \varphi(\hat{x})}{|\nabla \varphi(\hat{x})|} \otimes \frac{\nabla \varphi(\hat{x})}{|\nabla \varphi(\hat{x})|} \right) D^2 \varphi(\hat{x}) \right] \leq f(\hat{x}).$$

Notice that

$$\text{trace} \left[ \left( I + (p-2) \frac{\nabla \varphi(\hat{x})}{|\nabla \varphi(\hat{x})|} \otimes \frac{\nabla \varphi(\hat{x})}{|\nabla \varphi(\hat{x})|} \right) D^2 \varphi(\hat{x}) \right] \leq \mathcal{M}_{(p-1),1}^+(D^2 \varphi(\hat{x}))$$

and  $|\nabla \varphi(\hat{x})|^{p-2} \geq (|\nabla \varphi(\hat{x})|^2 + \epsilon)^{\frac{p-2}{2}}$  since  $p \in (1, 2)$ . In the case

$$\mathcal{M}_{(p-1),1}^+(D^2 \varphi(\hat{x})) < 0,$$

we have

$$-(|\nabla \varphi(\hat{x})|^2 + \epsilon)^{\frac{p-2}{2}} \cdot \mathcal{M}_{(p-1),1}^+(D^2 \varphi(\hat{x})) \leq f(\hat{x})$$

while in the case  $\mathcal{M}_{(p-1),1}^+(D^2 \varphi(\hat{x})) \geq 0$ , we have

$$-(|\nabla \varphi(\hat{x})|^2 + \epsilon)^{\frac{p-2}{2}} \cdot \mathcal{M}_{(p-1),1}^+(D^2 \varphi(\hat{x})) \leq 0 \leq f(\hat{x}). \quad \square$$

Next, we provide the proof of Theorem 3.2 in the case  $p \in (1, 2)$ .

*Proof of Theorem 3.2 in the case  $p \in (1, 2)$ .* First, we observe that we are not under the scope of Theorem 2.5 since the regularization process in the proof of that result only applies to equations which can be expressed as

$$\mathcal{F}(\nabla u, D^2 u) = -\text{div}(F(\nabla u)) \leq f(x),$$

for some continuous functional  $\mathcal{F} : \mathbb{R}^n \times S^n \rightarrow \mathbb{R}$ . Since the functional giving rise to equation (3.3) is not continuous when  $p \in (1, 2)$ , we will instead use Lemma 3.3 and think of  $u$  as a subsolution of

$$-(|\nabla u(x)|^2 + \epsilon)^{\frac{p-2}{2}} \cdot \mathcal{M}_{(p-1),1}^+(D^2 u(x)) \leq f^+(x) \quad \text{in } \Omega.$$

At this point, it is possible to perform the same regularizations as in the proof of Theorem 2.5 and assume  $u$  to be a  $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$  subsolution of (3.6).

Let  $x_0 \in \Omega$  be such that  $M = u(x_0) - \sup_{\partial\Omega} u^+ = \sup_{\Omega} u - \sup_{\partial\Omega} u^+$ , and consider the “regularized” mapping

$$F(\xi) = \left( |\xi|^2 + \epsilon \right)^{\frac{p-2}{2}} \cdot \xi$$

Now, we follow the arguments in the first part of the proof of Theorem 2.5 to get the following expression:

$$\begin{aligned} |B_1(0)| \cdot \left[ \left( \left( \frac{M}{d} \right)^2 + \epsilon \right)^{\frac{p-2}{2}} \left( \frac{M}{d} \right) \right]^n \\ \leq \int_{\Gamma^+(u)} \left| \det \left[ D \left( \left( |\nabla u(x)|^2 + \epsilon \right)^{\frac{p-2}{2}} \nabla u(x) \right) \right] \right| dx. \end{aligned}$$

Now, we use that for any  $x \in \Gamma^+(u)$ , we have,

$$\begin{aligned} \left| \det \left[ D \left( \left( |\nabla u(x)|^2 + \epsilon \right)^{\frac{p-2}{2}} \nabla u(x) \right) \right] \right| &= \\ &= \det \left[ - \left( |\nabla u(x)|^2 + \epsilon \right)^{\frac{p-2}{2}} D^2 u(x) \right] \\ &\quad \times \det \left[ I + (p-2) \frac{\nabla u(x)}{\left( |\nabla u(x)|^2 + \epsilon \right)^{\frac{1}{2}}} \otimes \frac{\nabla u(x)}{\left( |\nabla u(x)|^2 + \epsilon \right)^{\frac{1}{2}}} \right] \\ &\leq \left[ \frac{- \left( |\nabla u(x)|^2 + \epsilon \right)^{\frac{p-2}{2}} \cdot \text{trace}(D^2 u(x))}{n} \right]^n \leq \left( \frac{f^+(x)}{n(p-1)} \right)^n, \end{aligned}$$

where we have used in the last inequality that  $\mathcal{M}_{(p-1),1}^+(D^2 u(x)) = (p-1) \cdot \text{trace}(D^2 u(x))$  for any  $x \in \Gamma^+(u)$ . We conclude that

$$|B_1(0)| \cdot \left[ \left( \left( \frac{M}{d} \right)^2 + \epsilon \right)^{\frac{p-2}{2}} \left( \frac{M}{d} \right) \right]^n \leq \frac{1}{n^n (p-1)^n} \int_{\Gamma^+(u)} |f^+(x)|^n dx.$$

Finally, letting  $\epsilon \rightarrow 0$ , we get the desired result.  $\square$

#### 4. FULLY NONLINEAR WITH $p$ -LAPLACIAN GROWTH

In this section we intend to give an example of the fact that the ABP estimate can be proved even if the operator is not in divergence form, whenever it belongs to a class of equations bounded by some suitable extremal operators that contains some divergence-form representative.

We show how to get ABP-type results for Fully nonlinear operators with  $p$ -Laplacian growth, of the form

$$(4.1) \quad \mathcal{F}(\nabla u(x), D^2 u(x)) = f(x) \quad \text{in } \Omega$$

with  $\mathcal{F} : \mathbb{R}^n \times S^n \rightarrow \mathbb{R}$  satisfying the following structural hypothesis:

- (F1) *Continuity*:  $\mathcal{F} : (\mathbb{R}^n \setminus \{0\}) \times S^n \rightarrow \mathbb{R}$  is continuous.
- (F2) *Homogeneity*:  $\mathcal{F}(t\xi, \mu X) = |t|^\alpha \mu \cdot \mathcal{F}(\xi, X)$  for some  $\alpha > 0$  and for all  $t \in \mathbb{R} \setminus \{0\}$  and  $\mu \in \mathbb{R}^+$ .
- (F3) *Ellipticity*: There exist constants  $0 < \theta \leq \Theta$  such that for all  $X, Y \in S^n$ , and for every  $\xi \in \mathbb{R}^n \setminus \{0\}$ ,

$$-|\xi|^\alpha \mathcal{M}_{\theta, \Theta}^+(Y) \leq \mathcal{F}(\xi, X + Y) - \mathcal{F}(\xi, X) \leq -|\xi|^\alpha \mathcal{M}_{\theta, \Theta}^-(Y)$$

where  $\mathcal{M}_{\theta, \Theta}^\pm$  are the extremal Pucci operators with ellipticity constants  $0 < \theta \leq \Theta$ , defined in (3.4).

This kind of operators have been studied in a series of papers by Birindelli and Demengel, see [4] and the references therein. Moreover, in the elliptic case, the ABP estimate has been recently obtained in [17, 28].

As it was mentioned in the introduction, our approach is different, and builds on the  $p$ -Laplacian case, using the spatially-nonlinear mapping  $F(\xi) = |\xi|^\alpha \xi$ . The essential difference of point of view between [17, 28] and our approach comes from the different interpretation of the extremal operators associated to problems of type (4.1), since one can either think of the gradient term as part of the right-hand side of a uniformly elliptic problem, or keep the gradient term as a main part of a degenerate operator related to a  $p$ -Laplacian. The main advantage of our approach is that it allows to get parabolic results as a natural extension of the elliptic results and with little extra effort. Notice that the presence of the parabolic term  $u_t$  prevents from splitting the operator as before.

Notice that we only consider the degenerate case  $\alpha \geq 0$  for simplicity. One could treat the singular case  $-1 < \alpha < 0$  perturbing the nonlinear mapping in a similar way to Section 3.

**Theorem 4.1.** *Let  $\Omega \subset \mathbb{R}^n$  a bounded domain,  $0 < \theta \leq \Theta$  and  $\alpha \geq 0$ . Consider  $f \in L^n(\Omega) \cap \mathcal{C}(\Omega)$  and  $u \in \mathcal{C}(\overline{\Omega})$  is a viscosity solution of*

$$(4.2) \quad -|\nabla u(x)|^\alpha \cdot \mathcal{M}_{\theta, \Theta}^+(D^2 u(x)) \leq f(x) \quad \text{in } \Omega.$$

*Then, the ABP estimate holds, that is,*

$$(4.3) \quad \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C d \|f^+\|_{L^n(\Gamma^+(u))}^{\frac{1}{\alpha+1}}$$

where  $d = \text{diam}(\Omega)$  and

$$C = \left( \frac{\alpha + 1}{n^n \theta^n |B_1(0)|} \right)^{\frac{1}{n(\alpha+1)}}.$$

Analogously, whenever  $u \in \mathcal{C}(\overline{\Omega})$  is a viscosity solution of

$$(4.4) \quad -|\nabla u(x)|^\alpha \cdot \mathcal{M}_{\theta, \Theta}^-(D^2 u(x)) \geq f(x) \quad \text{in } \Omega,$$

we have the following estimate,

$$(4.5) \quad \sup_{\Omega} u^- \leq \sup_{\partial\Omega} u^- + C d \|f^-\|_{L^n(\Gamma^+(-u))}^{\frac{1}{\alpha+1}}$$

where  $d, C$  are constants defined as before.

