MULTIPLICITY OF SOLUTIONS TO UNIFORMLY ELLIPTIC FULLY NONLINEAR EQUATIONS WITH CONCAVE-CONVEX RIGHT HAND SIDE

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ABSTRACT. We deal with existence, nonexistence and multiplicity of solutions to the model problem

$$\text{(P)}\qquad \begin{cases} F(\nabla u_\lambda, D^2 u_\lambda) &= \lambda u_\lambda^q + u_\lambda^r, \quad \text{in } \Omega,\\ u_\lambda &> 0, \quad \text{in } \Omega,\\ u_\lambda &= 0, \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, F is a 1-homogeneous fully nonlinear uniformly elliptic operator, $\lambda > 0$, $0 < q < 1 < r < \hat{r}$ and $\hat{r}$ the critical exponent in a sense to be made precise.

We set up a general framework for F in which there exists a positive threshold Λ for existence and nonexistence. Moreover a result on multiplicity is obtained for $0 < \lambda < \Lambda$.

The main difficulty comes from the viscosity setting required for this kind of operators. We also use some degree-theoretic arguments.

The abstract result is applied to several examples, including Pucci's extremal operators, concave (convex) operators and a class of Isaacs operators.

Dedicated to Luis A. Caffarelli in his 60th birthday, with our admiration and friendship.

1. INTRODUCTION

Let Ω be a smooth bounded domain, $0 < q < 1 < r$ and $\lambda > 0$, and consider $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ satisfying the following structural hypothesis,

$(F1)$ *Uniform ellipticity:* There exist constants $0 < \theta \le \Theta$ such that for all $X, Y \in S^n$ with $Y \ge 0$,

$$-\Theta \operatorname{trace}(Y) \le F(\xi, X+Y) - F(\xi, X) \le -\theta \operatorname{trace}(Y)$$

for every $\xi \in \mathbb{R}^n$.

$(F2)$ *H*omogeneity: $F(t\xi, tX) = t \cdot F(\xi, X)$ for all $t > 0$. We further assume $F(0,0) = 0$.

$(F3)$ *Structure condition:* There exists $\gamma > 0$ such that, for all $X, Y \in S^n$, and $\xi_1, \xi_2 \in \mathbb{R}^n$,

$$\mathcal{P}^-_{\theta,\Theta}(X-Y) - \gamma\,|\xi_1 - \xi_2| \le F(\xi_1, X) - F(\xi_2, Y) \le \mathcal{P}^+_{\theta,\Theta}(X-Y) + \gamma\,|\xi_1 - \xi_2|,$$

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where $\mathcal{P}^{\pm}_{\theta,\Theta}$ are the extremal Pucci operators defined as

$$\mathcal{P}^{+}_{\theta,\Theta}(X) = -\theta \sum_{\lambda_i>0} \lambda_i(X) - \Theta \sum_{\lambda_i<0} \lambda_i(X),$$
$$\mathcal{P}^{-}_{\theta,\Theta}(X) = -\Theta \sum_{\lambda_i>0} \lambda_i(X) - \theta \sum_{\lambda_i<0} \lambda_i(X),$$

with $\lambda_i(X)$, $i = 1, \dots n$, the eigenvalues of X. Indeed,

$$\mathcal{P}^{-}_{\theta,\Theta}(X) = \inf_{A\in\mathcal{A}_{\theta,\Theta}} \{-\mathrm{trace}\,(AX)\}, \qquad \mathcal{P}^{+}_{\theta,\Theta}(X) = \sup_{A\in\mathcal{A}_{\theta,\Theta}} \{-\mathrm{trace}\,(AX)\}$$

for $\mathcal{A}_{\theta,\Theta} = \left\{A \in S^n : \ \theta|\xi|^2 \le \langle A\xi, \xi\rangle \le \Theta|\xi|^2 \ \ \forall \xi \in \mathbb{R}^n\right\}$. Notice that the structure condition (F3) amounts to uniform ellipticity when $\xi_1 = \xi_2$.

Remark 1. The results we are going to quote from [10] apply to our framework with their proofs unchanged, since they are based on the ABP estimate, also available in our problem by hypotheses (F1) and (F3), see for instance [12].

Assuming $(F1)$, $(F2)$, $(F3)$, in [13] was established that there is a number $\Lambda \in (0, \infty)$ such that the problem

$$\left\{\begin{array}{rcl} F(\nabla u_\lambda, D^2 u_\lambda) & = & \lambda u_\lambda^q + u_\lambda^r, \quad \text{in } \Omega, \\ u_\lambda & > & 0, \quad \text{in } \Omega, \\ u_\lambda & = & 0, \quad \text{on } \partial\Omega, \end{array}\right. \tag{1}$$

with $0 < q < 1 < r$ has at least one nontrivial viscosity solution for $\lambda < \Lambda$ and no nontrivial solution for $\lambda > \Lambda$. Notice that the result in [13] holds without any restriction on the size of r.

Definition 2. Given $F : \mathbb{R}^n \times S^n \to \mathbb{R}$, define $G(X) = F(0, X)$. We will say that the operator $G : S^n \to \mathbb{R}$ *s*atisfies a Liouville-type result in $\mathbb{R}^n$ up to s whenever $v \equiv 0$ is the unique non-negative viscosity solution of

$$G(D^2 v) = v^r, \quad \text{in } \mathbb{R}^n, \tag{2}$$

for any $1 < r < s$. Finally, we define the *critical exponent* for problem (1) as,

$$\hat{r} = \sup\{s \,|\, s \in \mathbb{R} \text{ and } G \text{ satisfies a Liouville-type result in } \mathbb{R}^n \text{ for any } 1 < r < s\,\}.$$

Notice that $(F1)$ and $(F2)$ imply that G is uniformly elliptic with constants $0 < \theta < \Theta$ and 1-homogeneous.

In Section 3 we will see that blow-up arguments lead to the following problems for the operator $G(X)$,

$$G(D^2 v) = v^r \quad \text{and} \quad 0 \le v(x) \le v(0) = 1 \quad \text{in } \mathbb{R}^n, \tag{3}$$

and

$$\left\{\begin{array}{l} G(D^2 v) = v^r, \quad \text{in } \mathbb{R}^n_+ \\ 0 \le v(x) \le v(0, \dots, 0, s) = 1, \quad \text{in } \mathbb{R}^n_+ \\ v = 0, \quad \text{on } \partial\mathbb{R}^n_+, \end{array}\right. \tag{4}$$

for some $s > 0$ and $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$. Consequently, we will refer to G as the *blow-up* operator hereafter.

Our goal in the present work will be to study the existence of a second nontrivial solution for every $\lambda \in (0, \Lambda)$ provided $r < \hat{r}$. The proof involves uniform L^∞ estimates and topological arguments.

We assume the following extra hypotheses on F in order to get the L^∞ estimates.

$(F4)$ $G(Q^t X Q) = G(X)$ where $G(X) = F(0, X)$ as above and $Q \in O(n) = \{Q \in S^n : \ Q \cdot Q^t = Id\}$.

$(F5)$ Problems (3), (4) have no positive solution.

The main result of the paper is the following *abstract* theorem.

Theorem 3. *Consider $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ satisfying $(F1) - (F5)$, and let $0 < q < 1 < r < \hat{r}$. Then, there exist $\Lambda \in \mathbb{R}$, $0 < \Lambda < \infty$ such that the problem*

$$\left\{ \begin{array}{rcl} F(\nabla u_\lambda, D^2 u_\lambda) &=& \lambda u_\lambda^q + u_\lambda^r, \quad in \ \ \Omega, \\ u_\lambda &>& 0, \quad in \ \ \Omega, \\ u_\lambda &=& 0, \quad on \ \ \partial\Omega, \end{array} \right.$$

(i) *has no positive solution for $\lambda > \Lambda$,*
(ii) *has at least one positive solution for $\lambda = \Lambda$,*
(iii) *has at least two positive solutions for every $\lambda \in (0, \Lambda)$.*

In some sense, these results bring to the fully nonlinear framework the well-known results on global existence and multiplicity of solutions in[2] and [3, 20] (and the references therein) for the semilinear and quasilinear setting, respectively.

The proof of Theorem 3 follows some ideas of Ambrosetti et al. in [3] and involves the mentioned uniform L^∞ a-priori estimates.

These kind of bounds are obtained following the blow-up technique of Gidas-Spruck in [21] which leads to a contradiction with $(F5)$ (see also [18], where different techniques involving topological and variational arguments are developed to get a priori uniform L^∞ estimates). As a general fact, it is difficult to verify $(F5)$; in this direction we have the following results.

Theorem 4. *Let v be a nontrivial non-negative bounded (viscosity) solution of*

$$\left\{ \begin{array}{rcl} G(D^2 v) &=& f(v), \quad in \ \ \mathbb{R}^n_+ \\ v &\geq& 0, \quad in \ \ \mathbb{R}^n_+ \\ v &=& 0, \quad on \ \ \partial\mathbb{R}^n_+, \end{array} \right. \tag{5}$$

where f is locally Lipschitz continuous function, $f(0) \geq 0$ and $G : S^n \to \mathbb{R}$ is uniformly elliptic with constants $0 < \theta < \Theta$ and 1-homogeneous. Furthermore, suppose that

$$\begin{gathered} G(Q^t X Q) = G(X) \ for \ Q = (q_{ij})_{1 \leq i,j \leq n}, \ \ a \ matrix \ with \\ q_{ij} = \delta_{ij} \ \ if \ neither \ \ i \ nor \ \ j = n, \ \ and \ q_{ij} = -\delta_{ij} \ \ otherwise. \end{gathered} \tag{6}$$

Then, v is monotonic in the x_n variable:

$$\frac{\partial v}{\partial x_n} > 0 \quad in \ \mathbb{R}^n_+.$$

The non-existence of solutions to problem (4) follows from the above theorem (see Section 4 and [5], [27]).

Concerning (3), the following result for general uniformly elliptic nonlinearities G is proved in [16].

Theorem 5 (Theorem 4.1 in [16])**.** *Let $G : \mathcal{S}^n \to \mathbb{R}$ be a uniformly elliptic operator in the sense that there exist constants $0 < \theta \leq \Theta$ such that for all $X, Y \in S^n$ with $Y \geq 0$,*

$$-\Theta \, trace(Y) \leq G(X + Y) - G(X) \leq -\theta \, trace(Y).$$

Assume $G(0) = 0$ and $\beta = \frac{\theta}{\Theta}(n-1) + 1 > 2$, and let $v \in \mathcal{C}(\mathbb{R}^n)$ be a viscosity solution of

$$\left\{ \begin{array}{rcl} G(D^2 v) &\geq& v^r, \quad in \ \ \mathbb{R}^n, \\ v &\geq& 0, \quad in \ \ \mathbb{R}^n. \end{array} \right.$$

Then, if $0 < r \leq \beta/(\beta - 2)$, we have $v \equiv 0$.

Notice that, even though the exponent $\beta/(\beta-2)$ could not be maximal for a precise nonlinearity G, Theorem 5 provides a lower bound for the critical exponent, valid for the whole class of uniformly elliptic operators. Further restrictions on F allow one to improve the range of exponents. For example, the maximal range for the linear equation is known to be

$$1 < r < 2^* - 1 = \frac{n+2}{n-2},$$

where $2^* = 2n/(n-2)$ is the Sobolev exponent (see [21]).

Remark 6. We would like to stress that the monotonicity property in Theorem 4 holds whenever $r > 1$, hence without restriction on the growth of r. This motivates the fact that the critical exponent in Definition 2 comes from the Liouville result for (3) alone.

Remark 7. Hypotheses $(F4)$, $(F5)$ are only used in the proof of the uniform L^∞ estimates, Proposition 12, in order to carry out the blow-up argument.

The paper is organized as follows. Section 2 is devoted to some preliminary results that will be used in the sequel. Then, in Section 3, the proof of Theorem 3 is given. The blow-up argument is presented there and uniform L^∞ a-priori bounds are stated under hypotheses $(F1)-(F5)$. Next, the existence of a second solution to (1) is proved using a-priori bounds and theoretical degre arguments.

