LIMITS AS $p \to \infty$ OF p-LAPLACIAN PROBLEMS WITH A SUPERDIFFUSIVE POWER-TYPE NONLINEARITY: POSITIVE AND SIGN-CHANGING SOLUTIONS

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ABSTRACT. We investigate the asymptotic behaviour as $p \to \infty$ of sequences of solutions of the equation
$$\begin{cases} -\Delta_p u &= \lambda\,|u|^{q(p)-2}u & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases}$$
where $\lambda > 0$ and $q(p) > p$ with $\lim_{p\to\infty} q(p)/p = Q \geq 1$. We are interested in the characterization of such limits as viscosity solutions of a PDE problem. Both positive and sign-changing solutions are considered.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. We consider the equation

$$\begin{cases} -\Delta_p u &= \lambda_p |u|^{q(p)-2}u & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, $\lambda_p > 0$ and $q(p) > p$ with $Q = \lim_{p\to\infty} \dfrac{q(p)}{p} \geq 1$. We are interested in the convergence as $p \to \infty$ of sequences of solutions to (1) and in the characterization of such limits as viscosity solutions to a PDE problem.

Limits as $p \to \infty$ of sequences of solutions of problem (1) have already been considered in the case where $Q < 1$ and the solutions are positive (see [5]), while the case $q(p) = p$ (eigenvalue problem) has been treated in [12] for the first eigenfunction and in [11] for the second eigenfunction.

Our aim is to contribute to the completeness of the theory with the study of the remaining cases: the case $Q > 1$, both for positive and sign-changing solutions, and the case $Q = 1$ with $q(p) > p$. Our results complement those already known in the literature in the subdiffusive and eigenvalue cases but are essentially different in nature, mostly due to the lack of comparison results as in [5] and [12].

In the case $Q > 1$ we prove that there exist sequences of solutions of (1) converging uniformly to a viscosity solution of the problem

$$\begin{cases} F_\Lambda(u, \nabla u, D^2 u) &= 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases}$$

where

$$F_\Lambda(s, \xi, X) = \begin{cases} \min\{|\xi| - \Lambda s^Q, -X\xi \cdot \xi\} & \text{if } s > 0 \\ -X\xi \cdot \xi & \text{if } s = 0 \\ \max\{-\Lambda |s|^{Q-1}s - |\xi|, -X\xi \cdot \xi\} & \text{if } s < 0, \end{cases}$$

and $\lim_{p\to\infty} \lambda_p^{1/p} = \Lambda$, assuming that such a limit exists.

In particular, positive solutions of (1) will converge in the viscosity sense, as $p \to \infty$, to a solution of

$$\begin{cases} \min\{|\nabla u| - \Lambda u^Q, -\Delta_\infty u\} &=& 0 & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_\infty u = \sum_{i,j=1}^{n} u_{x_i x_j} u_{x_i} u_{x_j} = \langle D^2 u \cdot \nabla u, \nabla u \rangle$.

The analysis of the case $Q = 1$ is more delicate since, in order to get $\infty-$eigenfunctions as a limit of solutions of (1), we will need a very precise control of the solutions in terms of the behaviour of the particular sequence λ_p. Let us denote by $\Lambda_1(\Omega)$ the first eigenvalue of the infinity Laplacian. For a sequence $\{u_{\lambda_p,p}\}_p$ of positive solutions of (1) we prove (see Theorem 6.1) that

(i) If $\Lambda > \Lambda_1(\Omega)$, then $\lim\limits_{p \to \infty} ||u_{\lambda_p,p}||_\infty = 0$.

(ii) If $\Lambda < \Lambda_1(\Omega)$, then $\lim\limits_{p \to \infty} ||u_{\lambda_p,p}||_\infty = \infty$.

Furthermore, we obtain corresponding results for least energy nodal solutions (sign changing) of (1) and $\Lambda_2(\Omega)$, the second eigenvalue of the infinity Laplacian (see Theorem 6.2).

A key point in the proof of convergence will be the Morrey estimate

$$\|u\|_{L^\infty(\Omega)} \le C_p \cdot \|\nabla u\|_{L^p(\Omega)}, \tag{2}$$

for which we provide an explicit expression of the constant satisfying

$$\lim_{p \to \infty} C_p = \Lambda_1(\Omega)^{-1}.$$

This fact is crucial in the case $Q = 1$. It is worth mentioning here that in [16], Theorem 2.E, it is proved that Morrey's inequality holds with constant

$$C_{\mathrm{T},p} = n^{-\frac{1}{p}} |B_1(0)|^{-\frac{1}{n}} \left(\frac{p-1}{p-n}\right)^{1-\frac{1}{p}} |\Omega|^{\frac{1}{n}-\frac{1}{p}},$$

which is optimal if $\Omega = B_R(x_0)$ as the functions

$$u_a(x) = a \cdot \left(R^{\frac{p-n}{p-1}} - |x-x_0|^{\frac{p-n}{p-1}}\right),$$

(with $a \in \mathbb{R}$) yield $\|u_a\|_{L^\infty(\Omega)} = C_{\mathrm{T},p} \cdot \|\nabla u_a\|_{L^p(\Omega)}$. However, if $\Omega \ne B_R(x_0)$ it is easy to see that C_p does better than $C_{T,p}$ for large p, since $\lim\limits_{p \to \infty} C_p < \lim\limits_{p \to \infty} C_{\mathrm{T},p}$ (see Remark 3.4).