Next, we provide the parabolic counterpart of Theorem 4.1.

**Theorem 4.2.** *Let  $Q \subset \mathbb{R}^{n+1}$  a bounded domain,  $0 < \theta \leq \Theta$  and  $\alpha \geq 0$ . Consider  $f \in L^{n+1}(Q)$  and  $u \in \mathcal{C}(\overline{Q})$  is a viscosity solution of*

$$(4.6) \quad u_t - |\nabla u(x, t)|^\alpha \cdot \mathcal{M}_{\theta, \Theta}^+(D^2 u(x, t)) \leq f(x, t) \quad \text{in } Q.$$

Then, the ABP estimate holds, that is,

$$(4.7) \quad \sup_Q u \leq \sup_{\partial_p Q} u^+ + C d^{\frac{(\alpha+1)n}{1+(\alpha+1)n}} \|f^+\|_{L^{n+1}(\Gamma_p^+(u))}^{\frac{n+1}{(\alpha+1)n+1}}$$

where  $d = \text{diam}(\Omega)$  and

$$C = \left( \frac{(\alpha+1)(1+(\alpha+1)n)}{|B_1(0)|(n+1)^{n+1}\theta^n} \right)^{\frac{1}{(\alpha+1)n+1}}.$$

Analogously, whenever  $u \in \mathcal{C}(\overline{Q})$  is a viscosity solution of

$$(4.8) \quad u_t - |\nabla u(x, t)|^\alpha \cdot \mathcal{M}_{\theta, \Theta}^-(D^2 u(x, t)) \geq f(x, t) \quad \text{in } Q,$$

in the viscosity sense, we have the following estimate,

$$(4.9) \quad \sup_Q u^- \leq \sup_{\partial_p Q} u^- + C d^{\frac{(\alpha+1)n}{1+(\alpha+1)n}} \|f^-\|_{L^{n+1}(\Gamma_p^+(-u))}^{\frac{n+1}{(\alpha+1)n+1}}$$

where  $d, C$  are the same constants as before.

We only sketch the proof of Theorem 4.2 as the proof of Theorem 4.1 is similar.

*Proof of Theorem 4.2.* Proceeding as in the proof of Theorem 2.8, we assume  $u \in \mathcal{C}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  and we suppose that  $\sup_{\partial_p Q} u^+ \leq 0$ . Let  $(x_0, t_0) \in Q$  be such that,

$$M = \sup_Q u = u(x_0, t_0).$$

We consider the nonlinear map

$$\begin{aligned} \mathcal{F} : \mathbb{R}^{n+1} &\rightarrow \mathbb{R}^{n+1} \\ (\xi, h) &\mapsto (\xi |\xi|^\alpha, h). \end{aligned}$$

We have that,

$$(4.10) \quad |\mathcal{F}(D)| = \int_0^M \left| B_{(\frac{h}{d})^{\alpha+1}}(0) \right| dh = \frac{|B_1(0)| M^{(\alpha+1)n+1}}{(1+(\alpha+1)n) d^{(\alpha+1)n}}.$$

Moreover, following the arguments of the proof of Theorem 2.8, we get

$$\begin{aligned}
|\mathcal{F}(D)| &\leq \left| \mathcal{F} \left( \Phi \left( \Gamma_{p, \frac{M}{d}}^+(u) \right) \right) \right| \\
&= \iint_{\Gamma_{p, \frac{M}{d}}^+(u)} u_t(x, t) |\det(D(|\nabla u|^\alpha \nabla u))| dx dt \\
(4.11) \quad &\leq \frac{\alpha+1}{\theta^n} \iint_{\Gamma_{p, \frac{M}{d}}^+(u)} u_t(x, t) \cdot \det(-|\nabla u|^\alpha \theta D^2 u) dx dt \\
&\leq \frac{\alpha+1}{\theta^n} \int_{\Gamma_{p, \frac{M}{d}}^+(u)} \left[ \frac{u_t - |\nabla u|^\alpha \mathcal{M}_{\theta, \Theta}^+(D^2 u(x, t))}{n+1} \right]^{n+1} dx \\
&\leq \frac{\alpha+1}{(n+1)^{n+1} \theta^n} \|f^+\|_{L^{n+1}(\Gamma_p^+(u))}^{n+1}.
\end{aligned}$$

Putting together (4.10) and (4.11) we get (4.7).  $\square$

## 5. THE MEAN CURVATURE OPERATOR

In this section, we provide an example in which hypothesis (F3) in Section 2 does not hold. This framework is related to hypothesis (F3') in Remark 2.2 except that in this case the function  $\psi$  is not invertible unless a size condition for  $f$  is imposed (as we will see, this size condition has a natural interpretation in terms of the total curvature that can be prescribed for a graph of a function  $u$ ).

Let  $S \subset \mathbb{R}^{n+1}$  be a hypersurface and suppose that in some coordinate system  $(x_1, \dots, x_n)$ , it is given by a graph  $x_{n+1} = u(x_1, \dots, x_n)$  of a regular function  $u$ . We will compute curvature with respect to the downwards directed normal,

$$(\nu(x), \nu_{n+1}(x)) = \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}}, \frac{-1}{\sqrt{1 + |\nabla u(x)|^2}} \right)$$

It can be shown (see for instance [26, Appendix to Chapter 14] or [41]) that the eigenvalues of the matrix

$$(5.1) \quad \nabla \nu(x) = \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right)$$

in these coordinates, denoted  $(\kappa_1, \dots, \kappa_n)$ , are the *Principal Curvatures* of  $S$  at  $x$  (with respect to the downwards directed normal).

The *Mean Curvature* at  $x$  is defined as,

$$H(x) = \frac{1}{n}(\kappa_1 + \dots + \kappa_n).$$

It follows that,

$$H(x) = \frac{1}{n} \operatorname{trace} \left( \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right) = \frac{1}{n} \operatorname{div} \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right).$$

We say that  $S$  is a *minimal surface* in  $\Omega$  if  $H(x) = 0$  for every  $x \in \Omega$ .

Conversely, one can consider the problem of the *Prescribed Mean Curvature*,

$$(5.2) \quad \operatorname{div} \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) = n H(x), \quad \text{for all } x \in \Omega.$$

that is, the problem of finding a graph  $u$  with a prescribed mean curvature  $H(x)$  at every point  $x \in \Omega$ . Following the ideas in the foregoing, we prove an ABP-type estimate for (5.2).

**Theorem 5.1.** *Let  $\Omega \subset \mathbb{R}^n$  a bounded domain and  $f \in L^n(\Omega) \cap \mathcal{C}(\Omega)$ . Consider  $u \in \mathcal{C}(\bar{\Omega})$  which satisfies*

$$(5.3) \quad -\operatorname{div} \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \leq f(x) \quad \text{in } \Omega$$

*in the viscosity sense. Then, the following ABP-type estimate holds,*

$$(5.4) \quad \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + d \frac{\|f^+\|_{L^n(\Gamma^+(u))}}{\sqrt{C^2 - \|f^+\|_{L^n(\Gamma^+(u))}^2}}$$

*(where  $d = \operatorname{diam}(\Omega)$ ) provided*

$$(5.5) \quad \|f^+\|_{L^n(\Gamma^+(u))} < C = n |B_1(0)|^{\frac{1}{n}}.$$

*Analogously, whenever  $u \in \mathcal{C}(\bar{\Omega})$  is a solution of*

$$-\operatorname{div} \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \geq f(x) \quad \text{in } \Omega$$

*in the viscosity sense, we have the following estimate*

$$\sup_{\Omega} u^- \leq \sup_{\partial\Omega} u^- + d \frac{\|f^-\|_{L^n(\Gamma^+(-u))}}{\sqrt{C^2 - \|f^-\|_{L^n(\Gamma^+(-u))}^2}}$$

*provided*

$$(5.6) \quad \|f^-\|_{L^n(\Gamma^+(-u))} < C$$

*where  $d, C$  are constants defined as before.*

Before proceeding with the proof of the theorem, some comments are in order. First, notice that, given the problem of finding a graph  $u$  with prescribed mean curvature  $H(x)$  in  $\Omega$ , that is, (5.2), we read off from Theorem 5.1 the following estimate for the height of the graph,

$$\begin{aligned} \inf_{\partial\Omega}(-u^-) - d \frac{\|H^+\|_{L^n(\Gamma^+(-u))}}{\sqrt{|B_1(0)|^{\frac{2}{n}} - \|H^+\|_{L^n(\Gamma^+(-u))}^2}} &\leq u(x) \\ &\leq \sup_{\partial\Omega} u^+ + d \frac{\|H^-\|_{L^n(\Gamma^+(u))}}{\sqrt{|B_1(0)|^{\frac{2}{n}} - \|H^-\|_{L^n(\Gamma^+(u))}^2}}, \end{aligned}$$

whenever,  $\|H^-\|_{L^n(\Gamma^+(u))}, \|H^+\|_{L^n(\Gamma^+(-u))} < |B_1(0)|^{\frac{1}{n}}$ .