Section 4 is devoted to prove Theorem 4 following [5] and [27].

Finally, Section 5 is devoted to applications of the abstract framework described in Sections 3 and 4. The examples considered fulfill hypotheses $(F1)-(F5)$ and include the model case where G is a Pucci extremal operator (subsection 5.1) –which include the laplacian as a particular case– and then, concave (convex) operators (subsection 5.2) and a class of Isaacs operators, which are neither concave nor convex (subsection 5.3). We impose hypothesis $(F4)$ in all the cases except when G is a Pucci operator, where the condition is built-in.

2. Preliminaries

2.1. **Eigenvalues.** Under hypotheses $(F1)$, $(F2)$ and $(F3)$, Theorem 8 in [6] holds (here $\mathcal{C}^2$ regularity of $\partial\Omega$ is required). Hence, we know that there exists a principal eigenvalue λ_1 for F defined as

$$\lambda_1 = \sup\{\lambda \mid \exists v > 0 \text{ in } \Omega \text{ s.t. } F(\nabla v, D^2 v) \geq \lambda v\}.$$

in the sense that $\lambda_1 < \infty$ and there exists a nontrivial solution (eigenfunction) to

$$\begin{cases} F(\nabla v, D^2 v) = \lambda\, v, & \text{in } \Omega, \\ v > 0, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Moreover, by definition of λ_1, we know that for every $\lambda > \lambda_1$ problem (7) does not have strictly positive solutions.

Other references for the existence of eigenvalues in the fully nonlinear setting are [27], [8] and the references therein.

The existence of such a principal eigenvalue and eigenfuntion is used in the proof of Theorem 3.

2.2. **Hopf's Lemma for uniformly elliptic equations.** We recall here the Hopf boundary lemma. An adaptation of the proof in [22, Section 3.2] can be found in [13]. For further refinements, see [25] (see also [4] and [28]).

Proposition 8 (Hopf's Lemma)**.** *Let Ω be a bounded domain and u a viscosity solution of*

$$F(\nabla u, D^2 u) \leq 0 \quad \text{in } \Omega,$$

where F satisfies $(F1)$, $(F2)$, and $(F3)$. In addition, let $x_0 \in \partial\Omega$ satisfy

i) *$u(x_0) > u(x)$ for all $x \in \Omega$.*
ii) *$\partial\Omega$ satisfies an interior sphere condition at x_0.*

Then, for every nontangential direction ξ pointing into Ω,

$$\lim_{t\to 0^+} \frac{u(x_0 + t\xi) - u(x_0)}{t} < 0.$$

2.3. **A strong comparison principle.** In the following result, we provide a strong comparison principle. The main feature is that, once standard comparison is proved, the new information is fed back in order to get strict comparison.

Proposition 9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and consider $f, g \in \mathcal{C}(\overline{\Omega})$ with $f \leq g$ in Ω, and $f > 0$ in Ω. Consider $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ verifying $(F1)$, $(F2)$ and $(F3)$. Finally, let $u, v \in \mathcal{C}(\overline{\Omega})$ such that*

$$F(\nabla u, D^2 u) \leq f(x), \qquad \text{and} \qquad F(\nabla v, D^2 v) \geq g(x), \quad \text{in } \Omega,$$

in the viscosity sense. Assume $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$. Furthermore, if $f < g$ in Ω, we have $u < v$ in Ω.

Proof. 1. We can assume $v > 0$ in Ω, since adding a constant to both u and v does not affect the problem. Now, let

$$v_\varepsilon(x) = (1 + \epsilon)\, v(x).$$

Indeed, by homogeneity,

$$F(\nabla v_\varepsilon, D^2 v_\varepsilon) \geq (1 + \varepsilon)\, g(x), \quad \text{and } u - v_\varepsilon \leq 0 \text{ on } \partial\Omega, \tag{8}$$

for ε small enough. Now, we argue by contradiction. Suppose that there exists $x_0 \in \overline{\Omega}$ such that

$$(u - v_\varepsilon)(x_0) = \max_{\overline{\Omega}} (u - v_\varepsilon) > 0.$$

Then, (8) implies $x_0 \notin \partial\Omega$. Now define

$$w(x, y) = u(x) - v_\epsilon(y) - \frac{\tau}{2}|x - y|^2$$

and denote (x_τ, y_τ) such that $w(x_\tau, y_\tau) = \max_{\overline{\Omega}\times\overline{\Omega}} w(x, y)$. Such pairs (x_τ, y_τ) satisfy

(1) $\lim_{\tau\to\infty} \tau\, |x_\tau - y_\tau|^2 = 0.$
(2) $\lim_{\tau\to\infty} w(x_\tau, y_\tau) = w(\hat{x}, \hat{x}) = \max_{\overline{\Omega}} (u - v)$, whenever $(\hat{x}, \hat{x})$ is an accumulation point of (x_τ, y_τ).

Properties (1) and (2) are well known and can be found, for example, in Lemma 3.1 in [15].

Hence, we can assume $x_\tau, y_\tau \to x_0$ as $\tau \to \infty$ hereafter without loss of generality. As a consequence, $x_\tau, y_\tau \in \Omega$ for every τ large enough; applying the Maximum Principle for semicontinuous functions (see for instance, [14] and [15]), there exist two symmetric matrices X_τ, Y_τ such that

$$\big(\tau(x_\tau - y_\tau), X_\tau\big) \in \overline{J}^{2+} u(x_\tau), \quad \text{and} \quad \big(\tau(x_\tau - y_\tau), Y_\tau\big) \in \overline{J}^{2-} v_\epsilon(y_\tau),$$

and $X_\tau \leq Y_\tau$ in the sense of matrices. By definition of viscosity sub- and supersolutions (see [15]), we have

$$F\big(\tau(x_\tau - y_\tau), X_\tau\big) \leq f(x_\tau),$$

and

$$F\big(\tau(x_\tau - y_\tau), Y_\tau\big) \geq (1+\varepsilon)g(y_\tau) \geq (1+\varepsilon)f(y_\tau).$$

Then, by degenerate ellipticity, we get

$$(1+\varepsilon)f(y_\tau) - f(x_\tau) \leq F\big(\tau(x_\tau - y_\tau), Y_\tau\big) - F\big(\tau(x_\tau - y_\tau), X_\tau\big) \leq 0.$$

Letting $\tau \to \infty$, by continuity we arrive at

$$0 < \epsilon \cdot f(x_0) \leq 0,$$

a contradiction. Thus, $u \leq v_\varepsilon$ in Ω, and, letting $\varepsilon \to 0$, we find $u \leq v$ in Ω.

2. We assume in the sequel that $f < g$ in Ω. If $u \neq v$ we have to prove $u < v$ in Ω. Since we have already proved that $u \leq v$ in Ω, suppose to the contrary that there exists $x_0 \in \Omega$ such that $u(x_0) = v(x_0)$, that is, x_0 is a maximum point of $u - v$. Consequently, x_0 is the only maximum point of $u(x) - v(x) - |x - x_0|^4$.

Consider

$$w(x,y) = u(x) - v(y) - |x-x_0|^4 - \frac{\tau}{2}|x-y|^2$$

and (x_τ, y_τ) such that $w(x_\tau, y_\tau) = \max_{\overline{\Omega}\times\overline{\Omega}} w(x,y)$ as before. By properties (1) and (2) above, we have $x_\tau, y_\tau \to x_0$ as $\tau \to \infty$. As a consequence, $x_\tau, y_\tau \in \Omega$ for every τ large enough. Reasoning as above, we can find two symmetric matrices X_τ, Y_τ such that

$$\big(\tau(x_\tau - y_\tau), X_\tau\big) \in \overline{J}^{2+}\left(u(x_\tau) - |x_\tau - x_0|^4\right), \quad \text{and} \quad \big(\tau(x_\tau - y_\tau), Y_\tau\big) \in \overline{J}^{2-} v(y_\tau),$$

and $X_\tau \leq Y_\tau$ in the sense of matrices. As a consequence,

$$\begin{aligned}&\big(\tau(x_\tau - y_\tau) + 4|x_\tau - x_0|^2(x_\tau - x_0),\\ &\qquad X_\tau + 4|x_\tau - x_0|^2 Id + 4(x_\tau - x_0)\otimes(x_\tau - x_0)\big) \in \overline{J}^{2+}u(x_\tau),\end{aligned}$$

By definition of viscosity sub- and supersolution (see [15]), we have

$$\begin{aligned} g(x_\tau) - f(y_\tau) \leq\, & F(\tau(x_\tau - y_\tau), Y_\tau) \\ & - F\big(\tau(x_\tau - y_\tau) + 4|x_\tau - x_0|^2(x_\tau - x_0), \\ & X_\tau + 4|x_\tau - x_0|^2 Id + 4(x_\tau - x_0)\otimes(x_\tau - x_0)\big) \\ \leq\, & \mathcal{P}^-_{\theta,\Theta}(Y_\tau - X_\tau) + O(|x_\tau - x_0|^2) \leq O(|x_\tau - x_0|^2). \end{aligned}$$

as $\tau \to 0$. Letting $\tau \to 0$ we get $0 < g(x_0) - f(x_0) \leq 0$ by hypothesis, and we are done. □

3. The abstract result. Proof of Theorem 3

For the reader's convenience we recall the main result already stated in the introduction.

Theorem 3. *Consider $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ satisfying $(F1) - (F5)$, and let $0 < q < 1 < r < \hat{r}$. Then, there exists $\Lambda \in \mathbb{R}$, $0 < \Lambda < \infty$ such that the problem*

$$\left\{\begin{array}{rcll} F(\nabla u_\lambda, D^2 u_\lambda) &=& \lambda u_\lambda^q + u_\lambda^r, & \text{in } \Omega, \\ u_\lambda &>& 0, & \text{in } \Omega, \\ u_\lambda &=& 0, & \text{on } \partial\Omega, \end{array}\right. \tag{9}$$

(i) *has no positive solution for $\lambda > \Lambda$,*
(ii) *has at least one positive solution for $\lambda = \Lambda$,*
(iii) *has at least two positive solutions for every $\lambda \in (0, \Lambda)$.*

The proof is divided into several steps, organized as subsections.

3.1. **Non-existence for large λ.**

Theorem 10. *For λ large enough, problem* (9) *has no solution in the viscosity sense.*

Proof. Fix $\mu > \lambda_1$ and consider

$$\lambda_0 = \mu^{\frac{r-q}{r-1}}(r-1)\left(\frac{(1-q)^{1-q}}{(r-q)^{r-q}}\right)^{\frac{1}{r-1}}.$$

In order to reach a contradiction, suppose that there exists $\lambda > \lambda_0$ such that the problem (9) has a solution u_λ. Then, we have

$$F(\nabla u_\lambda, D^2 u_\lambda) = \lambda\, u_\lambda^q + u_\lambda^r > \mu u_\lambda \quad \text{in } \Omega, \tag{10}$$

in the viscosity sense. In fact, it is enough to demonstrate that

$$\min_{t\in\mathbb{R}^+} \Phi_\lambda(t) > \mu \quad \text{where} \quad \Phi_\lambda(t) = \lambda\, t^{q-1} + t^{r-1}.$$

It is easy to check that

$$\frac{d}{dt}\Phi_\lambda(t) = 0 \quad \Leftrightarrow \quad t_\lambda = \left(\frac{\lambda(1-q)}{(r-1)}\right)^{\frac{1}{r-q}},$$

which, indeed, is a minimum. Since $\Phi_\lambda(t) \to \infty$ both as $t \to 0$ and $t \to \infty$, it is a global minimum. Then,

$$\Phi_\lambda(t_\lambda) = \lambda^{\frac{r-1}{r-q}} \frac{(r-q)\,(1-q)^{\frac{q-1}{r-q}}}{(r-1)^{\frac{r-1}{r-q}}} > \mu$$

by our selection of λ. On the other hand, define $\psi = \delta\varphi_1$, where φ_1 is a solution of (7). Hopf's Lemma (Proposition 8) implies that there exists $\delta > 0$ such that $\psi \leq u_\lambda$. Then,

$$F(\nabla\psi, D^2\psi) = \lambda_1\,\psi < \mu\, u_\lambda \qquad \text{in } \Omega, \tag{11}$$

by definition of μ.