Finally, we consider in Section 8 the issue of symmetry of positive limit solutions of the limit problem posed in a ball. It is interesting to point out that this result is related to a uniqueness property, namely, we prove that a properly scaled cone is the unique positive limit solution of our problem.

The paper is organized as follows. In Section 2 we provide some background for problem (1) in the case $p < \infty$. Then, in Section 3 we provide the proof of Morrey's estimate taking care of the explicit expression of the involved constant. Next, in Sections 4 and 5 uniform estimates for solutions of (1) are provided. As a consequence we deduce uniform convergence of a subsequence and non-degeneracy of the limit. In Section 6 we address the case $q(p) > p$ and $Q = 1$. The limit problem is given in Section 7. Finally, in Section 8 we show symmetry and uniqueness of positive limit solutions of the limit problem when the domain is a a ball.

2. Preliminaries on the case $p < \infty$

In this section we will present some properties of the equation

$$\begin{cases} -\Delta_p u &=& \lambda |u|^{q-2} u & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

where $\lambda > 0$, $1 < p < +\infty$, $1 < q < +\infty$.

It is important to note that equation (3) can be interpreted both in the variational and viscosity frameworks and consequently two natural notions of solution are found. Nevertheless, continuous weak solutions of (3) are also viscosity solutions, as stated in the following result. The proof follows similarly to Lemma 1.8 in [12] (see also [4]).

Lemma 2.1. *For $p \geq 2$, every continuous weak solution of (3) is a viscosity solution of the same problem, rewritten as*

$$\begin{cases} F_p(\nabla u, D^2 u) & = & \lambda \, |u|^{q-2} u & in \, \Omega \\ u & = & 0 & on \, \partial\Omega \end{cases} \tag{4}$$

where

$$F_p(\xi, X) = -\text{trace}\left(\left(Id + (p-2)\frac{\xi \otimes \xi}{|\xi|^2} \right) X \right) \cdot |\xi|^{p-2}.$$

In the sequel we will always choose the most suitable form of our problem between (1) and (4) without any further reference.

We will treat now the problem (3) from the variational point of view. The critical points of the functional

$$\varphi_p(v) = \frac{1}{p} \int_\Omega |\nabla v|^p \, dx - \frac{\lambda}{q} \int_\Omega |v|^q \, dx$$

are weak solutions and, as we have already said, viscosity solutions, of equation (3). We are interested in two main cases, namely $q = p$ and $q > p$.

2.1. The case $q = p$.
This case corresponds to the so-called *eigenvalue problem*. A number $\lambda \in \mathbb{R}$ is called *eigenvalue* if there exists a function $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, called *eigenfunction*, which solves the equation. It turns out that there exists a sequence of eigenvalues $\{\lambda_k(p;\Omega)\}_{k=1}^\infty$ with $\lambda_1(p;\Omega) < \lambda_2(p;\Omega)$ and $\lambda_k(p;\Omega) \to +\infty$ as $k \to +\infty$ (see again [8]). It is worth pointing out that it is not known if the mentioned sequence contains all possible eigenvalues.

The first eigenvalue can be characterized as the infimum of the Rayleigh quotient associated to the problem:

$$\lambda_1(p;\Omega) = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}. \tag{5}$$

The first eigenvalue is simple, which means that there exists only one eigenfunction e_1, up to a multiplicative factor; moreover, e_1 has constant sign (see for instance [1, 3]).

Higher eigenvalues can be obtained through the following minimax principle. Let us define the *Krasnoselskii genus* of a set $A \subseteq W_0^{1,p}(\Omega)$ as

$$\gamma(A) = \min\left\{ k \in \mathbb{N} \, | \, \exists f : A \mapsto \mathbb{R}^k \setminus \{0\}, \, f \text{ continuous and odd} \right\}.$$

Define

$$\Gamma_k = \left\{ A \subseteq W_0^{1,p}(\Omega) \, | \, A \text{ symmetric}, \, A \cap \{||v||_p = 1\} \text{ compact}, \, \gamma(A) \geq k \right\}.$$

Then,

$$\lambda_k(p;\Omega) = \inf_{A \in \Gamma_k} \sup_{u \in A} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}.$$

Higher eigenfunctions must be sign-changing. Moreover, one can prove (see [2]) that $\lambda_2(p;\Omega)$ is the smallest eigenvalue which admits a sign-changing eigenfunction.

The following result about the behaviour as $p \to \infty$ of the first and second eigenvalues of the p−Laplacian holds (see [12] and [11]).