It is interesting to mention that it is possible to find some results similar to Theorem 5.1 already in Bakel'man [3, Section II.6], see also [26, Sections 10.2 and 10.5] for a modern treatment. More precisely, they find the following estimate for  $u \in \mathcal{C}(\overline{\Omega}) \cap W_{loc}^{2,n}(\Omega)$ , solution of (5.2),

$$(5.7) \quad \sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C(n, \|H\|_{L^n(\Omega)}) \cdot \text{diam}(\Omega),$$

provided  $\|H\|_{L^n(\Omega)} < |B_1(0)|^{\frac{1}{n}}$ , which is a slightly stronger assumption than (5.5), (5.6). In [26], the constant  $C$  is not made explicit, nevertheless, after some computations, it can be checked that (5.7) essentially coincides with our estimate.

We think that it is worth comparing the techniques and proofs with ours. The essential difference is that in [3, 26], the operator is considered as a perturbation of a linear operator, while we consider the operator as a whole by means of a nonlinear mapping, intrinsic to the geometry of the equation. By considering the Mean Curvature operator as a perturbation of a linear operator, it is necessary to introduce adequate weights to handle the extra terms, as explained in the introduction to this paper. In our case, the structure of the equation, encoded in the nonlinear mapping, is part of the argument from the first moment. However, it is important to stress that the main advantage of our approach is that it also allows to treat the parabolic cases in a very coherent way, see Theorems 5.5 and 5.4 below, where flows by mean curvature are considered.

*Remark 5.2.* It is noticeable that, although the approaches are different, the restriction on the total mean curvature that can be prescribed, is the same in both cases: the measure of the unit ball. The reason for this is made apparent thinking about the case when  $\Omega = B_R(0)$  and the curvature  $H$  is constant and positive. Taking  $f = nH$  one readily sees that the size condition implies  $H < R^{-1}$ , which is equivalent to saying that  $u$  is a graph.

*Proof of Theorem 5.1.* As usual, we can assume  $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$  and let  $M = \sup_{\Omega} u - \sup_{\partial\Omega} u^+$ . Consider the mapping,

$$(5.8) \quad F(\xi) = \frac{\xi}{\sqrt{1 + |\xi|^2}}.$$

Now, proceeding as in the proof of Theorem 2.5, we get,

$$(5.9) \quad \begin{aligned} |B_1(0)| \cdot \left( \frac{M}{\sqrt{d^2 + M^2}} \right)^n &= |F(B_{M/d}(0))| \\ &\leq \int_{\Gamma^+(u)} \left| \det \left( \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right) \right| dx. \end{aligned}$$

We have that,

$$\nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) = \frac{1}{\sqrt{1 + |\nabla u(x)|^2}} \left( I - \frac{\nabla u(x) \otimes \nabla u(x)}{1 + |\nabla u(x)|^2} \right) D^2 u(x).$$

We observe that the matrix

$$I - \frac{\nabla u(x) \otimes \nabla u(x)}{1 + |\nabla u(x)|^2}$$

has eigenvalues  $(1 + |\nabla u(x)|^2)^{-1}$  (simple) and 1 (with multiplicity  $n - 1$ ). Hence,

$$\begin{aligned} &\int_{\Gamma^+(u)} \left| \det \left( \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right) \right| dx \\ &\leq \int_{\Gamma^+(u)} \left( -n^{-1} \cdot \operatorname{div} \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right)^n dx \leq n^{-n} \|f^+\|_{L^n(\Gamma^+(u))}^n. \end{aligned}$$

From this estimate, and (5.9), we deduce,

$$|B_1(0)| \cdot \left( \frac{M}{\sqrt{d^2 + M^2}} \right)^n \leq n^{-n} \|f^+\|_{L^n(\Gamma^+(u))}^n.$$

Then, using (5.5), we get (5.4).  $\square$

*Remark 5.3.* From the above proof, we read off the following estimate

$$n |B_1(0)|^{\frac{1}{n}} \frac{M}{\sqrt{d^2 + M^2}} \leq \|f^+\|_{L^n(\Gamma^+(u))},$$

where  $M = \sup_{\Omega} u - \sup_{\partial\Omega} u$ , independently of the size of  $\|f^+\|_{L^n(\Gamma^+(u))}$ . This estimate could be useful in estimating the measure of the upper contact set of  $u$ .

Next, we provide an ABP estimate for the Mean Curvature Flow, when the normal velocity is parallel to the mean curvature vector. We consider the case of a graph evolving by mean curvature, see Ecker and Huisken [20]. For the level set approach to the question, see Chen, Giga and Goto [13] and the series of papers by Evans and Spruck [23].

It is interesting to point out that in the proof we use the intrinsic “mean curvature” mapping in combination with the use of weights, as explained in the introduction.

**Theorem 5.4.** *Let  $Q \subset \mathbb{R}^{n+1}$  a bounded domain and  $f \in L^{n+1}(Q) \cap \mathcal{C}(Q)$ . Consider  $u \in \mathcal{C}(\overline{Q})$  which satisfies*

$$u_t(x, t) - \sqrt{1 + |\nabla u(x, t)|^2} \cdot \operatorname{div} \left( \frac{\nabla u(x, t)}{\sqrt{1 + |\nabla u(x, t)|^2}} \right) \leq f(x, t)$$

in the viscosity sense. Then the ABP estimate holds, that is

$$(5.10) \quad \sup_Q u \leq \sup_{\partial_p Q} u^+ + \frac{2d \|f^+\|_{L^{n+1}(\Gamma_p^+(u))}}{\sqrt{C^2 d^{\frac{2}{n+1}} - \|f^+\|_{L^{n+1}(\Gamma_p^+(u))}^2}}$$

provided

$$(5.11) \quad \|f^+\|_{L^{n+1}(\Gamma_p^+(u))} < C d^{\frac{1}{n+1}} \quad \text{with } C = (n+1) \cdot |B_1(0)|^{\frac{1}{n+1}}.$$

where  $d = \operatorname{diam}(\Omega)$  and  $\partial_p Q$  denotes the parabolic boundary of  $Q$ .

Analogously, whenever  $u \in \mathcal{C}(\overline{Q})$  is solution of

$$u_t(x, t) - \sqrt{1 + |\nabla u(x, t)|^2} \cdot \operatorname{div} \left( \frac{\nabla u(x, t)}{\sqrt{1 + |\nabla u(x, t)|^2}} \right) \geq f(x, t)$$

in the viscosity sense, we have the following estimate

$$\sup_Q u^- \leq \sup_{\partial_p Q} u^- + \frac{2d \|f^-\|_{L^{n+1}(\Gamma_p^+(-u))}}{\sqrt{C^2 d^{\frac{2}{n+1}} - \|f^-\|_{L^{n+1}(\Gamma_p^+(-u))}^2}}$$

whenever

$$\|f^-\|_{L^{n+1}(\Gamma_p^+(-u))} < C d^{\frac{1}{n+1}}$$

where  $d, C$  are the same constants as before.

*Proof.* Proceeding as in the proof of Theorem 2.8, we assume  $u \in \mathcal{C}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  and we suppose that  $\sup_{\partial_p Q} u^+ \leq 0$ . Let  $(x_0, t_0) \in Q$ , be such that  $M = \sup_Q u = u(x_0, t_0)$ , and consider the nonlinear map,

$$\begin{aligned} \mathcal{F} : \mathbb{R}^{n+1} &\rightarrow \mathbb{R}^{n+1} \\ (\xi, h) &\mapsto \left( \xi \left( 1 + |\xi|^2 \right)^{-\frac{1}{2}}, h \right). \end{aligned}$$

Following the notation and arguments in the proof of Theorem 2.8, we have,

$$(5.12) \quad |\mathcal{F}(D)| \leq \left| \mathcal{F} \left( \Phi \left( \Gamma_{p, \frac{M}{d}}^+(u) \right) \right) \right|.$$

To estimate  $|\mathcal{F}(D)|$  we consider a positive function  $g$  defined by

$$g(h) = \frac{1}{\sqrt{1 + \left(\frac{h}{d}\right)^2}}.$$

We observe that  $0 < g \leq 1$  and then,

(5.13)

$$\begin{aligned} |\mathcal{F}(D)| &\geq \iint_{\mathcal{F}(D)} g(h) d\xi dh = |B_1(0)| \int_0^M \frac{\left(\frac{h}{d}\right)^n}{\left(1 + \left(\frac{h}{d}\right)^2\right)^{\frac{n+1}{2}}} dh \\ &\geq |B_1(0)| \int_{\frac{M}{2}}^M \frac{\left(\frac{h}{d}\right)^n}{\left(1 + \left(\frac{h}{d}\right)^2\right)^{\frac{n+1}{2}}} dh \geq \frac{|B_1(0)|}{2^{n+1}d^n} \frac{M^{n+1}}{\left(1 + \left(\frac{M}{2d}\right)^2\right)^{\frac{n+1}{2}}}. \end{aligned}$$