By construction, we have $0 < \psi \leq u_\lambda$, where ψ and u_λ satisfy (10) and (11). Hence, we can apply the iteration method to get v, satisfying $\psi \leq v \leq u$, a viscosity solution of

$$F(\nabla v, D^2 v) = \mu v.$$

Hence, v is a positive solution of (7), which is a contradiction with the definition of λ_1. □

3.2. **Existence of one solution for** (9) **with** $0 < \lambda \leq \Lambda$**.** In [13] it is proved that there exists a threshold $\Lambda > 0$ such that problem (9) has at least one positive solution for $0 < \lambda < \Lambda$ and no positive solution for $\lambda > \Lambda$. Here we extend the result to the critical value $\lambda = \Lambda$.

Proposition 11. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, and suppose that $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ satisfies $(F1), (F2)$ and $(F3)$. Let $0 < q < 1 < r < \hat{r}$ and $\lambda > 0$. Then there exists a number $\Lambda > 0$ such that problem* (9) *has at least one solution for every $\lambda \in (0, \Lambda]$.*

Proof. Given $\lambda < \Lambda$, let $u_\lambda > 0$ be the solution to (9) found in [13]. As is pointed out in Remark 1, it is possible to follow the arguments in Proposition 4.10 in [10] in order to get Krylov-Safonov $\mathcal{C}^\alpha$ estimates. Namely, there exists a positive constant C such that

$$\|u_\lambda\|_{\mathcal{C}^\alpha(\Omega)} \leq C, \quad \text{uniformly in } \lambda,$$

since the right hand side of the equation in (9) is uniformly bounded by Proposition 12,

$$\lambda\, u_\lambda^q + u_\lambda^r \le \Lambda \|u_\lambda\|_\infty^q + \|u_\lambda\|_\infty^r \le C.$$

Consequently, applying the Arzela-Ascoli Theorem, we can find a sequence $\{u_{\lambda_j}\}$ with $\lambda_j \to \Lambda$ as $j \to \infty$ which converges uniformly to some u_Λ. Such u_{λ_j} are viscosity solutions to problem (9) with $\lambda = \lambda_j$; hence, we can pass to the limit in the viscosity sense to find that u_Λ is a solution to problem (9) with $\lambda = \Lambda$.

In order to prove that $u_\Lambda > 0$, notice that by construction $\|u_{\lambda_j}\|_\infty > c > 0$ uniformly in j, so $\|u_\Lambda\|_\infty > 0$. Then we get $u_\Lambda > 0$ from the weak Harnack inequality (see [10]) □

Thus, statements (i) and (ii) in Theorem 3 are proved. The rest of this section is devoted to prove the existence of a second solution in $(0, \Lambda)$.

3.3. **Existence of a second solution in** $(0, \Lambda)$.

3.3.1. L^∞ *estimates: Blow-up argument.* We present first the blow-up method in [21] adapted to the viscosity setting. We summarize the arguments in the following result.

Proposition 12. *Let* $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ *satisfy* $(F1)-(F5)$ *and let* u *be a nontrivial viscosity solution of problem* (9) *with* $0 < q < 1 < r < \hat{r}$. *Then there exists a constant* $C > 0$ *independent of* λ *and* u *such that* $\|u\|_{L^\infty} \le C$.

For the proof, we proceed by contradiction. Since we aim to prove that $u(x) \le C$ with $C = C(r, \Omega)$ independent of u, suppose to the contrary that there exists a sequence $\{u_k\}_k$ of positive solutions to

$$\left\{ \begin{array}{rcl} F(\nabla u_k, D^2 u_k) &=& \lambda u_k^q + u_k^r, \quad \text{in } \Omega, \\ u_k(x) &=& 0, \quad \text{on } \partial\Omega, \end{array} \right. \tag{12}$$

and a sequence of points $\{z_k\}_k \subset \Omega$ such that

$$M_k = \sup_\Omega u_k = u_k(z_k) \longrightarrow \infty \qquad \text{as } k \to \infty.$$

Without loss of generality we can assume $z_k \to \hat{z} \in \overline{\Omega}$ as $k \to \infty$. There are two cases to be considered, either $\hat{z} \in \Omega$ or $\hat{z} \in \partial\Omega$.

CASE 1: $\hat{z} \in \Omega$. Let $2d = \text{dist}(z, \partial\Omega)$ in the sequel. Now, consider

$$y = \frac{x - z_k}{\mu_k}, \qquad\qquad x = z_k + \mu_k y,$$

where

$$\mu_k^{\frac{2}{r-1}} M_k = 1.$$

We can then define

$$v_k(y) = \mu_k^{\frac{2}{r-1}} u_k(x). \tag{13}$$

Lemma 13. *For* k *large enough, the function* $v_k(y)$ *defined in* (13) *is a viscosity solution of*

$$F\big(\mu_k \nabla_y v_k(y), D_y^2 v_k(y)\big) = \lambda\, \mu_k^{\frac{2(r-q)}{r-1}} v_k^q(y) + v_k^r(y) \qquad \textit{in } B_{d/\mu_k}(0). \tag{14}$$

Proof. In order to prove that v_k is a viscosity solution of (14), we treat first the subsolution case.

Consider $\phi \in \mathcal{C}^2$, $y_0 \in B_{d/\mu_k}(0)$ such that $v_k - \phi$ has a local maximum at y_0. Indeed, we can suppose without loss of generality that ϕ touches v_k from above at y_0, that is,

$$(v_k - \phi)(y) \le (v_k - \phi)(y_0) = 0,$$

for all y in a neighborhood of y_0. Define

$$\Phi(x) = \mu_k^{\frac{-2}{r-1}} \cdot \phi\left(\frac{x - z_k}{\mu_k}\right).$$

Then, Φ touches u_k from above at $x_0 = z_k + \mu_k y_0 \in \Omega$ (it is here where $y_0 \in B_{d/\mu_k}(0)$ plays a role), namely

$$u_k(x_0) = u_k(z_k + \mu_k y_0) = \mu_k^{\frac{-2}{r-1}} v_k(y_0) = \mu_k^{\frac{-2}{r-1}} \phi(y_0) = \Phi(x_0),$$

and

$$u_k(x) = u_k(z_k + \mu_k y) = \mu_k^{\frac{-2}{r-1}} v_k(y) \le \mu_k^{\frac{-2}{r-1}} \phi(y) = \Phi(x),$$

for all x in a neighborhood of x_0.

We can compute the derivatives of $\Phi(x)$ in terms of those of $\phi(y)$

$$\begin{aligned} \nabla_x \Phi(x_0) &= \mu_k^{\frac{-r-1}{r-1}} \nabla_y \phi(y_0), \\ D_x^2 \Phi(x_0) &= \mu_k^{\frac{-2r}{r-1}} D_y^2 \phi(y_0). \end{aligned}$$

Since u_k is a viscosity subsolution of (12), by homogeneity, we get

$$F\big(\mu_k \nabla_y \phi(y_0), D_y^2 \phi(y_0)\big) \le \lambda\, \mu_k^{\frac{2(r-q)}{r-1}} v_k^q(y_0) + v_k^r(y_0),$$

which is what we aimed for. The supersolution case is analogous. □

We can fix $R > 0$ and suppose without loss of generality (taking k large enough) that $B_R(0) \subset B_{d/\mu_k}(0)$.

Our hypotheses on the uniform ellipticity and structure of F imply that

$$\mathcal{P}^-_{\theta,\Theta}(D^2 v_k) - \gamma \mu_k |\nabla v_k| \le \lambda\, \mu_k^{\frac{2(r-q)}{r-1}} v_k^q(y) + v_k^r(y), \quad \text{in } B_{d/\mu_k}(0),$$

and

$$\mathcal{P}^+_{\theta,\Theta}(D^2 v_k) + \gamma \mu_k |\nabla v_k| \ge \lambda\, \mu_k^{\frac{2(r-q)}{r-1}} v_k^q(y) + v_k^r(y), \quad \text{in } B_{d/\mu_k}(0).$$

In particular, since $\|v_k\|_{L^\infty(B_R)} = 1$ by construction, and $\mu_k \to 0$ as $k \to \infty$, we can fix $\varepsilon > 0$ and find for k large enough that

$$\mathcal{P}^-_{\theta,\Theta}(D^2 v_k) - \gamma |\nabla v_k| \le 1 + \varepsilon, \quad \text{in } B_{d/\mu_k}(0),$$

and

$$\mathcal{P}^+_{\theta,\Theta}(D^2 v_k) + \gamma |\nabla v_k| \ge -(1 + \varepsilon), \quad \text{in } B_{d/\mu_k}(0).$$

Hence, from the Harnack inequality (which follows from the ABP estimate, available by $(F3)$, see [10] and [12]), we get uniform $\mathcal{C}^\beta$ estimates (see [10]),

$$\|v_k\|_{\mathcal{C}^\beta(\overline{B}_{R/2})} \le C(n, R, \beta, \gamma, 1 + \varepsilon), \tag{15}$$

for some $0 < \beta < 1$.

Then, we apply the Arzela-Ascoli Theorem and conclude that there exists a subsequence v_{k_j} and a limit function v such that

$$\lim_{k_j \to \infty} v_{k_j} = v, \quad \text{uniformly in } B_R(0), \text{ and } v(0) = 1.$$

Indeed, we can consider any $R_1 > R$ and apply the arguments above to the subsequence v_{k_j} in $B_{R_1}(0)$. Then, we get a new subsequence $v_{k_{j_1}}$ such that

$$\lim_{k_{j_1} \to \infty} v_{k_{j_1}} = v, \quad \text{uniformly in } B_{R_1}(0), \text{ and } v(0) = 1.$$

Notice that, since $\{v_{k_{j_1}}\}_{k_{j_1}} \subset \{v_{k_j}\}_{k_j}$ the limits of both subsequences coincide in $B_R(0)$.

We can consider an increasing sequence of radii $\{R_j\}_j$ and iterate this procedure to get a diagonal subsequence v_k such that

$$\lim_{k\to\infty} v_k = v, \quad \text{uniformly in } B_R(0)\ \forall R>0, \text{ and } v(0)=1. \tag{16}$$

Finally, we take limits in the viscosity sense in (14), which is the content of the following lemma.

Lemma 14. *The limit $v(y)$ in (16) is a viscosity solution of*

$$\begin{cases} G\big(D_y^2 v(y)\big) = v^r(y), & \text{in } \mathbb{R}^n, \\ 0\le v(y)\le v(0)=1, & \text{in } \mathbb{R}^n, \end{cases}$$

where $G(X)=F(0,X)$.

Proof. Consider $\phi\in\mathcal{C}^2$ and y_0 such that $v-\phi$ has a strict local maximum at y_0, that is,

$$(v-\phi)(y) < (v-\phi)(y_0),$$

for all $y\neq y_0$ in a neighborhood of y_0.

Fix $R>0$ such that $y_0\in B_R(0)$. By uniform convergence in compact sets, we deduce that there exist a sequence of points $y_k\to y_0$ as $k\to\infty$ such that $v_k-\phi$ has a local maximum at y_k, that is,

$$(v_k-\phi)(y)\le (v_k-\phi)(y_k),$$

for all $y\neq y_k$ near y_k. Without loss of generality, we can further suppose $y_k\in B_R(0)\subset B_{d/\mu_k}(0)$ for every $k>k_0$.