Proposition 2.2. *Let $\lambda_1(p;\Omega)$ and $\lambda_2(p;\Omega)$ be respectively the first and second eigenvalue of the $p-$Laplacian. Define*

$$\Lambda_1(\Omega) = \left(\max_{x\in\Omega} \operatorname{dist}(x,\partial\Omega) \right)^{-1}$$

and

$$\Lambda_2(\Omega) = \left(\sup\{r : \text{ there are two disjoint balls } B_1, B_2 \subseteq \Omega \text{ of radius } r\} \right)^{-1}.$$

Then,

$$\lim_{p\to\infty} \left(\lambda_1(p;\Omega) \right)^{1/p} = \Lambda_1(\Omega) \qquad and \qquad \lim_{p\to\infty} \left(\lambda_2(p;\Omega) \right)^{1/p} = \Lambda_2(\Omega).$$

2.2. **The case $q > p$.** We define the first variation of φ_p at u in direction v

$$\mathrm{d}\varphi_p(u)(v) = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v - \lambda \int_\Omega |u|^{q-2} uv$$

and the *Nehari manifold*

$$\mathscr{N}_p = \{u \in W_0^{1,p}(\Omega) \setminus \{0\} \,|\, \mathrm{d}\varphi_p(u)(u) = 0\}.$$

It is obvious that all the nontrivial critical points of the functional belong to $\mathscr{N}_p$. We also set

$$\mathscr{N}_p^+ = \{u \in \mathscr{N}_p \,|\, u \geq 0\} \quad \text{and} \quad \mathscr{N}_p^- = \{u \in \mathscr{N}_p \,|\, u \leq 0\}.$$

Let us denote with $u^+ = \max\{0,u\}$ and $u^- = \min\{0,u\}$ the positive and the negative part of u respectively. We introduce the *nodal Nehari set*

$$\mathscr{M}_p = \{u \in \mathscr{N}_p \,|\, u^+ \in \mathscr{N}_p^+, u^- \in \mathscr{N}_p^-\}.$$

Then, $\mathscr{M}_p$ consists only of sign-changing functions and contains all sign-changing critical points of φ_p. It can be proved (see [9]) that the infima $\inf_{v\in\mathscr{N}_p^+} \varphi_p(v)$, $\inf_{v\in\mathscr{N}_p^-} \varphi_p(v)$ and $\inf_{v\in\mathscr{M}_p} \varphi_p(v)$ are attained, and that the corresponding minimum points are a positive, a negative and a sign-changing solution of (3) respectively. The following facts, whose proof can be found in [9], will be useful later.

Proposition 2.3. *For every $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, there exists a unique number $t_p^* > 0$ such that $t_p^* u \in \mathscr{N}_p$. Moreover,*

$$t_p^* = \left(\frac{\int_\Omega |\nabla u|^p}{\lambda \int_\Omega |u|^q} \right)^{\frac{1}{q-p}}$$

and

$$\varphi_p(t_p^* u) = \max_{t>0} \varphi_p(tu).$$

Corollary 2.4. *For every $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, the numbers $t_p^+, t_p^- > 0$ such that $t_p^+ u^+ + t_p^- u^- \in \mathscr{M}_p$ are uniquely defined.*

3. SOME CONSEQUENCES OF THE MORREY ESTIMATE

In order to prove the results mentioned in Section 1, Morrey's inequality with an explicit expression of the constant involved will be an important tool. The following result will be used profusely in the sequel.

Proposition 3.1. *Assume $n < p < \infty$ and $u \in W_0^{1,p}(\Omega)$. Then,*

$$\|u\|_{L^\infty(\Omega)} \leq C_p \cdot \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}}$$

holds with constant

$$C_p = p \, |B_1(0)|^{-\frac{1}{p}} n^{-\frac{n(p+1)}{p^2}} (p-1)^{\frac{n(p-1)}{p^2}} (p-n)^{\frac{n}{p^2}-1} \lambda_1(p;\Omega)^{\frac{n-p}{p^2}}.$$

Remark 3.2. Notice that $\lim\limits_{p\to\infty} C_p = \Lambda_1(\Omega)^{-1}$. This fact will be crucial in the sequel.

As a first step in the proof of Proposition 3.1, we will review the well-known proof of Morrey's estimates in [6] tracking down the precise dependence on p of the constants involved.

Lemma 3.3. *Assume $n < p < \infty$ and $u \in W_0^{1,p}(\Omega)$. Then u has a $\mathscr{C}^\gamma(\Omega)$ version, where $\gamma = 1 - \frac{n}{p}$, and the following estimates hold:*

1. *L^∞-estimate:*

$$\|u\|_{L^\infty(\Omega)} \leq C_p^* \cdot \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}},$$ (6)

with

$$C_p^* = \frac{1}{|B_1(0)|^{\frac{1}{p}}} \left[\frac{1}{n^{\frac{1}{p}}} \left(\frac{p-1}{p-n} \right)^{1-\frac{1}{p}} + \lambda_1(p,\Omega)^{-\frac{1}{p}} \right].$$

2. *Hölder continuity:*

$$\frac{|u(x) - u(y)|}{|x-y|^\gamma} \leq \tilde{C}_p \cdot \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}},$$ (7)

where

$$\tilde{C}_p = \frac{2C}{|\partial B_1(0)|^{\frac{1}{p}}} \left(\frac{p-1}{p-n} \right)^{1-\frac{1}{p}}$$

and C is a constant depending only on n.