Now, to estimate  $|\mathcal{F}(\Phi(\Gamma_{p, \frac{M}{d}}^+(u)))|$  we consider another different weight,

$$\eta(\xi) = \frac{1}{\left(1 - |\xi|^2\right)^{\frac{n}{2}}}$$

that satisfies  $\eta(F(\xi)) = (1 + |\xi|^2)^{\frac{n}{2}}$  and  $\eta(\xi) \geq 1$  whenever  $|\xi| < 1$ , which is true for every  $(\xi, h) \in \mathcal{F}(\Phi(\Gamma_{p, \frac{M}{d}}^+(u)))$ . Then,

$$\begin{aligned} \left| \mathcal{F}\left(\Phi\left(\Gamma_{p, \frac{M}{d}}^+(u)\right)\right) \right| &\leq \iint_{\mathcal{F}\left(\Phi\left(\Gamma_{p, \frac{M}{d}}^+(u)\right)\right)} \eta(\xi) d\xi dh \\ &\leq \iint_{\Gamma_{p, \frac{M}{d}}^+(u)} u_t(x, t) \det \left( -\sqrt{1 + |\nabla u|^2} \nabla \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \right) dx dt \\ &\leq \frac{1}{(n+1)^{n+1}} \iint_{\Gamma_{p, \frac{M}{d}}^+(u)} \left( u_t - \sqrt{1 + |\nabla u|^2} \operatorname{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \right)^{n+1} dx dt \\ &\leq \frac{1}{(n+1)^{n+1}} \|f^+\|_{L^{n+1}(\Gamma_{p, \frac{M}{d}}^+(u))}^{n+1}. \end{aligned}$$

The last estimate, together with (5.12) and (5.13) implies,

$$\frac{|B_1(0)|}{2^{n+1}d^n} \left(1 + \left(\frac{M}{2d}\right)^2\right)^{-\frac{n+1}{2}} M^{n+1} \leq \frac{1}{(n+1)^{n+1}} \|f^+\|_{L^{n+1}(\Gamma_p^+(u))}^{n+1}.$$

Then, using (5.11), we get (5.10).  $\square$

Finally, we show an ABP estimate for a parabolic version of the prescribed mean curvature equation which describes mean curvature flow in the  $e_{n+1}$  direction. The proof follows in the same way as that of Theorem 5.4 just setting the weight  $\eta(\xi) = 1$ .

**Theorem 5.5.** *Let  $Q \subset \mathbb{R}^{n+1}$  a bounded domain and  $f \in L^{n+1}(Q)$ . Consider  $u \in \mathcal{C}(\overline{Q})$  which satisfies*

$$u_t(x, t) - \operatorname{div} \left( \frac{\nabla u(x, t)}{\sqrt{1 + |\nabla u(x, t)|^2}} \right) \leq f(x, t)$$

*in the viscosity sense. Then the following ABP-type estimate holds,*

$$(5.14) \quad \sup_Q u \leq \sup_{\partial_p Q} u^+ + \frac{2d \|f^+\|_{L^{n+1}(\Gamma_p^+(u))}}{\sqrt{C^2 d^{\frac{2}{n+1}} - \|f^+\|_{L^{n+1}(\Gamma_p^+(u))}^2}}$$

*provided*

$$(5.15) \quad \|f^+\|_{L^{n+1}(\Gamma_p^+(u))} < C d^{\frac{1}{n+1}} \quad \text{with } C = (n+1) \cdot |B_1(0)|^{\frac{1}{n+1}}$$

*where  $d = \operatorname{diam}(\Omega)$  and  $\partial_p Q$  denotes the parabolic boundary of  $Q$ .*

*Analogously, whenever  $u \in \mathcal{C}(\overline{Q})$  is a solution of*

$$u_t - \operatorname{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \geq f(x, t)$$

*in the viscosity sense, we have the following estimate*

$$\sup_Q u^- \leq \sup_{\partial_p Q} u^- + \frac{2d \|f^-\|_{L^{n+1}(\Gamma_p^+(-u))}}{\sqrt{C^2 d^{\frac{2}{n+1}} - \|f^-\|_{L^{n+1}(\Gamma_p^+(-u))}^2}}$$

*with*

$$\|f^-\|_{L^{n+1}(\Gamma_p^+(-u))} < C d^{\frac{1}{n+1}}$$

*where  $d, C$  are constants defined as before.*

## 6. NON-DIVERGENCE FORM EQUATIONS. AN ABP ESTIMATE FOR THE PRESCRIBED $m$ -CURVATURE EQUATIONS

In this section, we extend our results to non-divergence form equations of the type,

$$S_m(\lambda[DF(\nabla u)]) = f(x) \quad \text{in } \Omega$$

where  $S_m$  is the  $m$ -th symmetric polynomial in  $n$  variables,  $\lambda[A]$  denotes the vector of eigenvalues of a matrix  $A$ , and  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a field as in previous sections (for instance in the hypotheses of Section 2). Some particular examples are the  $m$ -Hessian equations (see [9, 43] and the references therein), obtained taking  $F$  as the identity, the  $p$ - $m$ -Hessian equations (see [44, Section 4]), built on the  $p$ -Laplacian mapping  $F(\xi) = |\xi|^{p-2}\xi$ , or the prescribed  $m$ -curvature equations.

In the particular case of  $m$ -Hessian equations some arguments below can be simplified. More precisely, when  $m > n/2$ , estimates for smooth functions

can be easily extended to continuous functions using the Aleksandrov-type theorem in [12] which asserts that continuous  $m$ -convex functions are twice differentiable almost everywhere, whenever  $m > n/2$ .

For the sake of clarity, we restrict ourselves to the case of the prescribed curvature equations. As in Section 5, let  $S \subset \mathbb{R}^{n+1}$  be a hypersurface given by a smooth graph,  $x_{n+1} = u(x_1, \dots, x_n)$ . Analogously to the problem of the prescribed mean curvature, already considered in Section 5, one could consider prescribing other curvatures of the graph of  $u$ , for instance, the *Gauss curvature*,

$$K(x) = \kappa_1 \cdots \kappa_n = \det \left( \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right) = \frac{\det D^2 u(x)}{(1 + |\nabla u(x)|^2)^{\frac{n+2}{2}}}.$$

This yields the problem of the *prescribed Gauss curvature*,

$$(6.1) \quad \frac{\det D^2 u(x)}{(1 + |\nabla u(x)|^2)^{\frac{n+2}{2}}} = f(x) \quad \text{for all } x \in \Omega.$$

Moreover, if  $n \geq 3$ , we have other functions of the principal curvatures given by the elementary symmetric polynomials, which are called *higher order mean curvatures* or *Weingarten curvatures*. We denote by  $H_m$  the  $m$ th mean curvature which, properly normalized, is given by the  $m$ th symmetric polynomial in  $n$  variables:

$$H_m(x) = \binom{n}{m}^{-1} \cdot S_m(\kappa_1, \dots, \kappa_n) = \binom{n}{m}^{-1} \sum_{1 \leq i_1 < \dots < i_m \leq n} \kappa_{i_1} \cdots \kappa_{i_m}.$$

In particular,  $H_1$  is the Mean curvature,  $H_n$  is the Gauss curvature and  $H_2$  is the *Scalar curvature*. Similarly to (5.3) and (6.1) one can consider the problem of the *prescribed  $m$ -curvature*,

$$(6.2) \quad S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right] \right) = \binom{n}{m} \cdot H_m(x) \quad \text{for all } x \in \Omega,$$

where  $\lambda[A]$  denotes the vector of eigenvalues of a matrix  $A$  (see [9, 10, 27, 39, 40, 41, 42]).

It is important to notice that, while problem (5.3) is always elliptic, this is not the case for problem (6.1) and, more generally, for problem (6.2). This is the reason why we have chosen to give a separate treatment to problem (5.3) in Section 5.

More precisely, problem (6.2) is not elliptic unless we impose an admissibility condition on  $u$ , which we recall next.

For  $m \in \{1, \dots, n\}$ , we define the *admissible set*  $\mathcal{K}_m$  as the component of  $\{\lambda \in \mathbb{R}^n : S_m(\lambda) > 0\} \subset \mathbb{R}^n$ , containing the positive cone  $\mathcal{K}^+ = \{\lambda \in \mathbb{R}^n : \lambda_i > 0, i = 1, \dots, n\}$ . The set  $\mathcal{K}_m$  is a convex cone with vertex at the origin, and is characterized by

$$\mathcal{K}_m = \{\lambda \in \mathbb{R}^n : S_j(\lambda) > 0, \quad j = 1, \dots, m\}.$$

Moreover, the following chain of inclusions holds,

$$\mathcal{K}^+ = \mathcal{K}_n \subset \dots \subset \mathcal{K}_{m+1} \subset \mathcal{K}_m \subset \dots \subset \mathcal{K}_1.$$

Now, we recall the notion of  $m$ -admissibility for  $\mathcal{C}^2$  functions.