Then, since v_k is a viscosity solution of (14), we have

$$F\big(\mu_k\nabla\phi(y_k), D^2\phi(y_k)\big)\le \lambda\,\mu_k^{\frac{2(r-q)}{r-1}}\, v_k^q(y_k)+v_k^r(y_k). \tag{17}$$

Letting $k\to\infty$ in (17) we arrive at

$$F\big(0, D^2\phi(y_0)\big)\le v^r(y_0),$$

and we have proved that v is a viscosity subsolution. The supersolution case is symmetric. □

We conclude the argument in this case pointing out that the statement in Lemma 14 contradicts our assumption $(F5)$.

CASE 2: $\hat z\in\partial\Omega$. In this case the reduction argument leads to a problem in either $\mathbb{R}^n$ or a half space $\mathbb{R}^n_+$. Without loss of generality, we can assume that $\hat z=0$. In this way, the tangent space to $\partial\Omega$ at the origin is given by $\langle x,\xi\rangle=0$, for some fixed $\xi\in\mathbb{R}^n$. Moreover, after a rotation, we can suppose that $\xi=(0,...,0,1)$. Consider μ_k defined as in Case 1, see (13), and define the scaled function

$$v_k(y)=\mu_k^{\frac{2}{r-1}} u(z_k'+\mu_k y', z_k^{(n)}+\mu_k y^{(n)})$$

where $z_k=(z_k', z_k^{(n)})$, $y=(y',y^{(n)})$, with $z_k', y'\in\mathbb{R}^{n-1}$ and $z_k^{(n)}, y^{(n)}\in\mathbb{R}$.

Consider $d_k=\big|z_k^{(n)}/\mu_k\big|+o(1)$ as $k\to\infty$, which correspond to the distance from the maximum of v_k to the boundary of $\Omega_k=\frac{1}{\mu_k}(\Omega-(z_k', z_k^{(n)}))$.

Then, we have to consider the following alternatives depending on the behavior of d_k:

(1) $\{d_k\}$ is unbounded. In this case, passing to the limit in a similar way to the Case 1, we arrive to the equation $G\big(D_y^2 v(y)\big) = v^r(y)$ in $\mathbb{R}^n$, with $0\le v(y)\le v(0)=1$, and we reach a contradiction.

(2) $\{d_k\}$ is bounded. We can take a subsequence, if necessary, such that $d_k\to s\ge 0$.

In the second case there are two alternatives to be considered. If $s = 0$ we get a contradiction with the continuity of the limit function v, since on the one hand $v(0) = 1$, and on the other hand $v(y) = 0$ for any $y \in \mathbb{R}^n$ verifying $\langle y, \xi\rangle = 0$, in particular for $y = 0$.

If $s > 0$, we reach the problem

$$\begin{cases} G(D_y^2 v(y)) = v^r(y), \quad v \geq 0, \quad \text{in } \mathbb{R}^n_+, \\ 0 \leq v(y) \leq v(0, ..., 0, s) = 1, \quad \text{in } \mathbb{R}^n_+, \\ v(x', 0) = 0, \quad x' \in \mathbb{R}^{n-1}. \end{cases} \tag{18}$$

Then, by construction, v in (18) has a maximum at $(0, \ldots, 0, s)$. This implies $\nabla v(0, \ldots, 0, s) = 0$ and in particular

$$\frac{\partial v}{\partial x_n}(0, \ldots, 0, s) = 0,$$

which contradicts our assumption $(F5)$ and concludes the proof of Proposition 12.□

3.3.2. *Theoretical degree arguments.* Once L^∞ estimates have been proved, we proceed to the proof of the existence of a second solution using degree theoretic arguments.

Fix $\mu \in (0, \Lambda)$ and consider $0 < \lambda_m < \mu < \lambda_M < \Lambda$. Define $v_{\lambda_M}, w_{\lambda_m}$ to be viscosity solutions to

(19)

$$\begin{cases} F(\nabla v_{\lambda_M}, D^2 v_{\lambda_M}) &= \lambda_M \quad \text{in } \Omega, \\ v_{\lambda_M} &> 0 \quad \text{in } \Omega, \\ v_{\lambda_M} &= 0 \quad \text{on } \partial\Omega. \end{cases} \qquad \begin{cases} F(\nabla w_{\lambda_m}, D^2 w_{\lambda_m}) &= \lambda_m\, d(x), \quad \text{in } \Omega, \\ w_{\lambda_m} &> 0, \quad \text{in } \Omega, \\ w_{\lambda_m} &= 0, \quad \text{on } \partial\Omega, \end{cases}$$

respectively, where $d(x)$ is the normalized distance to the boundary, that is,

$$d(x) = \frac{\operatorname{dist}(x, \partial\Omega)}{\|\operatorname{dist}(\cdot, \partial\Omega)\|_\infty}.$$

It is easy to check (see [13]) that, for $t > 0$ sufficiently small, the function

$$\underline{u} = t\, w_{\lambda_m}$$

is a viscosity solution to

$$\begin{cases} F(\nabla \underline{u}, D^2 \underline{u}) &\leq \lambda_m\, \underline{u}^q + \underline{u}^r, \quad \text{in } \Omega, \\ \underline{u} &> 0, \quad \text{in } \Omega, \\ \underline{u} &= 0, \quad \text{on } \partial\Omega. \end{cases} \tag{20}$$

In addition, there exists $T(\lambda_M) > 0$ such that

$$\overline{u} = T(\lambda_M)\, v_{\lambda_M},$$

is a viscosity solution to

$$\begin{cases} F(\nabla \overline{u}, D^2 \overline{u}) &\geq \lambda_M\, \overline{u}^q + \overline{u}^r, \quad \text{in } \Omega, \\ \overline{u} &> 0, \quad \text{in } \Omega, \\ \overline{u} &= 0, \quad \text{on } \partial\Omega. \end{cases}$$

We can assume without loss of generality that $t < T(\lambda_M)$. Indeed, since in the viscosity sense,

$$F(\nabla \underline{u}, D^2 \underline{u}) \leq t\, \lambda_m < T(\lambda_M)\, \lambda_M = F(\nabla \overline{u}, D^2 \overline{u}),$$

we can apply Proposition 9 in order to get

$$\underline{u} < \overline{u} \quad \text{in } \Omega.$$

We define

$$X := \{v \in \mathcal{C}^1(\Omega) : \; v = 0 \text{ on } \partial\Omega, \; v > 0 \text{ in } \Omega\}$$

endowed with the $\mathcal{C}^1$ topology, and

$$K_\mu(v) := F^{-1}(f_\mu(v)),$$

where $f_\mu(v) = \mu v^q + v^r$ for simplicity.

The main properties of K_μ are the following:

(1) $K_\mu : X \to X$ (see [10] and Remark 1).

(2) $\underline{u} < K_\mu(\underline{u})$ and $\overline{u} > K_\mu(\overline{u})$ in Ω.
Moreover if $v \in K_\mu(\underline{u})$, we have

$$F(\nabla \underline{u}, D^2\underline{u}) \leq \lambda_m\, \underline{u}^q + \underline{u}^r < \mu\, \underline{u}^q + \underline{u}^r = F(\nabla v, D^2 v)$$

in the viscosity sense. By proposition 9, $\underline{u} < v = K_\mu(\underline{u})$ in Ω. The second inequality, $\overline{u} > K_\mu(\overline{u})$, follows in a similar way.

(3) K_μ is compact in $\mathcal{C}^1$ (see [10] and Remark 1).

Now, define $\chi \subset X$ as

$$\chi = \{v \in X : \underline{u} \leq v \leq \overline{u}\}.$$

Indeed, $K_\mu : \chi \to \chi$. To see this, let $v \in \chi$, that is, $v \in X$ such that $\underline{u} \leq v \leq \overline{u}$. We are going to show that $\underline{u} < K_\mu(v) < \overline{u}$. Let $w = K_\mu(v)$; then,

$$F(\nabla \underline{u}, D^2\underline{u}) \leq \lambda_m\, \underline{u}^q + \underline{u}^r < \mu\, v^q + v^r = F(\nabla w, D^2 w).$$

Again, Proposition 9 implies $\underline{u} < w$ in Ω and hence $\underline{u} < K_\mu(v)$. The other inequality follows analogously.

By the $\mathcal{C}^{1,\alpha}$ estimates in [10] (see Remark 1) and the above computations, we see that K_μ is compact. Moreover $\overline{K_\mu(\chi)} \subset \chi$ is a compact set in X. Thus, the Schauder fixed point theorem implies that there exists $u_\mu \in \chi$ such that $K_\mu(u_\mu) = u_\mu$, i.e., a solution to problem (9) with $\lambda = \mu$.

If u_μ is not the unique fixed point in χ we are done. Otherwise, by Proposition 9, it is easy to show that $\underline{u} < u_\mu < \overline{u}$ in Ω. Moreover, we have the following result.

Lemma 15. *There exists $\varepsilon > 0$ such that $u_\mu + \varepsilon \mathcal{B}_1(0) \subset \chi$, where $\mathcal{B}_1(0)$ denotes the unit ball centered at 0 in X.*

Proof. For $\delta > 0$ sufficiently small, we define

$$\Omega_\delta := \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) < \delta\}.$$

Previously we have proved that $\underline{u} < u_\mu < \overline{u}$; then using the $\mathcal{C}^{1,\alpha}$ estimates given in [10] (see also remark 1), there exists a constant $C > 0$ such that $\overline{u}(x) < C\mathrm{dist}(x, \partial\Omega)$ for any $x \in \Omega_\delta$. By Hopf's Boundary Lemma 8 it is easy to conclude the existence of a constant $c > 0$ such that $c\,\mathrm{dist}(x, \partial\Omega) < \underline{u}(x)$ for any $x \in \Omega_\delta$ (for $\delta > 0$ possibly smaller than before, depends on the geometric properties of the domain).

Let $x_0 \in \partial\Omega$ and $\nu_{x_0} = \nu(x_0)$ the outward unitary normal to $\partial\Omega$ at x_0. By the aforementioned $\mathcal{C}^{1,\alpha}$ estimates, we obtain

(21)
$$\exists\, t_0 = t_{x_0} > 0 \quad \text{such that} \quad \overline{u}(x_0 - t\nu_{x_0}) < C\,\mathrm{dist}(x_0 - t\nu_{x_0}, \partial\Omega), \quad \forall\, 0 < t < t_0.$$

Observe that $x_0 \in \partial\Omega$ is arbitrary, so we can cover $\partial\Omega$ by $\displaystyle\bigcup_{x \in \partial\Omega} B_{t_x}(x)$, for t_x defined as in (21). By the compactness of $\partial\Omega$, there exists $m \in \mathbb{N}$, $x_j \in \partial\Omega$ and $t_j = t_{x_j}$, for $j = 1, ..., m$, verifying (21) and such that $\partial\Omega \subset \bigcup_{j=1}^m B_{t_j}(x_j)$. As a consequence, there exists $\tilde{t} > 0$ such that

$$\overline{u}(x - t\nu_x) < C\mathrm{dist}(x - t\nu_x, \partial\Omega), \quad \forall\, 0 < t < \tilde{t} \leq \min_{j=1,\dots,m} t_j \quad \text{and} \quad \forall\, x \in \partial\Omega. \tag{22}$$

Thus, we have proved that $\overline{u}(x) < C\mathrm{dist}(x, \partial\Omega)$ for any $x \in \Omega_{\tilde{t}}$. It is not difficult to show that $\overline{u}(x) < C\mathrm{dist}(x, \partial\Omega)$ for any $x \in \Omega \setminus \Omega_{\tilde{t}}$ with C greater than before if necessary. These arguments prove that there exists a positive constant C verifying

$$\overline{u}(x) < C\text{dist}(x, \partial\Omega), \quad \forall\, x \in \Omega. \tag{23}$$

Arguing in a similar way, using the Hopf Boundary Lemma 8, instead of the gradient estimates, we obtain the existence of a constant $c > 0$ such that

$$c\,\text{dist}(x, \partial\Omega) < \underline{u}(x) \quad \forall\, x \in \Omega. \tag{24}$$

Using (23), (24) and $\mathcal{C}^{1,\alpha}$ estimates, one can interpolate the distance times a small positive constant in the inequalities $\underline{u} < u_\mu < \overline{u}$, in the following sense: there exists $0 < \varepsilon << 1$ such that

$$\underline{u}(x) + \varepsilon\text{dist}(x, \partial\Omega) < u_\mu(x) < \overline{u}(x) - \varepsilon\text{dist}(x, \partial\Omega), \quad \forall\, x \in \Omega. \tag{25}$$

We prove for example the first inequality: $\underline{u}(x) + \varepsilon\text{dist}(x, \partial\Omega) < u_\mu(x)$ for all $x \in \Omega$. Indeed one takes $x_0 \in \partial\Omega$, t_0 and ν_{x_0} as before and defines $w = \underline{u} - u_\mu$.