Proof. We suppose hereafter that $u \in \mathscr{C}^1(\Omega) \cap \mathscr{C}_0(\overline{\Omega})$ since our conclusions apply to $W_0^{1,p}(\Omega)$ by density. We also suppose the function u extended by zero to the whole space $\mathbb{R}^n$. We will consider such an extension without making any further reference.

1. Fix $s \in [0,r]$ and $w \in \partial B_1(0)$. Then

$$|u(x+sw) - u(x)| = \left| \int_0^s \frac{d}{dt} u(x+tw) \, dt \right| = \left| \int_0^s \nabla u(x+tw) \cdot w \, dt \right| \leq \int_0^s |\nabla u(x+tw)| \, dt.$$

Integrating over $\partial B_1(0)$

$$\int_{\partial B_1(0)} |u(x+sw) - u(x)| d\sigma \leq \int_0^s \int_{\partial B_1(0)} |\nabla u(x+tw)| \, d\sigma \, dt$$

$$= \int_0^s \int_{\partial B_1(0)} |\nabla u(x+tw)| \, t^{n-1} d\sigma \frac{1}{t^{n-1}} \, dt \leq \int_{B_r(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, dy.$$

Multiplying by s^{n-1} and integrating over $[0,r]$, we obtain

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y) - u(x)| \, dy \leq \frac{1}{n|B_1(0)|} \int_{B_r(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, dy.$$ (8)

2. Now, we will estimate $|u(x)|$ for fixed $x \in \mathbb{R}^n$. From estimate (8)

$$|u(x)| = \frac{1}{|B_1(x)|} \int_{B_1(x)} |u(x)| \, dy \leq \frac{1}{|B_1(x)|} \int_{B_1(x)} |u(x) - u(y)| dy + \frac{1}{|B_1(x)|} \int_{B_1(x)} |u(y)| dy$$

$$\leq \frac{1}{n|B_1(0)|} \int_{B_1(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, dy + \frac{1}{|B_1(x)|} \int_{B_1(x)} |u(y)| dy.$$

Applying Hölder's inequality, we have

$$
\begin{aligned}
|u(x)| \;\leq\; & \frac{1}{n|B_1(0)|}\left(\int_{B_1(x)}\frac{1}{|x-y|^{\frac{p(n-1)}{(p-1)}}}\,dy\right)^{1-\frac{1}{p}}\left(\int_{B_1(x)}|\nabla u(y)|^p\,dy\right)^{\frac{1}{p}} \\
& +\frac{1}{|B_1(0)|^{\frac{1}{p}}}\left(\int_{B_1(x)}|u(y)|^p\,dy\right)^{\frac{1}{p}} \\
\;\leq\; & \frac{1}{n|B_1(0)|}\left(|\partial B_1(0)|\left(\frac{p-1}{p-n}\right)\right)^{1-\frac{1}{p}}\left(\int_\Omega|\nabla u(y)|^p\,dy\right)^{\frac{1}{p}} \\
& +\frac{1}{|B_1(0)|^{\frac{1}{p}}}\left(\int_\Omega|u(y)|^p\,dy\right)^{\frac{1}{p}}.
\end{aligned}
$$

Applying Poincaré's inequality (recall from (5) that the optimal constant is $\lambda_1(p,\Omega)^{-1/p}$) we arrive at

$$
|u(x)|\leq\left[\frac{1}{n|B_1(0)|}\left(|\partial B_1(0)|\left(\frac{p-1}{p-n}\right)\right)^{1-\frac{1}{p}}+\frac{\lambda_1(p,\Omega)^{-\frac{1}{p}}}{|B_1(0)|^{\frac{1}{p}}}\right]\left(\int_\Omega|\nabla u(y)|^p\,dy\right)^{\frac{1}{p}}
$$

and (6) follows, using that $|\partial B_1(0)|=n|B_1(0)|$.

3.   Let $\gamma=1-n/p$ and consider $x,y\in\Omega$. Define $W=B_r(x)\cap B_r(y)$, where $r=|x-y|$. Then

$$
\begin{aligned}
|u(x)-u(y)| \;=\; & \frac{1}{|W|}\int_W|u(x)-u(y)|\,dz \\
\;\leq\; & \frac{1}{|W|}\int_W|u(x)-u(z)|\,dz+\frac{1}{|W|}\int_W|u(z)-u(y)|\,dz. \qquad (9)
\end{aligned}
$$

Next, we choose a positive number C such that $|B_r(x)|\leq C|W|$ and we compute

$$
\begin{aligned}
\frac{1}{|W|}\int_W|u(x)\;-\;u(z)|\,dz\leq & \frac{|B_r(x)|}{|W|}\frac{1}{|B_r(x)|}\int_{B_r(x)}|u(x)-u(z)|\,dz \\
\;\leq\; & \frac{C}{n|B_1(0)|}\int_{B_r(x)}\frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy \\
\;\leq\; & \frac{C}{n|B_1(0)|}\left(\int_{B_r(x)}\frac{1}{|x-y|^{\frac{p(n-1)}{(p-1)}}}\,dy\right)^{1-\frac{1}{p}}\left(\int_{B_r(x)}|\nabla u(y)|^p\,dy\right)^{\frac{1}{p}} \\
\;\leq\; & \frac{C}{n|B_1(0)|}\left(|\partial B_1(0)|\,r^{\frac{p-n}{p-1}}\frac{p-1}{p-n}\right)^{1-\frac{1}{p}}\left(\int_\Omega|\nabla u(y)|^p\,dy\right)^{\frac{1}{p}} \\
\;=\; & \frac{C}{|\partial B_1(0)|^{\frac{1}{p}}}\left(\frac{p-1}{p-n}\right)^{1-\frac{1}{p}}\left(\int_\Omega|\nabla u(y)|^p\,dy\right)^{\frac{1}{p}}r^\gamma.
\end{aligned}
$$