**Definition 6.1.** Let  $m \in \{1, \dots, n\}$  and let  $\Omega \subset \mathbb{R}^n$  be a domain. A function  $u \in \mathcal{C}^2$  is called  $m$ -admissible (respectively strictly  $m$ -admissible) in  $\Omega$  if and only if

$$\lambda \left[ \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right] \in \bar{\mathcal{K}}_m \quad (\text{resp. } \mathcal{K}_m)$$

for each  $x \in \Omega$ . Equivalently,  $u$  is  $m$ -admissible in  $\Omega$  if and only if

$$S_j \left( \lambda \left[ \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right] \right) \geq 0 \quad \text{in } \Omega \quad (\text{resp. } > 0)$$

for each  $j = 1, \dots, m$ .

There is a geometrical meaning of the above definition. As the principal curvatures of the graph of  $u$  are defined to be the eigenvalues of (5.1) and  $S_j(\kappa_1, \dots, \kappa_n)$  is the  $j$ th mean curvature of the graph of  $u$  at  $x$  (up to a normalization), a function  $u$  will be  $m$ -admissible provided the  $j$ th mean curvatures of its graph are nonnegative for each  $j = 1, \dots, m$  at every  $x \in \Omega$ . In the particular case  $m = n$ , the notion of  $n$ -admissibility amounts to the usual notion of convexity.

The importance of the notion of  $m$ -admissibility comes from the fact that problem (6.2) is degenerate elliptic for  $m$ -admissible functions (resp. uniformly elliptic for strictly  $m$ -admissible functions), see [9]. The  $m$ -admissibility condition is a key point in existence results.

In existence issues, a corresponding admissibility condition on the domain  $\Omega$  is usually required (see for instance [41]). Namely, a domain  $\Omega$  is  $m$ -admissible whenever the  $j$ th mean curvatures of  $\partial\Omega$  are nonnegative for each  $j = 1, \dots, m$ . However our estimates do not require this condition.

With this considerations, given a bounded domain  $\Omega \subset \mathbb{R}^n$ , our model equation reads:

$$(6.3) \quad \begin{cases} S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right] \right) = f(x) & \text{in } \Omega, \\ u \text{ is } m\text{-admissible in } \Omega. \end{cases}$$

Our aim in this section is to prove an ABP-type estimate for problem (6.3) analogous to Theorem 5.1. We would like to point out that the  $m$ -admissibility of  $u$  in  $\Omega$  implies immediately the bound  $\sup_{\Omega} u \leq \sup_{\partial\Omega} u$  via the Maximum Principle (see for instance [40]). The main point of our estimate is that, building on the ideas in the proof of Theorem 5.1, we do not need to require any  $m$ -admissibility condition from  $u$  (see Remark 6.3).

We state the results for classical solutions and will consider later the approximation question. Notice that the extremal cases  $m = 1$  and  $m = n$  are included.

**Theorem 6.2.** *Let  $m \in \{1, \dots, n\}$ ,  $\Omega \subset \mathbb{R}^n$  a bounded domain and  $f \in \mathcal{C}(\Omega) \cap L^{\frac{n}{m}}(\Omega)$ . Define,*

$$(6.4) \quad C = \binom{n}{m}^{\frac{1}{m}} |B_1(0)|^{\frac{1}{n}}.$$

*Consider  $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$  which satisfies,*

$$(6.5) \quad (-1)^m \cdot S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right] \right) \leq f(x) \quad \text{in } \Omega.$$

*Then, the following ABP-type estimate holds*

$$(6.6) \quad \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + d \frac{\|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(u))}^{\frac{1}{m}}}{\sqrt{C^2 - \|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(u))}^{\frac{2}{m}}}},$$

*for  $d = \text{diam}(\Omega)$ , provided  $\|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(u))} < C^m$ .*

*Analogously, whenever  $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$  satisfies,*

$$(6.7) \quad S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right] \right) \leq f(x) \quad \text{in } \Omega,$$

*we have the following estimate*

$$(6.8) \quad \sup_{\Omega} u^- \leq \sup_{\partial\Omega} u^- + d \frac{\|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(-u))}^{\frac{1}{m}}}{\sqrt{C^2 - \|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(-u))}^{\frac{2}{m}}}},$$

*provided  $\|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(-u))} < C^m$ .*

**Remark 6.3.** 1. Whenever  $m$  is even, (6.5) and (6.7) coincide so we get both estimates (6.6) and (6.8) from the same inequality.

2. In the above result we do not require  $u$  to be  $m$ -admissible. The reason is that in the contact sets  $\Gamma^+(\pm u)$  all the principal curvatures of the graph of  $u$  have the same sign so, in particular,  $u$  is  $m$ -admissible in the contact sets, see (6.12).

3. Notice that when  $u$  is  $m$ -admissible, the proof can be sharpened to obtain the trivial estimate  $\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+$ .

In the estimate of the determinant of (5.1) from above we will use equation (6.2) in combination with the following generalization of the inequality between the arithmetic and geometric means, a matrix version of the well-known Maclaurin inequalities for positive real numbers. We provide the proof in Appendix A.

**Lemma 6.4.** *Let  $A, B$  symmetric positive semidefinite  $n \times n$  matrices. Then, for  $m \in \{1, \dots, n\}$ , we have,*

$$(6.9) \quad \det(AB)^{\frac{1}{n}} \leq \left( \binom{n}{m}^{-1} \cdot S_m(\lambda[AB]) \right)^{\frac{1}{m}}.$$

Now, we proceed with the proof of Theorem 6.2.

*Proof of Theorem 6.2.* We prove (6.6) as the proof of (6.8) is similar. Let  $M = \sup_{\Omega} u - \sup_{\partial\Omega} u^+$  and consider the same mapping as in Theorem 5.1, that is,

$$(6.10) \quad F(\xi) = \frac{\xi}{\sqrt{1 + |\xi|^2}}.$$

Proceeding exactly as in the proof of Theorem 5.1, we get,

$$(6.11) \quad |B_1(0)| \cdot \left( \frac{M}{\sqrt{d^2 + M^2}} \right)^n \leq \int_{\Gamma^+(u)} \left| \det \left( \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right) \right| dx.$$

We observe that, since the function  $u$  is concave in the upper contact set, all its principal curvatures are non-positive. Namely,

$$(6.12) \quad \lambda \left[ \nabla \left( \frac{-\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right] \in \bar{\mathcal{K}}^+ \subset \bar{\mathcal{K}}_m \quad \forall x \in \Gamma^+(u).$$

Hence, using Lemma 6.4 and equation (6.5), we get,

$$\begin{aligned} & \int_{\Gamma^+(u)} \left| \det \left( \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right) \right| dx \\ & \leq \int_{\Gamma^+(u)} \binom{n}{m}^{-\frac{n}{m}} \left( (-1)^m \cdot S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right] \right) \right)^{\frac{n}{m}} dx \\ & \leq \int_{\Gamma^+(u)} \binom{n}{m}^{-\frac{n}{m}} (f^+(x))^{\frac{n}{m}} dx = \binom{n}{m}^{-\frac{n}{m}} \|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(u))}^{\frac{n}{m}}. \end{aligned}$$

From this estimate, and (6.11), we deduce,

$$|B_1(0)| \cdot \left( \frac{M}{\sqrt{d^2 + M^2}} \right)^n \leq \binom{n}{m}^{-\frac{n}{m}} \|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(u))}^{\frac{n}{m}}.$$

Then, using that  $\|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(u))} < C^m$ , with  $C$  as in (6.4), we get (6.6).  $\square$

As in Section 5, we point out that, given the problem of finding a graph  $u$  with prescribed  $m$ -curvature  $H_m(x)$  in  $\Omega$ , that is, (6.2), we read off from Theorem 6.2 the following estimate for the height of the graph. Notice that, again, we do not assume any  $m$ -admissibility of  $u$ .

**Corollary 6.5.** *Let  $\Omega \subset \mathbb{R}^n$  a bounded domain and  $m \in \{1, \dots, n\}$ . Given  $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ , solution of (6.2), the following estimate holds,*

$$\begin{aligned} \inf_{\partial\Omega} (-u^-) - d \frac{\|H_m^+\|_{L^{\frac{n}{m}}(\Gamma^+(-u))}^{\frac{1}{m}}}{\sqrt{|B_1(0)|^{\frac{2}{n}} - \|H_m^+\|_{L^{\frac{n}{m}}(\Gamma^+(-u))}^{\frac{2}{m}}}} &\leq u(x) \\ &\leq \sup_{\partial\Omega} u^+ + d \frac{\|((-1)^m H_m)^+\|_{L^{\frac{n}{m}}(\Gamma^+(u))}^{\frac{1}{m}}}{\sqrt{|B_1(0)|^{\frac{2}{n}} - \|((-1)^m H_m)^+\|_{L^{\frac{n}{m}}(\Gamma^+(u))}^{\frac{2}{m}}}}, \end{aligned}$$

whenever  $\|((-1)^m H_m)^+\|_{L^{\frac{n}{m}}(\Gamma^+(u))}, \|H_m^+\|_{L^{\frac{n}{m}}(\Gamma^+(-u))} < |B_1(0)|^{\frac{m}{n}}$ .

*Remark 6.6.* If  $m$  is even, as in the Scalar Curvature case  $H_2$ , it is enough in order to prove Corollary 6.5 to have less or equal instead of the full equality in (6.2). The other inequality is not used in the proof.