Claim:

$$\begin{cases} \mathcal{P}^-_{\theta,\Theta}(D^2 w) - \gamma\,|\nabla w| \le 0, & \text{in } \Omega, \\ w(x) < 0 = w(x_0), & \forall x \in \Omega. \end{cases} \tag{26}$$

By Hopf's Lemma (Proposition (8)), there exists $\varepsilon_0 > 0$ such that $w(x_0 - t\nu_{x_0}) < -\varepsilon_0\text{dist}(x_0 - t\nu_{x_0}, \partial\Omega)$ for any $0 < t < t_0$, so by continuity one can take $\delta_0 > 0$ such that $w(x) < -\varepsilon_0\text{dist}(x, \partial\Omega)$ for all $x \in B_{\delta_0}(x_0) \cap \Omega$. Using that $x_0 \in \partial\Omega$ is arbitrary together with compactness arguments as before, we conclude $w(x) < -\varepsilon\text{dist}(x, \partial\Omega)$ for all $x \in \Omega$ for some $\varepsilon > 0$ small enough. By similar arguments we can prove the second inequality in (25).

It only remains to prove the claim. To this end, we adapt the proof method of Theorem 5.3 in [10].

Fix H and H_1 such that $\overline{H}_1 \subset H \subset \overline{H} \subset \Omega$. Let us denote

$$u(x) = \underline{u}(x) \qquad \text{and} \qquad v(x) = u_\mu(x)$$

for simplicity. Consider their sup- and inf-convolutions (see for instance chapter 5 in [10]), respectively,

$$u^\varepsilon(x) = \sup_{y \in \overline{H}} \left\{ u(y) + \varepsilon - \frac{1}{\varepsilon}\,|y - x|^2 \right\}, \qquad \text{for } x \in H,$$

and

$$v_\varepsilon(x) = \inf_{y \in \overline{H}} \left\{ v(y) - \varepsilon + \frac{1}{\varepsilon}\,|y - x|^2 \right\}, \qquad \text{for } x \in H.$$

Indeed, $u^\varepsilon, v_\varepsilon$ are, respectively, viscosity solutions to

$$F(\nabla u^\varepsilon, D^2 u^\varepsilon) \le f_{\lambda_m}\big(u + c^2\,\varepsilon + o(\varepsilon)\big) \qquad \text{in } H_1 \tag{27}$$

and

$$F(\nabla v_\varepsilon, D^2 v_\varepsilon) \ge f_\mu\big(v - c^2\,\varepsilon + o(\varepsilon)\big) \qquad \text{in } H_1 \tag{28}$$

where $c = \max\{\|\nabla u\|_\infty, \|\nabla v\|_\infty\}$ (recall that $u, v \in \mathcal{C}^{1,\alpha}$).

We show the details in the case of u^ε. Let $\phi \in \mathcal{C}^2$, and $\hat{x} \in H$ such that $(u^\varepsilon - \phi)(x) \le (u^\varepsilon - \phi)(\hat{x}) = 0$ for all $x \in B = \{y \in H : |y - \hat{x}| < \text{dist}(\hat{x}^*, \partial H)\}$, with $\hat{x}^* \in \overline{H}$ such that

$$u^\varepsilon(\hat{x}) = \sup_{y \in \overline{H}} \left\{ u(y) + \varepsilon - \frac{1}{\varepsilon}\,|y - \hat{x}|^2 \right\} = u(\hat{x}^*) + \varepsilon - \frac{1}{\varepsilon}\,|\hat{x}^* - \hat{x}|^2.$$

Define $\Phi(y) = \phi(y + \hat{x} - \hat{x}^*) + \frac{1}{\varepsilon}\,|\hat{x}^* - \hat{x}|^2 - \varepsilon$. Then, $u - \Phi$ has a local maximum at $\hat{x}^*$, that is, $(u - \Phi)(y) \le (u - \Phi)(\hat{x}^*) = 0$. Notice that

$$|\hat{x}^* - \hat{x}| \le \|\nabla u\|_\infty \cdot \varepsilon \tag{29}$$

and consequently $\hat{x}^* \in H$ for ε small enough. Since $x \in B$, we then have $y = x - \hat{x} + \hat{x}^* \in H$.

Hence, since u satisfies $F(\nabla u, D^2 u) \le f_{\lambda_m}(u)$ in the viscosity sense, we have by the definition of viscosity solution

$$F(\nabla\Phi(\hat{x}^*), D^2\Phi(\hat{x}^*)) \le f_{\lambda_m}\big(u(\hat{x}^*)\big)$$

and, by the definition of Φ,

$$F(\nabla\phi(\hat{x}), D^2\phi(\hat{x})) \le f_{\lambda_m}\big(u(\hat{x}^*)\big).$$

Finally, since $u \in \mathcal{C}^{1,\alpha}$, its Taylor's expansion and (29) yields

$$u(\hat{x}^*) \le u(\hat{x}) + \|\nabla u\|_\infty \cdot |\hat{x}^* - \hat{x}| + o(|\hat{x}^* - \hat{x}|) \le u(\hat{x}) + c^2\varepsilon + o(\varepsilon).$$

Then we get

$$F(\nabla\phi(\hat{x}), D^2\phi(\hat{x})) \le f_{\lambda_m}\big(u(\hat{x}) + c^2\varepsilon + o(\varepsilon)\big).$$

and (27) is proved.

Continuing with the proof of the claim, we aim to prove that for ε small enough,

$$\mathcal{P}^-_{\theta,\Theta}\big(D^2(u^\varepsilon - v_\varepsilon)\big) - \gamma\,|\nabla(u^\varepsilon - v_\varepsilon)| \le 0 \quad \text{in } H_1, \tag{30}$$

in the viscosity sense. Then, since $H_1 \subset \Omega$ is arbitrary, and $u^\varepsilon - v_\varepsilon$ converges uniformly to $u - v$ (see [10]), we can pass to the limit in the viscosity sense in (30) to get (26).

For $\eta > 0$ small enough, we can fix $\varepsilon_0 > 0$ sufficiently small to ensure that for any $0 < \varepsilon < \varepsilon_0$,

$$f_{\lambda_m}\big(u(x) + c^2\,\varepsilon + o(\varepsilon)\big) - f_\mu\big(v(x) - c^2\,\varepsilon + o(\varepsilon)\big) \le -\eta, \qquad \forall x \in \overline{H} \tag{31}$$

where $f_\lambda(t) = \lambda t^q + t^r$. Clearly, ε_0 depends on $u, v, \nabla u, \nabla v$ and H.

Then, for $\varepsilon < \varepsilon_0$ we consider a paraboloid P touching $u^\varepsilon - v_\varepsilon$ from above at $x_0 \in H_1$. More precisely, we consider P verifying

$$\big((u^\varepsilon - v_\varepsilon) - P\big)(x) \le \big((u^\varepsilon - v_\varepsilon) - P\big)(x_0) = 0, \qquad \forall x \in B_r(x_0),$$

for $r > 0$ to be fixed. We want to prove $\mathcal{P}^-_{\theta,\Theta}\big(D^2P(x_0)\big) - \gamma\,|\nabla P(x_0)| \le 0$. To this end, take $\delta > 0$ and define

$$w(x) = v_\varepsilon(x) - u^\varepsilon(x) + P(x) + \delta\,|x - x_0|^2 - \delta\,r^2.$$

The regularity of u, v, imply

$$|\nabla w(x)| = |\nabla w(x) - \nabla w(x_0)| \le C \cdot |x - x_0|^\alpha < C \cdot r^\alpha \qquad \forall x \in B_r(x_0) \tag{32}$$

for some positive constants α and C depending on u, v, ε^{-1} and P. Hence, since P is fixed, we may assume that r is small enough to fulfill

$$\gamma\,\|D^2P\|\,r < \frac{\eta}{2}, \qquad \text{and} \qquad \gamma\,C\,r^\alpha < \frac{\eta}{2}. \tag{33}$$

as well as $B_{2r}(x_0) \subset H$.

We have $w \ge 0$ on $\partial B_r(x_0)$ and $w(x_0) < 0$. Using (b) in Theorem 5.1 in [10], we know that for any $x \in \overline{B}_r(x_0)$ there exists a convex paraboloid P^x of opening K which touches w from above at x in $B_r(x)$, where K is a constant independent of x.

Define the convex envelope of w in $B_r(x_0)$ as

$$\Gamma_w(x) = \sup_g \big\{g(x) : g \le w \text{ in } B_r(x_0),\ g \text{ convex in } B_r(x_0)\big\}.$$

The set $\{w = \Gamma_w\}$ is usually known as the lower contact set of w. We apply Lemma 3.5 in [10] to w in $B_r(x_0)$ to show that if $x \in \overline{B}_r(x_0) \cap \{w = \Gamma_w\}$, then P^x also touches Γ_w from above at x in $B_r(x)$. Indeed, since $w(x_0) < 0$, the aforementioned lemma yields

$$0 < \int_{B_r(x_0)\cap\{w=\Gamma_w\}} \det D^2\Gamma_w. \tag{34}$$

By (b) in Theorem 5.1 in [10], we know that there exists $A \subset B_r(x_0)$ such that $|B_r(x_0) \setminus A| = 0$, and $u^\varepsilon, v_\varepsilon$ (and hence w) are pointwise second order differentiable in A. In fact, by (c) in Theorem 5.1 in [10], equations (27) and (28) are satisfied pointwise in A.

Since Γ_w is convex and $\Gamma_w \le w$, we have that $D^2w(x) \ge 0$ for $x \in A\cap\{w = \Gamma_w\}$. It follows from (34) and $|B_r(x_0) \setminus A| = 0$ that

$$|\{w = \Gamma_w\} \cap A| > 0,$$

and hence, there is at least one point $x_1 \in \{w = \Gamma_w\} \cap A$. At such a point, we have

$$F\big(\nabla u^\varepsilon(x_1), D^2u^\varepsilon(x_1)\big) \le f_{\lambda_m}\big(u(x_1) + c^2\,\varepsilon + o(\varepsilon)\big),$$
$$F\big(\nabla v_\varepsilon(x_1), D^2v_\varepsilon(x_1)\big) \ge f_\mu\big(v(x_1) - c^2\,\varepsilon + o(\varepsilon)\big),$$

and $D^2w(x_1) \ge 0$. We deduce that

$$\begin{aligned}
&F\big(\nabla u^\varepsilon(x_1), D^2u^\varepsilon(x_1)\big) = \\
&\quad = F\big(\nabla v_\varepsilon(x_1) - \nabla w(x_1) + \nabla P(x_1) + 2\delta(x_1 - x_0), D^2v_\varepsilon(x_1) - D^2w(x_1) + D^2P + 2\delta I\big) \\
&\quad \ge F\big(\nabla v_\varepsilon(x_1) + \nabla P(x_1) + 2\delta(x_1 - x_0), D^2v_\varepsilon(x_1) + D^2P + 2\delta I\big) - \gamma\,|\nabla w(x_1)| \\
&\quad \ge F\big(\nabla v_\varepsilon(x_1), D^2v_\varepsilon(x_1)\big) - \gamma\,|\nabla w(x_1)| \\
&\qquad\quad + \mathcal{P}^-_{\theta,\Theta}(D^2P) - \gamma\,|\nabla P(x_1)| + \mathcal{P}^-_{\theta,\Theta}(2\delta I) - 2\gamma\delta|x_1 - x_0| \\
&\quad \ge F\big(\nabla v_\varepsilon(x_1), D^2v_\varepsilon(x_1)\big) - \gamma\,|\nabla w(x_1)| + \mathcal{P}^-_{\theta,\Theta}(D^2P) - \gamma\,|\nabla P(x_1)| - 2\delta(n\Theta + \gamma r).
\end{aligned}$$

Notice that

$$|\nabla P(x_1)| = \left|\nabla P(x_0) + D^2P\frac{x_1 - x_0}{|x_1 - x_0|}\,|x_1 - x_0|\right| \le |\nabla P(x_0)| + \|D^2P\|\,r$$

All the above inequalities together with (31), (32) and (33) yield,

$$\begin{aligned}
\mathcal{P}^-_{\theta,\Theta}(D^2P) - \gamma\,|\nabla P(x_0)| &\le f_{\lambda_0}\big(u(x_1) + c^2\,\varepsilon + o(\varepsilon)\big) - f_\mu\big(v(x_1) - c^2\,\varepsilon + o(\varepsilon)\big) \\
&\qquad + 2\delta(n\Theta + \gamma r) + \gamma\,\|D^2P\|\,r + \gamma\,Cr^\alpha \\
&\le 2\delta(n\Theta + \gamma r).
\end{aligned}$$

Letting $\delta \to 0$, we complete the proof of the *claim* and hence the Lemma 15. □

To complete the proof of the existence of a second fixed point in χ, we follow the arguments developed in [1] and, more precisely, in [3].