Taking the former estimate into (9) we get (7). $\square$

Next, we improve the constant in estimate (6) by means of a scaling argument. As a motivation, let us point out that estimate (6) is more accurate the bigger the domain Ω is, since $\lambda_1(p;\Omega)$ is decreasing with respect to Ω (see (5)).

Proof of Proposition 3.1.   1.   Fix $\eta>0$. First, we prove that estimate (6) holds with constant

$$
C_p^*(\eta)=\frac{\eta^{-\frac{n}{p}}}{|B_1(0)|^{\frac{1}{p}}}\left[\frac{1}{n^{\frac{1}{p}}}\left(\frac{p-1}{p-n}\right)^{1-\frac{1}{p}}\eta+\lambda_1(p,\Omega)^{-\frac{1}{p}}\right].
$$

To this aim, we define the rescaled domain

$$\Omega_\eta = \eta^{-1}\Omega = \left\{ x \in \mathbb{R}^n : y = \eta x \in \Omega \right\}$$

and the function $v : \Omega_\eta \to \mathbb{R}$ given by $v(x) = u(\eta x)$.

Notice that $u \in W_0^{1,p}(\Omega)$ implies $v \in W_0^{1,p}(\Omega_\eta)$. Hence, we can apply estimate (6) to v:

$$\|v\|_{L^\infty(\Omega_\eta)} \leq \frac{1}{|B_1(0)|^{\frac{1}{p}}} \left[\frac{1}{n^{\frac{1}{p}}} \left(\frac{p-1}{p-n} \right)^{1-\frac{1}{p}} + \lambda_1(p, \Omega_\eta)^{-\frac{1}{p}} \right] \left(\int_{\Omega_\eta} |\nabla v|^p \, dx \right)^{\frac{1}{p}}.$$

Now we analyze separately the dependence on η of each term in the above expression. From the characterization of $\lambda_1(p; \Omega)$ as a Rayleigh quotient, see (5), it follows

$$\lambda_1(p; \Omega_\eta) = \eta^p \cdot \lambda_1(p; \Omega).$$

Moreover, $\|v\|_{L^\infty(\Omega_\eta)} = \|u\|_{L^\infty(\Omega)}$ and $\|\nabla v\|_{L^p(\Omega_\eta)} = \eta^{1-\frac{n}{p}} \cdot \|\nabla u\|_{L^p(\Omega)}$. Putting together all these facts we obtain

$$\|u\|_{L^\infty(\Omega)} \leq C_p^*(\eta) \cdot \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}},$$

2. We will now refine the constant in the previous step, finding the value of η which gives the minimal constant. From the previous step, we know that, for every $\eta > 0$, estimate (6) holds with constant $C_p^*(\eta)$. It is easily seen that the only critical point of $C_p^*(\eta)$ as a function of η is

$$\eta^* = \frac{n^{1+\frac{1}{p}}}{(p-n)^{\frac{1}{p}} (p-1)^{1-\frac{1}{p}}} \cdot \lambda_1(p; \Omega)^{-\frac{1}{p}}.$$

Since $C_p^*(\eta) \to \infty$ both as $\eta \to 0$ and $\eta \to \infty$, η^* is a global minimum. It is then elementary to check that

$$C_p^*(\eta^*) = C_p = p \, |B_1(0)|^{-\frac{1}{p}} \, n^{-\frac{n(p+1)}{p^2}} (p-1)^{\frac{n(p-1)}{p^2}} (p-n)^{\frac{n}{p^2}-1} \lambda_1(p; \Omega)^{\frac{n-p}{p^2}}. \qquad \square$$

Remark 3.4. Following the notation in the proof of Proposition 3.1, notice that $C_p \leq C_p^*(\eta)$ for all $\eta > 0$; in particular $C_p \leq C_p^*(1) = C_p^*$, the constant in estimate (6). In addition, we point out that in [16], Theorem 2.E, it is proved that Morrey's inequality holds with constant

$$C_{\mathrm{T},p} = n^{-\frac{1}{p}} |B_1(0)|^{-\frac{1}{n}} \left(\frac{p-1}{p-n} \right)^{1-\frac{1}{p}} |\Omega|^{\frac{1}{n}-\frac{1}{p}}.$$

The constant is optimal if $\Omega = B_R(x_0)$. Indeed, the functions

$$u_a(x) = a \cdot \left(R^{\frac{p-n}{p-1}} - |x - x_0|^{\frac{p-n}{p-1}} \right),$$