It is possible to find in the literature some results related to our Corollary 6.5 in [40, 41]. However, the main point of our results is that we do not require  $u$  to be  $m$ -admissible, see Remark 6.3. Similar size conditions appear in [40, 42], which, as in the prescribed mean curvature case in Section 5 seems to reflect that  $u$  is given by a graph.

Finally, we extend the results in Theorem 6.2 to continuous functions by means of the notion of viscosity solution. Next, we recall from [41, 42] the definition of viscosity solution of

$$(6.13) \quad S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right] \right) = f(x) \quad \text{in } \Omega.$$

**Definition 6.7.** A function  $u \in \mathcal{C}(\Omega)$  is a viscosity subsolution of (6.13) (resp. supersolution) if for any  $m$ -admissible function  $\phi \in \mathcal{C}^2$  and  $x_0 \in \Omega$  such that  $u - \phi$  has a local maximum at  $x_0$  (resp. minimum), we have

$$S_m \left( \lambda \left[ \nabla \left( \frac{\nabla \phi(x_0)}{\sqrt{1 + |\nabla \phi(x_0)|^2}} \right) \right] \right) \geq f(x_0) \quad (\text{resp. } \leq f(x_0)).$$

A function  $u \in \mathcal{C}(\Omega)$  is a viscosity solution of (6.13) in  $\Omega$  if it is both a sub- and supersolution.

Notice that test functions in Definition 6.7 are required to be  $m$ -admissible since the notion of viscosity solution is based on the ellipticity of the operator and the Maximum Principle. Hence, in order to preserve coherence with the notion of classical solution, we must restrict the set of test functions to those for which the operator is elliptic.

We have the following consequences of the  $m$ -admissibility of test functions in Definition 6.13.

*Remark 6.8.* Let  $u \in \mathcal{C}(\Omega)$  be a viscosity supersolution of (6.13). From the definition we find that  $u$  is never tested on the set  $\{x \in \Omega : f(x) < 0\}$ , since at a point  $x_0 \in \Omega$  where a  $m$ -admissible function touches from below, necessarily  $f(x_0) \geq 0$ .

*Remark 6.9.* Let  $u \in \mathcal{C}(\Omega)$  be a viscosity subsolution of (6.13). Then,  $u$  is a viscosity subsolution of,

$$S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \right] \right) = f^+(x) \quad \text{in } \Omega.$$

In the following result we extend Theorem 6.2 to the viscosity setting. Since in the viscosity framework one has to be careful when multiplying an equation times -1, the presentation differs slightly from the regular case, Theorem 6.2.

**Theorem 6.10.** *Let  $\Omega \subset \mathbb{R}^n$  a bounded domain,  $m \in \{1, \dots, n\}$  and  $f \in \mathcal{C}(\Omega) \cap L^{\frac{n}{m}}(\Omega)$ . Define  $C$  as in (6.4). We distinguish two cases depending on  $m$ :*

1. *If  $m$  is odd and  $u \in \mathcal{C}(\overline{\Omega})$  is a viscosity subsolution of (6.13), then,*

$$(6.14) \quad \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+.$$

*Analogously, whenever  $u \in \mathcal{C}(\overline{\Omega})$  is a viscosity supersolution of (6.13), we have the following estimate,*

$$\sup_{\Omega} u^- \leq \sup_{\partial\Omega} u^- + d \frac{\|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(-u))}^{\frac{1}{m}}}{\sqrt{C^2 - \|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(-u))}^{\frac{2}{m}}}},$$

*for  $d = \text{diam}(\Omega)$ , provided  $\|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(-u))} < C^m$ .*

2. *If  $m$  is even and  $u \in \mathcal{C}(\overline{\Omega})$  is a viscosity supersolution of (6.13), then we have the following estimates,*

$$(6.15) \quad \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + d \frac{\|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(u))}^{\frac{1}{m}}}{\sqrt{C^2 - \|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(u))}^{\frac{2}{m}}}},$$

*provided  $\|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(u))} < C^m$ , and,*

$$\sup_{\Omega} u^- \leq \sup_{\partial\Omega} u^- + d \frac{\|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(-u))}^{\frac{1}{m}}}{\sqrt{C^2 - \|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(-u))}^{\frac{2}{m}}}},$$

*provided  $\|f^+\|_{L^{\frac{n}{m}}(\Gamma^+(-u))} < C^m$ .*

Notice the refinement in estimate (6.14). The reason is that some  $m$ -admissibility of  $u$  is built-in in Definition 6.13 through the test functions. Recall that the same estimate is a direct consequence of the Maximum Principle when  $u \in \mathcal{C}^2$  is  $m$ -admissible itself.

The proof of Theorem 6.10 is based on an double-approximation process as usual. In the following lemma, we recall the equations satisfied by the sup- and inf-convolution of  $u$ . We omit the proof as it is standard.

**Lemma 6.11.** *Let  $u \in C(\Omega)$  be a viscosity subsolution of (6.13). Then,*

$$u^\epsilon(x) = \sup_{y \in \Omega} \left\{ u(y) - \frac{|x - y|^2}{2\epsilon} \right\},$$

*is a viscosity subsolution of*

$$S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u^\epsilon}{\sqrt{1 + |\nabla u^\epsilon|^2}} \right) \right] \right) = \inf_{|x-y|^2 \leq 4\epsilon \|u\|_\infty} f^+(y) \quad \text{in } \Omega_{2(\epsilon \|u\|_\infty)^{1/2}}.$$

*Analogously, if  $u \in C(\Omega)$  is a viscosity supersolution of (6.13) then,*

$$u_\epsilon(x) = \inf_{y \in \Omega} \left\{ u(y) + \frac{|x - y|^2}{2\epsilon} \right\},$$

*is a viscosity supersolution of*

$$S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u_\epsilon}{\sqrt{1 + |\nabla u_\epsilon|^2}} \right) \right] \right) = \sup_{|x-y|^2 \leq 4\epsilon \|u\|_\infty} f(y) \quad \text{in } \Omega_{2(\epsilon \|u\|_\infty)^{1/2}}.$$

We will also need the following simple result in the proof of Theorem 6.10.

**Lemma 6.12.** *Let  $u \in \mathcal{C}^2$  and suppose that  $0 \geq D^2u \geq -\frac{1}{\epsilon}I$ . Let  $x \in \Omega$  such that  $\nabla u(x) \neq 0$ . Then,*

$$0 \geq \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \geq -\frac{1}{\epsilon}I.$$

*Proof.* We have,

$$\nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) = \frac{1}{\sqrt{1 + |\nabla u(x)|^2}} P P^t D^2u(x),$$

with  $P$  the matrix with columns  $\{w_j\}_{j=1,\dots,n}$  for

$$w_1 = (1 + |\nabla u|^2)^{-\frac{1}{2}} \frac{\nabla u}{|\nabla u|},$$

and  $w_j$   $j = 2, \dots, n$  an orthonormal basis of  $\text{span}(\nabla u)^\perp$ . Hence, for any  $\xi \in \mathbb{R}^n \setminus \{0\}$ ,

$$\begin{aligned} \left\langle \nabla \left( \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \xi, \xi \right\rangle &= \frac{1}{\sqrt{1 + |\nabla u(x)|^2}} \langle PP^t D^2 u(x) \xi, \xi \rangle \\ &= \frac{1}{\sqrt{1 + |\nabla u(x)|^2}} \langle Q^{-1} D^2 u(x) Q \eta, \eta \rangle \end{aligned}$$

where  $\eta = P^t \xi$  and  $Q^{-1} = P^t$ . Since  $\eta$  and  $\xi$  are in bijective correspondence and  $Q^{-1} D^2 u Q$  has the same eigenvalues as  $D^2 u$ , we get

$$\begin{aligned} 0 &\geq \frac{1}{\sqrt{1 + |\nabla u(x)|^2}} \langle Q^{-1} D^2 u(x) Q \eta, \eta \rangle \geq -\frac{1}{\epsilon \sqrt{1 + |\nabla u(x)|^2}} |\eta|^2 \\ &= -\frac{1}{\epsilon \sqrt{1 + |\nabla u(x)|^2}} \left( |\xi|^2 - \frac{\langle \nabla u, \xi \rangle^2}{1 + |\nabla u|^2} \right) \geq \frac{-|\xi|^2}{\epsilon}, \end{aligned}$$

and the result follows.  $\square$

Finally, we provide the proof of Theorem 6.10.

*Proof of Theorem 6.10.* We prove (6.14) since the other cases are similar. We assume that  $\sup_\Omega u = u(x_0) > 0$ , since otherwise there is nothing to prove. Observe that

$$R_0 = \frac{\sup_\Omega u - \sup_{\partial\Omega} u}{d}$$

is the maximal slope of a plane that touches  $u$  at an interior point of  $\Omega$ . We can fix  $a > \sup_{\partial\Omega} u^+$ , such that,

$$r_0(u) = \frac{\sup_\Omega u - a}{d}$$

is positive, and then fix  $r < r_0(u)$ . We claim that we can fix a compact set  $G \subset \Omega$ , such that,

$$(6.16) \quad \Gamma_r^+(u) \subset\subset G \subset\subset \Omega,$$

for  $\Gamma_r^+(u)$  defined as in Definition 2.4. In order to prove (6.16), notice that  $\hat{x} \in \Gamma_r^+(u)$  implies that there exists  $\xi \in B_r(0)$  such that  $u(x) \leq u(\hat{x}) + \langle \xi, x - \hat{x} \rangle$  for all  $x \in \Omega$ . Consequently,

$$\sup_\Omega u(x) - u(\hat{x}) \leq |\xi| d < rd.$$

Then,

$$u(\hat{x}) > a + (r_0(u) - r) d > \sup_{\partial\Omega} u^+ + (r_0(u) - r) d.$$

so (6.16) holds.