By the permanence and excision properties of degree, we have

$$\deg(I - K_\mu, u_\mu + \varepsilon\mathcal{B}_1(0), 0) = i(K_\mu, u_\mu + \varepsilon\mathcal{B}_1(0), \chi) = i(K_\mu, \chi, \chi) = 1. \tag{35}$$

On the other hand, we recall that problem (9) does not have any positive solution for $\lambda > \Lambda$. Moreover, since Proposition 12 provides uniform L^∞ estimates, the results in [10] yield uniform $\mathcal{C}^{1,\alpha}$ estimates, see also Remark 1. In particular, there exists $C > 0$ independent of λ such that, every $u > 0$ solution to problem (9) satisfies

$$\|u\|_{\mathcal{C}^1} \le C.$$

Take $\rho > C$. Clearly, there are no solution u to problem (9) with $\|u\|_{\mathcal{C}^1} = \rho$. By the homotopy invariance of the Leray-Schauder degree, we get

$$\deg(I - K_\mu, \rho\mathcal{B}_1(0), 0) = \deg(I - K_{\Lambda+\delta}, \rho\mathcal{B}_1(0), 0) = 0.$$

Now, by the excision property and (35) we deduce

$$\begin{aligned}&\deg\big(I-K_\mu,\rho\mathcal{B}_1(0)\setminus\{u_\mu+\varepsilon\mathcal{B}_1(0)\},0\big)\\&\quad=\deg\big(I-K_\mu,\rho\mathcal{B}_1(0),0\big)-\deg\big(I-K_\mu,u_\mu+\varepsilon\mathcal{B}_1(0),0\big)=-1.\end{aligned}$$

Hence, K_μ has another fixed point $\hat{u}_\mu\in\rho\mathcal{B}_1(0)\setminus\{u_\mu+\varepsilon\mathcal{B}_1(0)\}$.

It remains to show that the trivial solution $u=0$ has degree 0, i.e., $\deg\big(I-K_\mu,\varepsilon\mathcal{B}_1(0),0\big)=0$ for any $\varepsilon>0$ sufficiently small.

To this aim, notice that there exists $\tilde{\lambda}$ such that for any $\tau>0$ the problem

$$\left\{\begin{array}{rcl}F(\nabla u,D^2u)&=&\tilde{\lambda}u^q+u^r+\tau,\quad u>0,\quad \text{in }\Omega,\\ u&=&0,\quad \text{on }\partial\Omega\end{array}\right.\tag{36}$$

does not have any positive solution. Indeed, arguing as in the proof of Theorem 3.12 in [13], for a given $\delta>0$ it is easy to find $\tilde{\lambda}>0$ such that

$$\tilde{\lambda}u^q+u^r>(\lambda_1+\delta)u,$$

and hence, obviously,

$$\tilde{\lambda}u^q+u^r+\tau>(\lambda_1+\delta)u,$$

for any $\tau>0$. Following the arguments in the proof of Theorem 3.12 in [13], we get a positive *eigenfunction* associated to $\lambda_1+\delta$, a contradiction with the results on existence of eigenvalues in [6], already discussed in Subsection 2.1.

It follows that the homotopy

$$H(\tau,u)=u-F^{-1}\big(\tilde{\lambda}u^q+u^r+\tau\big)$$

is admissible and hence

$$\deg\big(I-K_{\tilde{\lambda}},\varepsilon\mathcal{B}_1(0),0\big)=\deg\big(H(0,\cdot),\varepsilon\mathcal{B}_1(0),0\big)=\deg\big(H(1,\cdot),\varepsilon\mathcal{B}_1(0),0\big)=0$$

for all $\varepsilon>0$. Then, again by homotopy, one finds

$$\deg\big(I-K_\mu,\varepsilon\mathcal{B}_1(0),0\big)=\deg\big(I-K_{\tilde{\lambda}},\varepsilon\mathcal{B}_1(0),0\big)=0.$$

As a consequence, there exists $\varepsilon>0$ such that $u=0$ is the unique nonnegative solution of (9) in $\varepsilon\mathcal{B}_1(0)$, which yields (iii) and finish the proof of Theorem 3. □

4. A monotonicity property. Proof of Theorem 4

Here we provide the proof of Theorem 4, a monotonicity result in the spirit of Theorem 3.1 in [27] and Corollary 1.3 in [5]. We recall the statement of the Theorem for the reader's convenience.

Theorem 4. *Let v be a nontrivial non-negative bounded (viscosity) solution of*

$$\left\{\begin{array}{rcl}G(D^2v)&=&f(v),\quad \text{in }\mathbb{R}^n_+\\ v&\ge&0,\quad \text{on }\mathbb{R}^n_+\\ v&=&0,\quad \text{in }\partial\mathbb{R}^n_+,\end{array}\right.\tag{37}$$

where f is a locally Lipschitz continuous function, $f(0)\ge 0$ and $G:S^n\to\mathbb{R}$ is uniformly elliptic with constants $0<\theta<\Theta$ and 1-homogeneous. Furthermore, suppose that

$$\begin{gathered}G(Q^tXQ)=G(X)\quad\text{for}\quad Q=(q_{ij})_{1\le i,j\le n}\quad\text{a matrix with}\\ q_{ij}=\delta_{ij}\quad\text{if neither }\ i\ \text{ nor }\ j=n\quad\text{and}\quad q_{ij}=-\delta_{ij}\quad\text{otherwise.}\end{gathered}\tag{38}$$

Then, v is monotonic in the x_n variable:

$$\frac{\partial v}{\partial x_n}>0\quad\text{in }\mathbb{R}^n_+.$$

Before going into the proof, let us recall some well-known general results in the form needed below (see for instance [9]).

Proposition 16 (Strong Maximum Principle). *Let Ω be a regular domain and let v be a non-negative viscosity solution to $\mathcal{P}^+_{\theta,\Theta}(D^2v) \geq c(x)v$ in Ω with $c(x) \in L^\infty$. Then, either v vanishes identically in Ω or $v(x) > 0$ for all $x \in \Omega$. Moreover, in the latter case for any $x_0 \in \partial\Omega$ such that $v(x_0) = 0$,*

$$\limsup_{t \to 0} \frac{v(x_0 - t\nu) - v(x_0)}{t} < 0,$$

where ν is the outer normal to $\partial\Omega$.

Proposition 17 (Maximum Principle in narrow domains). *Suppose that Ω lies between two parallel hyperplanes at a distance d. If d is small enough, depending only on bounds for the coefficient c, v is a viscosity solution to $\mathcal{P}^+_{\theta,\Theta}(D^2v) \geq c(x)v$ and $\liminf_{x \to \partial\Omega} v(x) \geq 0$, then we have $v \geq 0$ in Ω.*

We point out that the symmetry condition (38) is needed in the proof of Theorem 4 in order to be able to apply a *moving plane* method.

Proof of Theorem 4. Let v be a nontrivial solution to (37). By hypothesis, $0 \leq v \leq M$ for some constant M. We can rewrite our equation in the form

$$G(D^2v) - c(x)v \geq f(0) \geq 0, \quad \text{in } \Omega,$$

for

$$c(x) = \begin{cases} \dfrac{f\big(v(x)\big) - f(0)}{v(x)}, & \text{if } v(x) \neq 0, \\ 0, \quad \text{otherwise.} \end{cases}$$

Since v is bounded, and f is locally Lipschitz then $c(x) \in L^\infty$. As a consequence, the strong maximum principle (Proposition 16) yields $v > 0$.

We are going to use the moving plane method as in [27]. For each β, we define, as usual,

$$T_\beta = \{x \in \mathbb{R}^n_+ : \; x_n = \beta\}, \quad \Sigma_\beta = \{x \in \mathbb{R}^n_+ : \; 0 < x_n < \beta\},$$

and the functions

$$v_\beta(x) = v(y, 2\beta - x_n), \quad w_\beta(x) = v_\beta(x) - v(x), \quad x = (y, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_+,$$

defined in Σ_β.

First, we point out that

$$G\big(D^2 v_\beta(x)\big) = f\big(v_\beta(x)\big) \quad \text{in } \Sigma_\beta,$$

in the viscosity sense. Let us consider for example the subsolution case. Take $\phi \in \mathcal{C}^2$ and $x_0 = (y_0, x_n^0) \in \Sigma_\beta$ such that $v_\beta - \phi$ has a local maximum at x_0. Define $\phi_\beta(x) = \phi(y, 2\beta - x_n)$. It is easy to see that $v - \phi_\beta$ has a local maximum at $(y_0, 2\beta - x_n^0)$. Then, $D^2\phi_\beta(y, x_n) = QD^2\phi(y, 2\beta - x_n)Q$ where Q is a matrix with elements $q_{ij} = \delta_{ij}$ if neither i nor $j = n$ and $q_{ij} = -\delta_{ij}$ otherwise. Finally, by definition of v, we get

$$f\big(v_\beta(y_0, x_0)\big) \geq G\big(D^2\phi_\beta(y_0, 2\beta - x_n^0)\big) = G\big(QD^2\phi(y_0, x_n^0)Q\big) = G\big(D^2\phi(y_0, x_n^0)\big) \quad \text{in } \Sigma_\beta,$$

which is what we aimed for.

Next, we have to show that $w_\beta = v_\beta - v$ satisfies

$$\mathcal{P}^+_{\theta,\Theta}\big(D^2 w_\beta(x)\big) \geq c_\beta(x) w_\beta(x), \tag{39}$$

in the viscosity sense, where

$$c_\beta(x) = \begin{cases} \dfrac{f\big(v_\beta(x)\big) - f\big(v(x)\big)}{v_\beta(x) - v(x)}, & \text{if } v_\beta(x) \neq v(x) \\ 0, \quad \text{otherwise.} \end{cases}$$

Notice that, again since f is Lipschitz, we have $c_\beta(x) \in L^\infty$. The proof follows the ideas in [17].