(with $a \in \mathbb{R}$) yield

$$\|u_a\|_{L^\infty(\Omega)} = C_{\mathrm{T},p} \cdot \|\nabla u_a\|_{L^p(\Omega)}.$$

Since the principal eigenvalue of the p-Laplacian is explicitly known when $n = 1$ (see [13] and the references therein), namely,

$$\lambda_1\big(p; (a,b)\big) = (p-1) \cdot \left(\frac{2\pi}{p \cdot (b-a) \cdot \sin\left(\frac{\pi}{p}\right)} \right)^p.$$

one finds that $C_{\mathrm{T},p} < C_p$ and $\lim_{p \to \infty} C_{\mathrm{T},p} = \lim_{p \to \infty} C_p$ for $n = 1$. However, things change if $n \geq 2$ and Ω is not a ball; in that case it is easy to see that $C_p < C_{\mathrm{T},p}$ for p large enough. Indeed, let $R > 0$ be the radius of the largest ball inscribed in Ω; then

$$\lim_{p \to \infty} C_{\mathrm{T},p} = \left(\frac{|\Omega|}{|B_1(0)|} \right)^{\frac{1}{n}} > \left(\frac{|B_R(0)|}{|B_1(0)|} \right)^{\frac{1}{n}} = R = \Lambda_1(\Omega)^{-1} = \lim_{p \to \infty} C_p.$$

4. UPPER BOUNDS. CONVERGENCE IN THE CASE $Q > 1$

Let us introduce some notation. By $u_{\lambda,p}$ we will denote a positive solution of (1) with parameter $\lambda_p = \lambda$, while the notation $v_{\lambda,p}$ will stand for a least-energy nodal solution of the same problem. For simplicity, we set $u_{\lambda_1,p} = u_{\lambda_1(p;\Omega),p}$ and $v_{\lambda_2,p} = v_{\lambda_2(p;\Omega),p}$. By $e_{1,p}$ and $e_{2,p}$ we will denote a first and a second eigenfunction of the p-Laplacian respectively, such that $||e_{1,p}||_p = ||e_{2,p}||_p = 1$.

Lemma 4.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $q > p > n$. The positive solution $u_{\lambda_1,p}$, which solves (1) for $\lambda_p = \lambda_1(p;\Omega)$, satisfies the estimate*

$$||u_{\lambda_1,p}||_\infty \leq C_p \cdot \lambda_1(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}}, \tag{10}$$

while the least-energy nodal solution $v_{\lambda_2,p}$, which solves (1) for $\lambda_p = \lambda_2(p;\Omega)$, satisfies the estimate

$$||v_{\lambda_2,p}||_\infty \leq 2^{\frac{1}{p}} \cdot C_p \cdot \lambda_2(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}}, \tag{11}$$

where C_p is the constant in Proposition 3.1. Moreover, for every fixed m such that $p > m > n$ and $x,y \in \Omega$, we have

$$\frac{|u_{\lambda_1,p}(x) - u_{\lambda_1,p}(y)|}{|x-y|^{1-\frac{n}{m}}} \leq \tilde{C}_m \cdot |\Omega|^{\frac{1}{m}} \cdot \lambda_1(p;\Omega)^{\frac{1}{p}}. \tag{12}$$

and

$$\frac{|v_{\lambda_2,p}(x) - v_{\lambda_2,p}(y)|}{|x-y|^{1-\frac{n}{m}}} \leq 2^{\frac{1}{p}} \cdot \tilde{C}_m \cdot |\Omega|^{\frac{1}{m}} \cdot \lambda_2(p;\Omega)^{\frac{1}{p}}. \tag{13}$$

where $\tilde{C}_m$ is the constant in Lemma 3.3 with parameter m.

Proof. We will prove the lemma only for the case of least-energy nodal solutions. The case of positive solutions is even simpler and follows in a similar way.

1. First, from [9], Proposition 3.4, we have

$$\int_\Omega |\nabla v_{\lambda_2,p}|^p \, dx \leq (t_p^+)^p \int_\Omega |\nabla e_2^+|^p \, dx + (t_p^-)^p \int_\Omega |\nabla e_2^-|^p \, dx$$

where e_2 is a second eigenfunction of the p-Laplacian, and

$$t_p^+ = \left(\frac{\int_\Omega |\nabla e_2^+|^p \, dx}{\lambda_2(p;\Omega) \int_\Omega |e_2^+|^q \, dx} \right)^{\frac{1}{q-p}}, \quad t_p^- = \left(\frac{\int_\Omega |\nabla e_2^-|^p \, dx}{\lambda_2(p;\Omega) \int_\Omega |e_2^-|^q \, dx} \right)^{\frac{1}{q-p}}$$

as defined in Proposition 2.3. Using Hölder's inequality one obtains

$$t_p^+ \leq \left(\frac{\int_\Omega |e_2^+|^p \, dx}{\int_\Omega |e_2^+|^q \, dx} \right)^{\frac{1}{q-p}} \leq \left(\frac{|\Omega|^{\frac{q}{p}-1} \int_\Omega |e_2^+|^p \, dx}{\left(\int_\Omega |e_2^+|^p \, dx \right)^{\frac{q}{p}}} \right)^{\frac{1}{q-p}} = |\Omega|^{\frac{1}{p}} \left(\int_\Omega |e_2^+|^p \, dx \right)^{-\frac{1}{p}}$$