For  $\epsilon > 0$  let  $u^\epsilon$  be the sup-convolution of  $u$ , defined as,

$$u^\epsilon(x) = \sup_{y \in \Omega} \left\{ u(y) - \frac{|x - y|^2}{2\epsilon} \right\}.$$

By uniform convergence of  $u^\epsilon$  to  $u$ , for any fixed  $\theta \in (0, 1)$ , we have that,

$$r < \theta r_0(u) + (1 - \theta) r < r_0(u^\epsilon)$$

and,

$$\sup_{\partial\Omega} (u^\epsilon)^+ < \theta a + (1 - \theta) \sup_{\partial\Omega} u^+ < a,$$

for  $\epsilon$  small enough (depending on  $\theta$ ). Then, the former argument leads to,

$$u^\epsilon(\hat{x}) > \sup_{\partial\Omega} u^+ + \theta(r_0(u) - r)d$$

for  $\epsilon$  small enough. Hence,  $\Gamma_r^+(u^\epsilon) \subset\subset G \subset\subset \Omega$  for  $\epsilon < \epsilon_0$  small enough, with  $G$  as before.

Consider now,  $u_\eta^\epsilon$ , a standard mollification of  $u^\epsilon$ . By uniform convergence  $u_\eta^\epsilon \rightarrow u^\epsilon$  as  $\eta \rightarrow 0$ , we can repeat the argument above and find that,

$$\Gamma_r^+(u_\eta^\epsilon) \subset\subset G \subset\subset \Omega_{2\sqrt{\epsilon\|u\|_\infty}}.$$

for  $\epsilon < \epsilon_0$  and  $\eta$  small enough, where  $G$  is the same as before.

We now claim that,

$$B_r(0) \subset \nabla u_\eta^\epsilon(\Gamma_r^+(u_\eta^\epsilon)).$$

Indeed, take  $\xi \in B_r(0)$ , and consider the hyperplane  $l_\xi(x) = h + \langle \xi, x \rangle$  where  $h = \sup_{y \in \Omega} \{u_\eta^\epsilon(y) - \langle \xi, y \rangle\}$  (the Legendre transform of  $x$ ). Then,  $u_\eta^\epsilon(x) \leq l_\xi(x)$  in  $\Omega$  and  $u_\eta^\epsilon(z) = l_\xi(z)$  for some  $z \in \bar{\Omega}$ . We aim to prove that  $z \in \Omega$  so, suppose to the contrary that  $z \in \partial\Omega$ . We have that,

$$u_\eta^\epsilon(x_0) \leq l_\xi(x_0) = l_\xi(z) + \langle \xi, x_0 - z \rangle = u_\eta^\epsilon(z) + \langle \xi, x_0 - z \rangle < a + rd,$$

and by uniform convergence,

$$\sup_{\Omega} u = u(x_0) \leq a + rd < a + r_0(u)d = \sup_{\Omega} u,$$

so the claim is proved.

Clearly  $F(B_r(0)) \subset F(\nabla u_\eta^\epsilon(\Gamma_r^+(u_\eta^\epsilon)))$  and consequently

$$|F(B_r(0))| \leq |F(\nabla u_\eta^\epsilon(\Gamma_r^+(u_\eta^\epsilon)))|.$$

Now, since  $u_\eta^\epsilon \in \mathcal{C}^2$ , we get,

$$\begin{aligned} \left(\frac{r}{\sqrt{1+r^2}}\right)^n |B_1(0)| &= |F(B_r(0))| \leq |F(\nabla u_\eta^\epsilon(\Gamma_r^+(u_\eta^\epsilon)))| \\ &\leq \int_{\Gamma_r^+(u_\eta^\epsilon)} |\det(DF(\nabla u_\eta^\epsilon))| dx \\ &= \int_{\Gamma_r^+(u_\eta^\epsilon)} \left| \det \left( \nabla \left( \frac{\nabla u_\eta^\epsilon}{\sqrt{1+|\nabla u_\eta^\epsilon|^2}} \right) \right) \right| dx. \end{aligned}$$

Moreover,

$$\lambda \left[ \nabla \left( \frac{-\nabla u_\eta^\epsilon}{\sqrt{1 + |\nabla u_\eta^\epsilon|^2}} \right) \right] \in \overline{\mathcal{K}}^+ \subset \overline{\mathcal{K}}_m \quad \text{for every } x \in \Gamma_r^+(u_\eta^\epsilon),$$

since  $u_\eta^\epsilon$  is a concave function in  $\Gamma_r^+(u_\eta^\epsilon) \subset \Gamma^+(u_\eta^\epsilon)$ . Then, using Lemma 6.4, we get,

(6.17)

$$\begin{aligned} \left( \frac{r}{\sqrt{1 + r^2}} \right)^n |B_1(0)| &\leq \int_{\Gamma_r^+(u_\eta^\epsilon)} \det \left( \nabla \left( \frac{-\nabla u_\eta^\epsilon}{\sqrt{1 + |\nabla u_\eta^\epsilon|^2}} \right) \right) dx \\ &\leq \int_{\Gamma_r^+(u_\eta^\epsilon)} \binom{n}{m}^{-\frac{n}{m}} \left( (-1)^m S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u_\eta^\epsilon}{\sqrt{1 + |\nabla u_\eta^\epsilon|^2}} \right) \right] \right) \right)^{\frac{n}{m}} dx. \end{aligned}$$

Lemma 6.12 implies that

$$0 \geq \lambda \left[ \nabla \left( \frac{-\nabla u_\eta^\epsilon}{\sqrt{1 + |\nabla u_\eta^\epsilon|^2}} \right) \right] \geq -\frac{1}{\epsilon} \quad \text{on } \Gamma_r^+(u_\eta^\epsilon),$$

and by the Dominated Convergence Theorem, we can pass to the limit in  $\eta$  in (6.17), we obtain,

$$\begin{aligned} \left( \frac{r}{\sqrt{1 + r^2}} \right)^n |B_1(0)| &\leq \int_G \binom{n}{m}^{-\frac{n}{m}} \left( (-1)^m S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u^\epsilon}{\sqrt{1 + |\nabla u^\epsilon|^2}} \right) \right] \right) \right)^{\frac{n}{m}} dx. \end{aligned}$$

By Lemma 6.11, we have,

$$(-1)^m S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u^\epsilon}{\sqrt{1 + |\nabla u^\epsilon|^2}} \right) \right] \right) \leq 0 \quad \text{a.e. in } G,$$

so we can pass to the limit as  $\epsilon \rightarrow 0$  and finally we get (6.14).  $\square$

We conclude proving parabolic counterparts of the elliptic estimates in this Section, including *flow by higher mean curvatures*, combining the arguments in Section 5 with suitable “parabolic versions” of Lemma 6.4, whose proofs can be found in the Appendix.

**Lemma 6.13.** *Let  $A, B$  symmetric positive semidefinite  $n \times n$  matrices and  $z$  a non-negative real number. Then, for  $m \in \{1, \dots, n\}$ , we have,*

$$z \cdot \det(AB) \leq C \cdot \left( z + S_m(\lambda[AB]) \right)^{1 + \frac{n}{m}}$$

with,

$$C = \left( \frac{n-1}{m-1} \right)^{-\frac{n}{m}} \left( \frac{m}{m+n} \right)^{\frac{m+n}{m}}.$$

Next we provide the corresponding parabolic estimates. The proof follows as in Theorem 5.3 choosing,

$$g(h) = \left( \frac{h}{d} \right)^{m-1} \left( 1 + \left( \frac{h}{d} \right)^2 \right)^{-\frac{m}{2}},$$

and using Lemma 6.13 instead of Lemma 6.4.