To this aim, let $\phi \in \mathcal{C}^2$ such that $w_\beta - \phi$ has a local minimum at some point $x_0 \in \Sigma_\beta$. In other words, x_0 is a local maximum of $v - v_\beta + \phi$. As usual in the theory of viscosity solutions, introduce for every $\varepsilon > 0$

$$\Phi_\varepsilon(x,y) = v(x) - v_\beta(y) + \phi(x) - \frac{|x-y|^2}{\varepsilon^2} - |x-x_0|^4.$$

For ε small enough, Φ_ε attains a maximum in $\Sigma_\beta \times \Sigma_\beta$ at some point $(x_\varepsilon, y_\varepsilon) \in B_r(x_0) \times B_r(x_0)$ for some $r > 0$. Since x_0 is a local strict maximum of

$$x \mapsto v(x) - v_\beta(x) + \phi(x) - |x-x_0|^4$$

standard results of the theory of viscosity solutions (see [15]) yields $x_\varepsilon, y_\varepsilon \to x_0$ and $\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \to 0$ as $\varepsilon \to 0$.

In addition, defining $\psi(x,y) = -\phi(x) + \frac{|x-y|^2}{\varepsilon^2} + |x-x_0|^4$, the results in [15] imply that for any given $\alpha > 0$, there exist matrices $X, Y \in S^n$ such that

$$\begin{aligned} (\nabla_x \psi(x_\varepsilon, y_\varepsilon), X) &\in \overline{J}^{2,+} v(x_\varepsilon) \\ (-\nabla_y \psi(x_\varepsilon, y_\varepsilon), Y) &\in \overline{J}^{2,-} v_\beta(y_\varepsilon), \end{aligned} \tag{40}$$

and

$$-\left(\frac{1}{\alpha} + \|A\|\right) I \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \alpha A^2,$$

where $A = D^2\psi(x_\varepsilon, y_\varepsilon)$. From there, setting $\alpha = \varepsilon^2$, it is standard to see that

$$X - Y \le -D^2\phi(x_\varepsilon) + O(\varepsilon^2 + |x_\varepsilon - x_0|^2).$$

By definition (see [15]) of viscosity solutions and (40), we get

$$G(X) \le f\big(v(x_\varepsilon)\big) \quad \text{and} \quad G(Y) \ge f\big(v_\beta(y_\varepsilon)\big),$$

and subtracting in the previous inequalities we obtain

$$\begin{aligned} f\big(v_\beta(y_\varepsilon)\big) - f\big(v(x_\varepsilon)\big) &\le G(Y) - G(X) \\ &\le G\big(X + D^2\phi(x_\varepsilon) + O(\varepsilon^2 + |x_\varepsilon - x_0|^2)\big) - G(X) \\ &\le \mathcal{P}^+_{\theta,\Theta}(D^2\phi(x_\varepsilon)) + O(\varepsilon^2 + |x_\varepsilon - x_0|^2). \end{aligned}$$

Letting $\varepsilon \to 0$, we get (39).

Then, $w_\beta \ge 0$ in Σ_β if β is small enough, since $w_\beta \ge 0$ on $\partial\Sigma_\beta$ and hence we can apply the maximum principle in narrow domains (Proposition 17).

We define,

$$\beta^* = \sup\{\beta : \ w_\mu \ge 0 \text{ in } \Sigma_\mu \text{ for all } \mu < \beta\} > 0.$$

Using Hopf's Lemma, we conclude that $w_\beta > 0$ in Σ_β and

$$\frac{\partial v}{\partial x_n} = -\frac{1}{2}\frac{\partial w_\beta}{\partial x_n} > 0 \quad \text{on } T_\beta$$

for every $0 < \beta \le \beta^*$. If we prove that $\beta^* = \infty$, we have finished.

Suppose to the contrary that $\beta^* < \infty$. We can fix ε_0 small such the maximum principle holds for $G(\cdot) - c_\mu(x)$ in $\Sigma_{\beta^*+\varepsilon_0} \setminus \Sigma_{\beta^*-\varepsilon_0}$.

Claim: There exists $\delta_0 \in (0, \varepsilon_0]$ such that for each $\delta \in (0, \delta_0)$ we have $w_{\beta^*+\delta} \ge 0$ in $\Sigma_{\beta^*-\varepsilon_0} \setminus \Sigma_{\varepsilon_0}$.

Once the claim is proven, we can apply the maximum principle in narrow domains to

$$\mathcal{P}^+_{\theta,\Theta}\big(D^2 w_\beta(x)\big) \ge c_\beta(x) w_\beta(x) \qquad \text{in } \Sigma_{\beta^*+\delta} \setminus \Sigma_{\beta^*-\varepsilon_0} \cup \Sigma_{\varepsilon_0},$$

with c_β as before, and conclude that $w_{\beta^*+\delta} \geq 0$ in $\Sigma_{\beta^*+\delta}$, contradicting the maximality of β^*.

Hence, it remains to prove the claim. It follows in a similar way to Lemma 3.1 in [27]. We include the details for the reader's convenience.

Suppose that the claim were false, that is, that there exist sequences $\delta_m \to 0$ and $x^{(m)} = (y^{(m)}, x_n^{(m)}) \in \Sigma_{\beta^*-\varepsilon_0} \setminus \Sigma_{\varepsilon_0}$ such that

$$w_{\beta^*+\delta_m}(x^{(m)}) < 0. \tag{41}$$

We can suppose that $x_n^{(m)} \to x_n^0 \in [\varepsilon_0, \beta^* - \varepsilon_0]$ as $m \to \infty$.

We define the functions

$$v^{(m)}(y, x_n) = v(y + y^{(m)}, x_n)$$

and respectively

$$w_\beta^{(m)}(y, x_n) = v^{(m)}(y, 2\beta - x_n) - v^{(m)}(y, x_n).$$

Notice that

$$G\big(D^2 v^{(m)}\big) = f\big(v^{(m)}(x)\big).$$

in the viscosity sense. Then, it is standard to show (see for instance Proposition 4.11 in [10]) that there exists a subsequence and a limit $\tilde{v} \in \mathcal{C}$ such that $v^{(m)} \to \tilde{v}$ uniformly in compact sets as $m \to \infty$ and

$$G\big(D^2 \tilde{v}\big) = f\big(\tilde{v}(x)\big)$$

in the viscosity sense.

By the strong maximum principle (Proposition 16), we have that either $\tilde{v}$ is strictly positive in $\mathbb{R}^n_+$ or $\tilde{v} \equiv 0$ in $\mathbb{R}^n_+$.

Suppose first that $\tilde{v} > 0$ in $\mathbb{R}^n_+$. By what we have already shown, we know that $w_\beta^{(m)}(y, x_n) = w_\beta(y + y^{(m)}, x_n) > 0$ in Σ_β for all $\beta \leq \beta^*$. Hence the limit function $\tilde{w}^\beta = \lim_{m\to\infty} w_\beta^{(m)}$ is non-negative in Σ_β for all $\beta \leq \beta^*$.

So we can repeat the moving plane argument for $\tilde{v}$ and get $\tilde{\beta}^* \geq \beta^*$, where $\tilde{\beta}^*$ is to $\tilde{v}$ what β^* is to v. Since $\tilde{w}^\beta$ satisfies

$$\mathcal{P}^+_{\theta,\Theta}\big(D^2 \tilde{w}_\beta(x)\big) \geq \tilde{c}_\beta(x) \tilde{w}_\beta(x)$$

we can apply the strong maximum principle and get, as before, that $\tilde{w}_\beta > 0$ in Σ_β for all $\beta \leq \tilde{\beta}^*$. On the other hand, by continuity and (41), we have $\tilde{w}^{\beta^*}(0, x_n^0) = 0$ and $x_n^0 \in (0, \beta^* - \varepsilon_0]$, a contradiction.

Suppose next that $\tilde{v} \equiv 0$ in $\mathbb{R}^n_+$. We fix the rectangular domains

$$Q_1 = \left\{x \in \mathbb{R}^n_+ :\ -1 < x_1 < 1, \dots, -1 < x_{n-1} < 1, \varepsilon_0 < x_n < 2\beta^* + 1\right\},$$
$$Q_2 = \left\{x \in \mathbb{R}^n_+ :\ -2 < x_1 < 2, \dots, -2 < x_{n-1} < 2, \frac{\varepsilon_0}{2} < x_n < 2\beta^* + 2\right\}.$$

Since $v^{(m)}$ converges uniformly to zero in Q_2, we can suppose that $v^{(m)} \leq 1$ in Q_2 for m sufficiently small. We set

$$\alpha_m = v^{(m)}(0, x_n^{(m)}) \quad \text{and} \quad \overline{v}^{(m)} = \frac{v^{(m)}}{\alpha_m}.$$

Now, the function $\overline{v}^{(m)}$ satisfies

$$G\big(D^2 \overline{v}^{(m)}(x)\big) = \frac{f\big(v^{(m)}(x)\big)}{v^{(m)}(x)} \overline{v}^{(m)}(x), \qquad x \in\ Q_2. \tag{42}$$

The Harnack inequality (see Chapter 4 in [10]) implies

$$\sup_{Q_1} \overline{v}^{(m)} \leq C_1 \inf_{Q_1} \overline{v}^{(m)} \leq C_1.$$

Next, we recall that $w^{\beta^*} \geq 0$ in Σ_{β^*}, which implies

$$\overline{v}^{(m)}(y, x_n) \leq \overline{v}^{(m)}(y, 2\beta^* - x_n) \leq C_1, \quad \text{for} \quad (y, x_n) \in \Sigma_{\beta^*}.$$

Thus, $\|\overline{v}^{(m)}\|_{L^\infty(Q)} \leq C_1$, where

$$Q = \left\{x \in \mathbb{R}^n_+ : \ -1 < x_1 < 1, \ldots, -1 < x_{n-1} < 1, 0 < x_n < 2\beta^* + 1\right\},$$

hence, our $\mathcal{C}^\alpha$ estimates yields (up to a subsequence) that $\overline{v}^{(m)} \to \overline{v} \in \mathcal{C}$ uniformly in compact sets, and $\overline{v}$ is a viscosity solution of

$$G\big(D^2\overline{v}\big) \geq l\overline{v},$$

where $l = \lim_{t\to 0} f(t)/t$. By the strong maximum principle, either $\overline{v} \equiv 0$ in Q or $\overline{v} > 0$ in Q. The first possibility is excluded since $\overline{v}(0, x_n^0) = 1$.

We introduce the functions

$$z^\beta(y, x_n) = \overline{v}(y, 2\beta - x_n) - \overline{v}(y, x_n)$$

defined in $\Sigma_\beta \cap \overline{Q}$ for all $\beta \leq \beta^* + 1/2$. We have, by continuity,

$$z^{\beta^*} \geq 0 \quad \text{and} \quad z^{\beta^*}(0, x_n^0) = 0.$$

Since

$$\mathcal{P}^+_{\theta,\Theta}\big(D^2 z^\beta(x)\big) \geq l z^\beta(x),$$

the strong maximum principle, implies $z^{\beta^*} \equiv 0$ in $\Sigma_{\beta^*} \cap \overline{Q}$. This contradicts the fact that $\overline{v} = 0$ on $\{x_n = 0\}$ and $\overline{v} > 0$ on $\{x_n = 2\beta^*\}$. □

5. Applications of Theorem 3

In Section 3 we have developed an abstract framework for the global multiplicity result, Theorem 3. To make it more explicit, we have proved Theorem 3 under hypotheses $(F1) - (F5)$.

The present section is devoted to examples in which it is possible to prove Proposition 12, or in other words, examples for which hypotheses $(F4), (F5)$ hold. The monotonicity property in Section 4 holds in all the subsequent examples.

We treat first (subsection 5.1) the important example when G is a Pucci extremal operator, which include the Laplacian as a particular case.

In subsections 5.2, and 5.3 we treat concave (convex) operators and, respectively, a class of Isaacs operators, which are neither concave nor convex. We will always assume that the operators considered satisfy hypothesis $(F4)$, which, in fact, is built-in in the case of Pucci extremal operators.

It is worth comparing here these examples with the results in [2] where problem (1) for the operator $-\Delta$ is studied using variational methods.

The results in [2] are optimal, since the maximal range of exponents, $1 < r < 2^* - 1$, is known thanks to the results in [21]. However, the arguments (in particular those leading to the L^∞ estimates) are strongly dependent on the structure of the Laplacian.