and similarly for t_p^-. Substituting we obtain

$$\int_\Omega |\nabla v_{\lambda_2,p}|^p \, dx \leq |\Omega| \left(\int_\Omega |e_2^+|^p \, dx \right)^{-1} \left(\int_\Omega |\nabla e_2^+|^p \, dx \right)$$

$$+ |\Omega| \left(\int_\Omega |e_2^-|^p \, dx \right)^{-1} \left(\int_\Omega |\nabla e_2^-|^p \, dx \right) = 2 \cdot |\Omega| \cdot \lambda_2(p;\Omega)$$

so that

$$||\nabla v_{\lambda_2,p}||_p \leq 2^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}} \cdot \lambda_2(p;\Omega)^{\frac{1}{p}}. \tag{14}$$

Plugging the above estimate into Morrey's inequality (Proposition 3.1), we get (11).

2. For fixed $x,y \in \Omega$ and $m > n$ we have from Lemma 3.3 and Hölder's inequality with exponents p/m and $p/(p-m)$ that

$$\frac{|v_{\lambda_2,p}(x) - v_{\lambda_2,p}(y)|}{|x-y|^{1-\frac{n}{m}}} \leq \tilde{C}_m \cdot \left(\int_\Omega |\nabla v_{\lambda_2,p}|^m \, dx \right)^{1/m} \leq \tilde{C}_m \cdot |\Omega|^{\frac{1}{m}-\frac{1}{p}} \cdot \left(\int_\Omega |\nabla v_{\lambda_2,p}|^p \, dx \right)^{1/p}.$$

Using the inequality (14) we get (13). $\square$

Since the right hand sides in (10), (11), (12) and (13) can be bounded uniformly in p, we have the following convergence result.

Corollary 4.2. *Consider the sequences $\{u_{\lambda_1,p}\}_p$ and $\{v_{\lambda_2,p}\}_p$. Then, there exists a sub-sequence p_i and limit functions u_{Λ_1} and v_{Λ_2} with $\lim\limits_{i\to\infty} u_{\lambda_1,p_i} = u_{\Lambda_1}$ and $\lim\limits_{i\to\infty} v_{\lambda_2,p_i} = v_{\Lambda_2}$ uniformly.*

Remark 4.3. The limits u_{Λ_1} and v_{Λ_2} could depend on the particular subsequence we are considering. In the case of u_{λ_1,p_i} and Ω a ball, we will show in Section 8 a symmetry property for limits u_{Λ_1} that will imply uniqueness of the limit and, consequently, that not only a subsequence, but the whole sequence converges.

5. LOWER BOUNDS. NON-DEGENERACY OF THE LIMIT.

From Morrey's estimate (Proposition 3.1), we get the following lower bound which yields non-degeneracy of the limit as $p \to \infty$.

Lemma 5.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $q > p > n$. The function $u_{\lambda_1,p}$ positive solution of (1) with $\lambda_p = \lambda_1(p;\Omega)$ satisfies the estimate*

$$\|u_{\lambda_1,p}\|_\infty \geq \left[C_p \cdot \lambda_1(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}}\right]^{-\frac{1}{\frac{q}{p}-1}} > 0, \tag{15}$$

where C_p is the constant in Proposition 3.1.

Proof. The function $u_{\lambda_1,p}$ satisfies (1). Multiplying the equation by $u_{\lambda_1,p}$ and integrating by parts, we get

$$\int_\Omega |\nabla u_{\lambda_1,p}|^p \, dx = \lambda_1(p;\Omega) \cdot \int_\Omega |u_{\lambda_1,p}|^q \, dx.$$

By Proposition 3.1 and the equality above, we get

$$\|u_{\lambda_1,p}\|_\infty \leq C_p \cdot \left(\int_\Omega |\nabla u_{\lambda_1,p}|^p \, dx\right)^{1/p} = C_p \cdot \left(\lambda_1(p;\Omega) \cdot \int_\Omega |u_{\lambda_1,p}|^q \, dx\right)^{1/p} \leq$$

$$\leq C_p \cdot \lambda_1(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}} \cdot \|u_{\lambda_1,p}\|_\infty^{\frac{q}{p}},$$

and hence the result. $\square$

These arguments can be adapted to the family of least-energy nodal solutions as follows.

Lemma 5.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $q > p > n$. The function $v_{\lambda_2,p}$ least-energy nodal solution of (1) with $\lambda_p = \lambda_2(p;\Omega)$ satisfies the estimate*

$$\|v_{\lambda_2,p}\|_\infty \geq \left[\widehat{C}_p \cdot \lambda_2(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}}\right]^{-\frac{1}{\frac{q}{p}-1}} > 0, \tag{16}$$

where

$$\widehat{C}_p = p \, |B_1(0)|^{-\frac{1}{p}} \, n^{-\frac{n(p+1)}{p^2}} \, (p-1)^{\frac{n(p-1)}{p^2}} \, (p-n)^{\frac{n}{p^2}-1} \lambda_2(p;\Omega)^{\frac{n-p}{p^2}}. \tag{17}$$

Remark 5.3. Notice that $\lim\limits_{p\to\infty} \widehat{C}_p = \Lambda_2(\Omega)^{-1}$.