**Theorem 6.14.** *Let  $m \in \{1, \dots, n\}$ ,  $Q \subset \mathbb{R}^{n+1}$  a bounded domain and  $f \in \mathcal{C}(Q) \cap L^{1+\frac{n}{m}}(Q)$ . Define,*

$$C = \left( \frac{n-1}{m-1} \right)^{\frac{n}{m(n+m)}} \left( \frac{m}{n+m} \right)^{-\frac{1}{m}} |B_1(0)|^{\frac{1}{n+m}}.$$

*Consider  $u \in \mathcal{C}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  which satisfies,*

$$u_t + (-1)^m \cdot S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u(x, t)}{\sqrt{1 + |\nabla u(x, t)|^2}} \right) \right] \right) \leq f(x, t) \quad \text{in } Q.$$

*Then, the following ABP-type estimate holds,*

$$\sup_Q u \leq \sup_{\partial_p Q} u^+ + 2d \frac{\|f^+\|_{L^{1+\frac{n}{m}}(\Gamma_p^+(u))}^{\frac{1}{m}}}{\sqrt{C^2 d^{\frac{2}{n+m}} - \|f^+\|_{L^{1+\frac{n}{m}}(\Gamma_p^+(u))}^{\frac{2}{m}}}},$$

*for  $d = \text{diam}(Q)$ , provided  $\|f^+\|_{L^{1+\frac{n}{m}}(\Gamma_p^+(u))} < C^m d^{\frac{m}{n+m}}$  where  $d = \text{diam}(\Omega)$ .*

*Analogously, whenever  $u \in \mathcal{C}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  satisfies,*

$$u_t - S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u(x, t)}{\sqrt{1 + |\nabla u(x, t)|^2}} \right) \right] \right) \leq f(x, t) \quad \text{in } Q,$$

*we have the following estimate*

$$\sup_Q u^- \leq \sup_{\partial_p Q} u^- + 2d \frac{\|f^+\|_{L^{1+\frac{n}{m}}(\Gamma_p^+(-u))}^{\frac{1}{m}}}{\sqrt{C^2 d^{\frac{2}{n+m}} - \|f^+\|_{L^{1+\frac{n}{m}}(\Gamma_p^+(-u))}^{\frac{2}{m}}}},$$

*provided  $\|f^+\|_{L^{1+\frac{n}{m}}(\Gamma_p^+(u))} < C^m d^{\frac{m}{n+m}}$ .*

Finally, we provide a variant of Lemma 6.13 with homogeneity 1 in the spatial part, as well as the corresponding parabolic estimates.

**Lemma 6.15.** *Let  $A, B$  symmetric positive semidefinite  $n \times n$  matrices and  $z$  a non-negative real number. Then, for  $m \in \{1, \dots, n\}$ , we have,*

$$z \cdot \det(AB) \leq C \cdot \left( z + S_m(\lambda[AB])^{\frac{1}{m}} \right)^{n+1}$$

with,

$$C = \binom{n}{m}^{-n} \frac{n^n}{(n+1)^{n+1}}.$$

**Theorem 6.16.** *Let  $m \in \{1, \dots, n\}$ ,  $Q \subset \mathbb{R}^{n+1}$  a bounded domain and  $f \in \mathcal{C}(Q) \cap L^{n+1}(Q)$ . Define*

$$C = (n+1) |B_1(0)|^{\frac{1}{n+1}} \binom{n}{m}^{\frac{n}{n+1}} n^{-\frac{n}{n+1}}.$$

Consider  $u \in \mathcal{C}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  which satisfies

$$u_t - S_m \left( \lambda \left[ \nabla \left( \frac{\nabla u(x, t)}{\sqrt{1 + |\nabla u(x, t)|^2}} \right) \right] \right)^{\frac{1}{m}} \leq f(x, t) \text{ in } Q.$$

Then, the following ABP-type estimate holds

$$\sup_Q u \leq \sup_{\partial_p Q} u^+ + \frac{2d \|f^+\|_{L^{n+1}(\Gamma_p^+(u))}}{\sqrt{C^2 d^{\frac{2}{n+1}} - \|f^+\|_{L^{n+1}(\Gamma_p^+(u))}^2}}$$

for  $d = \text{diam}(\Omega)$ , provided

$$\|f^+\|_{L^{n+1}(\Gamma_p^+(u))} \leq C d^{\frac{1}{n+1}}.$$

#### APPENDIX A. PROOF OF LEMMAS 6.4, 6.13 AND 6.15

In this appendix, we prove Lemmas 6.4, 6.13 and 6.15. First, let us introduce some notation. Let  $1 \leq m \leq n$  and denote  $\{n_m\}$  the collection of all subsets of  $m$  elements chosen from the set  $1, \dots, n$ . The number of elements in  $\{n_m\}$  is  $\binom{n}{m}$ .

Let  $\sigma \in \{n_m\}$ . If in the matrix  $A$  all rows and columns are deleted, *except* those whose indices belong to  $\sigma$ , then the remaining  $m \times m$  principal minor will be denoted  $(A)_\sigma$ .  $[A]_\sigma$  denotes the determinant of  $(A)_\sigma$ .

Consider the following expansions of the characteristic polynomial of  $A$ ,

$$\begin{aligned} \sum_{m=0}^n (-1)^m S_m(\lambda[A]) \lambda^{n-m} &= \det(\lambda I - A) \\ (A.1) \qquad \qquad \qquad &= \sum_{m=0}^n \sum_{\sigma \in \{n_m\}} (-1)^m [A]_\sigma \lambda^{n-m}, \end{aligned}$$

Identifying powers of  $\lambda$ , we get an useful characterization of  $S_m$ ,

$$(A.2) \quad S_m(\lambda[A]) = \sum_{\sigma \in \{n_m\}} [A]_\sigma.$$

Moreover, it can be read from either (A.1) or (A.2) that  $S_m$  is invariant by orthogonal transformations, that is  $S_m(\lambda[A]) = S_m(\lambda[PAP^t])$  with  $PP^t = P^tP = I$ .

*Proof of Lemma 6.4.* Since the matrix  $A$  is symmetric, it can be written as  $A = PJP^t$  with  $P$  orthogonal and  $J$  a diagonal matrix. From the invariance of  $S_m$  by orthogonal matrices we get

$$S_m(\lambda[AB]) = S_m(\lambda[JP^tBP]) = \sum_{\sigma \in \{n_m\}} [J\tilde{B}]_\sigma = \sum_{\sigma \in \{n_m\}} \left( \prod_{i \in \sigma} \lambda_i \right) [\tilde{B}]_\sigma$$

where  $\tilde{B} = P^tBP$  symmetric and with the same eigenvalues as  $B$ , and  $\{\lambda_i\}_{1 \leq i \leq n}$  the eigenvalues of  $A$ . Since  $B$  is positive semi-definite, all the minors  $[\tilde{B}]_\sigma$  are non-negative.

Now, from the arithmetic-geometric mean inequality for positive numbers, we obtain,

$$(A.3) \quad \begin{aligned} \sum_{\sigma \in \{n_m\}} \left( \prod_{i \in \sigma} \lambda_i \right) [\tilde{B}]_\sigma &\geq \binom{n}{m} \left( \prod_{\sigma \in \{n_m\}} \left( \prod_{i \in \sigma} \lambda_i \right) [\tilde{B}]_\sigma \right)^{\frac{1}{\binom{n}{m}}} \\ &= \binom{n}{m} \left( \det(A)^{\binom{n-1}{m-1}} \prod_{\sigma \in \{n_m\}} [\tilde{B}]_\sigma \right)^{\frac{1}{\binom{n}{m}}}. \end{aligned}$$

Now, we consider two cases, either if  $\det(B) > 0$  or  $\det(B) = 0$ . Suppose first that  $\det(B) > 0$  (and hence  $B$  is actually positive definite). Then, from the Szász inequality (see [36]) we get,

$$\prod_{\sigma \in \{n_m\}} [\tilde{B}]_\sigma \geq \det(\tilde{B})^{\binom{n-1}{m-1}}.$$

Hence, we conclude from (A.3) that,

$$\sum_{\sigma \in \{n_m\}} \left( \prod_{i \in \sigma} \lambda_i \right) [\tilde{B}]_\sigma \geq \binom{n}{m} \cdot \det(AB)^{\frac{\binom{n-1}{m-1}}{\binom{n}{m}}} = \binom{n}{m} \cdot \det(AB)^{\frac{m}{n}},$$

and (6.9) is proved.

The case  $\det(B) = 0$  is easier. As all the minors  $[\tilde{B}]_\sigma$  are non-negative,

$$\prod_{\sigma \in \{n_m\}} [\tilde{B}]_\sigma \geq 0 = \det(\tilde{B})^{\binom{n-1}{m-1}},$$

and the conclusion follows.  $\square$

Next, we provide the proof of Lemma 6.13, since Lemma 6.15 follows in a similar way.

*Proof of Lemma 6.13.* Let  $p > 1$  and  $q$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . By Lemma 6.4, we have,

$$\frac{z}{p} + \frac{1}{q} \cdot \frac{S_m(\lambda[AB])}{\binom{n}{m}} \geq \frac{z}{p} + \frac{1}{q} \cdot \det(AB)^{\frac{m}{n}}.$$

Then, from the elementary inequality,

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q},$$

which holds for  $a, b \geq 0$  and  $p, q$  as above, we have that,

$$\frac{z}{p} + \frac{1}{q} \cdot \frac{S_m(\lambda[AB])}{\binom{n}{m}} \geq z^{\frac{1}{p}} \cdot \det(AB)^{\frac{m}{nq}}.$$

With the particular choice  $p = 1 + \frac{n}{m} \geq 2$ , we have,

$$z \cdot \det(AB) \leq \left( \frac{m}{m+n} \right)^{\frac{m+n}{m}} \cdot \left( z + \binom{n-1}{m-1}^{-1} \cdot S_m(\lambda[AB]) \right)^{1+\frac{n}{m}}.$$

Finally, applying this estimate to  $\alpha A$  and  $B$ , with  $\alpha = \binom{n-1}{m-1}^{\frac{1}{m}}$ , we get the result.  $\square$

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