The examples below lack important features of the Laplacian, such as the variational structure or classical regularity of the solutions (which is not known in some cases), which make necessary the use of the viscosity framework in Sections 3 and 4.

5.1. **Extremal Pucci operators.** In the precise context of F involving a Pucci extremal operator, that is

$$F(\xi, X) = \mathcal{P}^{\pm}_{\theta,\Theta}(X) + H(\xi),$$

with $H : \mathbb{R}^n \to \mathbb{R}$ homogeneous and Lipschitz continuous and such that $H(0) = 0$, it is clear that F satisfies $(F1)$-$(F4)$.

Concerning $(F5)$, the Liouville-type result in $\mathbb{R}^n$ by Cutri-Leoni (Theorem 5) applies. On the other hand, Quaas and Sirakov, following the ideas in [5], proved Theorem 4 in this case, with $G(\cdot) = \mathcal{P}^{\pm}_{\theta,\Theta}(\cdot)$.

Observe that $\mathcal{P}^{\pm}_{\theta,\Theta}$ admits $\mathcal{C}^{2,\alpha}$ estimates, in the sense that if the function u is a viscosity solution to the equation $\mathcal{P}^{\pm}_{\theta,\Theta}(D^2u) = g(x)$ in a ball B_{2R} and $g \in \mathcal{C}^{\alpha}$ for some $\alpha \in (0, 1)$, then $u \in \mathcal{C}^{2,\alpha}$ and moreover,

$$\|u\|_{\mathcal{C}^{2,\alpha}(B_R)} \le C\big(\|u\|_{L^\infty(B_{2R})} + \|g\|_{\mathcal{C}^{\alpha}(B_{2R})}\big)$$

for some constant $C > 0$. With the aim of these $\mathcal{C}^{2,\alpha}$-estimates the proof of Theorem 4 could be simplified.

The abstract existence result (Theorem 3) holds and reads as follows.

Theorem 18. *Consider $\Omega \subset \mathbb{R}^n$, $n \ge 3$, a smooth bounded domain, and set*

$$\hat{r} = \frac{\theta(n-1) + \Theta}{\theta(n-1) - \Theta}.$$

Let $H : \mathbb{R}^n \to \mathbb{R}$ be homogeneous and Lipschitz continuous. Then, for $0 < q < 1 < r \le \hat{r}$ (or $0 < q < 1 < r < \infty$ if $\theta(n-1) \le \Theta$), there exists $\Lambda \in \mathbb{R}$, $0 < \Lambda < \infty$ such that the problem

$$\left\{\begin{array}{rcl} \mathcal{P}^{\pm}_{\theta,\Theta}(D^2u) + H(\nabla u) & = & \lambda\, u^q + u^r, \quad \text{in } \Omega, \\ u & > & 0, \quad \text{in } \Omega, \\ u & = & 0, \quad \text{on } \partial\Omega \end{array}\right. \tag{43}$$

(i) *has no positive solution for $\lambda > \Lambda$,*
(ii) *has at least one viscosity positive solution for $\lambda = \Lambda$,*
(iii) *has at least two viscosity positive solutions for every $\lambda \in (0, \Lambda)$.*

As an important consequence of Theorem 4, Quaas and Sirakov ([27]) deduce a Liouville-type result in $\mathbb{R}^n_+$ for Pucci operators, adapting arguments in [5] to the nonlinear setting.

Theorem 19 (Theorem 1.5 in [27])**.** *Suppose $n \ge 3$ and set*

$$\hat{r} = \frac{\theta(n-2) + \Theta}{\theta(n-2) - \Theta}.$$

Then the problem

$$\left\{\begin{array}{rcl} \mathcal{P}^{\pm}_{\theta,\Theta}\big(D^2v\big) & = & v^r, \quad \text{in } \mathbb{R}^n_+ \\ v & = & 0, \quad \text{on } \partial\mathbb{R}^n_+ \end{array}\right. \tag{44}$$

does not have a nontrivial non-negative bounded solution, provided $1 < r \le \hat{r}$ (or $1 < r < \infty$ if $\theta(n-2) \le \Theta$).

Notice that, since the critical exponent in Theorem 19 is greater than the corresponding one in Theorem 5, it is the latter one which yields the critical exponent $\hat{r}$ in Theorem 18. This fact agrees with the information obtained from Theorem 4 in the blow-up argument.

Consequently, whenever the range of exponents r in the Liouville result in $\mathbb{R}^n$ is maximal, so it is in Theorem 18. Indeed, as we mentioned in the introduction, the maximal range for the Laplacian (the Pucci operator with $\theta = \Theta = 1$) is known to be

$$1 < p < 2^* - 1 = \frac{n+2}{n-2},$$

where $2^* = 2n/(n-2)$ is the critical Sobolev exponent. Notice that the exponent $\hat{r}$ in Theorem 5 is not optimal in this case.

In fact, in the radial case, Felmer and Quaas [19] proved a Liouville type result for solutions (instead of just supersolutions as in [16]) for a larger range of exponents $1 < r < r_*^+$. However, an explicit expression for r_*^+ in terms of θ, Θ, n is not known. When $\theta = \Theta$ one gets $r_*^+ = 2^* - 1$ as in [21] which is, as we have already mentioned, optimal also in the non-radial case. When $\theta < \Theta$, it is known that $r_*^+ > \max\{\hat{r}, 2^* - 1\}$. As far as we know, to establish the Liouville result in the range $\hat{r} < r < r_+^*$ is an open problem.

5.2. **Concave and convex operators.** Let $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ satisfy $(F1)$-$(F4)$, and suppose that the blow-up operator $G(X) = F(0, X)$ is concave (or convex).

The hypothesis on G being concave (or convex), yields the following regularity result by Evans and Krylov. For the proof we refer to [10], Section 8.1.

Theorem 20. *Let $G : S^n \to \mathbb{R}$ be a uniformly elliptic (with constants $0 < \theta < \Theta$) concave (convex) operator. If the function u is a viscosity solution to the equation*

$$G(D^2u) = g(x) \tag{45}$$

in a ball B_{2R} and $g \in \mathcal{C}^\alpha$ for some $\alpha \in (0,1)$, then $u \in \mathcal{C}^{2,\alpha}$ and moreover,

$$\|u\|_{\mathcal{C}^{2,\alpha}(B_R)} \le C\big(\|u\|_{L^\infty(B_{2R})} + \|g\|_{\mathcal{C}^\alpha(B_{2R})}\big).$$

In addition, if (45) *is satisfied in a regular domain and $u = 0$ on the boundary of the domain, then u satisfies a $\mathcal{C}^\alpha$ estimate up to the boundary.*

The Liouville-type result in $\mathbb{R}^n$ by Cutri-Leoni (Theorem 5) applies in this case. For the $\mathbb{R}^n_+$ case, we use Theorem 4 which implies that problem (4) does not have a solution and allows us to reach a contradiction and conclude the blow-up argument.

As in the case of Pucci operators, the classical regularity of solutions (Theorem 20) simplifies the proofs.

The arguments in Sections 3 and 4 shows that Theorem 3 reads as follows.

Theorem 21. *Consider $\Omega \subset \mathbb{R}^n$, $n \ge 3$, a smooth domain. Let $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ be an operator satisfying $(F1)$, with constants $0 < \theta \le \Theta$, and $(F2)$ to $(F4)$. In addition, suppose that $G(X) = F(0, X)$ is concave (convex). Finally, set*

$$\hat{r} = \frac{\theta(n-1) + \Theta}{\theta(n-1) - \Theta}.$$

Then for $0 < q < 1 < r \le \hat{r} < \infty$ (or $0 < q < 1 < r < \infty$ if $\theta(n-1) \le \Theta$), there exists $\Lambda \in \mathbb{R}$, $0 < \Lambda < \infty$ such that the problem

$$\begin{cases} F(\nabla u, D^2u) &= \lambda\, u^q + u^r, \quad in\ \Omega, \\ u &> 0, \quad in\ \Omega, \\ u &= 0, \quad on\ \partial\Omega, \end{cases}$$

(i) *has no positive solution for $\lambda > \Lambda$,*
(ii) *has at least one viscosity positive solution for $\lambda = \Lambda$,*
(iii) *has at least two viscosity positive solutions for every $\lambda \in (0, \Lambda)$.*

5.3. **A class of Isaacs operators.** Finally we intend to apply the above analysis to a class of operators which are neither convex nor concave and for which classical regularity is not known in general.

Consider the class of Isaacs operators

$$F(\xi, X) = \sup_{l\in\mathcal{L}} \inf_{k\in\mathcal{K}} \big\{L_{k,l}(\xi, X)\big\}, \tag{46}$$

where $\mathcal{K}$ and $\mathcal{L}$ are arbitrary sets of indexes and $L_{k,l}$ are of the form

$$L_{k,l}(\xi, X) = -\mathrm{trace}(A_{k,l}X) + H_{k,l}(\xi)$$

for $H_{k,l} : \mathbb{R}^n \to \mathbb{R}$ 1-homogeneous and Lipschitz continuous (all with the same constant), such that $H_{k,l}(0) = 0$ and $A_{k,l}$ is a family of matrices with the same ellipticity constants.

In addition, we assume that the symmetry hypothesis $(F4)$ holds for the operator $G(X) = F(0, X)$. More precisely, we require that for every $Q \in \mathcal{O}^n$, whenever $A_{k,l} \in S^n$ gives rise to an operator $L_{k,l}$, with $(k, l) \in \mathcal{K} \times \mathcal{L}$, the matrix $QA_{k.l}Q^t$ gives rise to $L_{\tilde{k},\tilde{l}}$ for some other pair $(\tilde{k}, \tilde{l}) \in \mathcal{K} \times \mathcal{L}$.

Classical regularity is not known in general for problems of the form (46). However in [11] classical regularity is proved for Isaacs operators of the particular form

$$F(\xi, X) = \min\big\{F^{\cap}(\xi, X),\ F^{\cup}(\xi, X)\big\}, \quad \forall \xi \in \mathbb{R}^n \text{ and } \ X \in S^n \tag{47}$$

with $F^{\cup} : \mathbb{R}^n \times S^n \to \mathbb{R}$ and $F^{\cap} : \mathbb{R}^n \times S^n \to \mathbb{R}$ uniformly elliptic operators respectively convex and concave in the matrix argument. Clearly F in (47) is neither concave nor convex. As before, a model operator satisfying all the hypotheses above could be

$$F(\nabla u, D^2u) = \min\big\{ \inf_{k\in\mathcal{K}} L_k u,\ \sup_{l\in\mathcal{L}} L_l u\big\}$$

where

$$L_k u = -\mathrm{trace}(A_k D^2u) + H_k(\nabla u).$$

Notice that, since classical regularity is not known in general, the viscosity setting becomes crucial in this case. Arguing as in Sections 3 and 4 we have the following result.

Theorem 22. *Consider $\Omega \subset \mathbb{R}^n$, $n \geq 3$, a smooth domain. Let $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ be an operator of the form* (46) *satisfying* $(F1)$, *with ellipticity constants* $0 < \theta \leq \Theta$, $(F2)$, $(F3)$ *and* $(F4)$. *Finally, set*

$$\hat{r} = \frac{\theta(n-1) + \Theta}{\theta(n-1) - \Theta}.$$

Then, for $0 < q < 1 < r \leq \hat{r}$ (or $0 < q < 1 < r < \infty$ if $\theta(n-1) \leq \Theta$), there exists $\Lambda \in \mathbb{R}$, $0 < \Lambda < \infty$ such that the problem

$$\begin{cases} F(\nabla u, D^2u) &= \lambda\, u^q + u^r, \quad \text{in } \Omega, \\ u &> 0, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{cases}$$

(i) *has no positive solution for $\lambda > \Lambda$,*
(ii) *has at least one viscosity positive solution for $\lambda = \Lambda$,*
(iii) *has at least two viscosity positive solutions for every $\lambda \in (0, \Lambda)$.*

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