Proof. The function $v_{\lambda_2,p}$ satisfies (1). Multiplying the equation by $v_{\lambda_2,p}^+$ and integrating by parts, we get

$$\int_\Omega |\nabla v_{\lambda_2,p}^+|^p \, dx = \lambda_2(p;\Omega) \cdot \int_\Omega |v_{\lambda_2,p}^+|^q \, dx, \tag{18}$$

and similarly

$$\int_\Omega |\nabla v_{\lambda_2,p}^-|^p \, dx = \lambda_2(p;\Omega) \cdot \int_\Omega |v_{\lambda_2,p}^-|^q \, dx.$$

Since $v_{\lambda_2,p}$ is sign-changing, $v_{\lambda_2,p}^+$, $v_{\lambda_2,p}^- \neq 0$. Let us consider the set

$$A = \left\{ u \in W_0^{1,p}(\Omega) \,|\, u = \alpha v_{\lambda_2,p}^+ + \beta v_{\lambda_2,p}^-, \, (\alpha, \beta) \neq (0,0) \right\}.$$

It is possible to prove that $\gamma(A) = 2$ as defined in Subsection 2.2. By definition of $\lambda_2(p;\Omega)$ we have

$$\lambda_2(p;\Omega) \leq \max_{u \in A} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx} = \max_{(\alpha,\beta) \neq (0,0)} \frac{|\alpha|^p \int_\Omega |\nabla v_{\lambda_2,p}^+|^p \, dx + |\beta|^p \int_\Omega |\nabla v_{\lambda_2,p}^-|^p \, dx}{|\alpha|^p \int_\Omega |v_{\lambda_2,p}^+|^p \, dx + |\beta|^p \int_\Omega |v_{\lambda_2,p}^-|^p \, dx} \leq$$

$$\leq \max \left\{ \frac{\int_\Omega |\nabla v_{\lambda_2,p}^+|^p \, dx}{\int_\Omega |v_{\lambda_2,p}^+|^p \, dx}, \frac{\int_\Omega |\nabla v_{\lambda_2,p}^-|^p \, dx}{\int_\Omega |v_{\lambda_2,p}^-|^p \, dx} \right\}.$$

Without loss of generality we can suppose

$$\lambda_2(p;\Omega) \leq \frac{\int_\Omega |\nabla v_{\lambda_2,p}^+|^p \, dx}{\int_\Omega |v_{\lambda_2,p}^+|^p \, dx}.$$

One can repeat the proof of Lemma 3.3 for the particular function $v_{\lambda_2,p}^+$ using the above inequality instead of Poincaré's in the end of Step 2 in order to obtain, for every $\eta > 0$,

$$\|v_{\lambda_2,p}^+\|_{L^\infty(\Omega)} \leq \widehat{C}_p^*(\eta) \cdot \left(\int_\Omega |\nabla v_{\lambda_2,p}^+|^p dx \right)^{\frac{1}{p}},$$

with

$$\widehat{C}_p^*(\eta) = \frac{\eta^{-\frac{n}{p}}}{|B_1(0)|^{\frac{1}{p}}} \left[\frac{1}{n^{\frac{1}{p}}} \left(\frac{p-1}{p-n} \right)^{1-\frac{1}{p}} \eta + \lambda_2(p,\Omega)^{-\frac{1}{p}} \right].$$

Proceeding as in Proposition 3.1, it can be checked that the estimate above holds with constant $\widehat{C}_p$ in (17) which satisfies $\widehat{C}_p \to \Lambda_2(\Omega)^{-1}$ as $p \to \infty$.

So, using (18), we obtain

$$\|v_{\lambda_2,p}^+\|_\infty \leq \widehat{C}_p \cdot \left(\int_\Omega |\nabla v_{\lambda_2,p}^+|^p \, dx \right)^{1/p} = \widehat{C}_p \cdot \left(\lambda_2(p;\Omega) \cdot \int_\Omega |v_{\lambda_2,p}^+|^q \, dx \right)^{1/p} \leq$$

$$\leq \widehat{C}_p \cdot \lambda_2(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}} \cdot \|v_{\lambda_2,p}^+\|_\infty^{\frac{q}{p}},$$

and hence

$$\|v_{\lambda_2,p}\|_\infty \geq \|v_{\lambda_2,p}^+\|_\infty \geq \left[\widehat{C}_p \cdot \lambda_2(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}} \right]^{-\frac{1}{\frac{q}{p}-1}} > 0. \qquad \square$$

Since the right-hand side in (15) converges to a positive quantity as $p \to \infty$, we deduce that any possible limit u_{Λ_1} in the spirit of Corollary 4.2 is nontrivial. In fact, we have the following result.

Theorem 5.4. *Suppose $Q > 1$, and let u_{Λ_1} be a uniform limit of the sequence $\{u_{\lambda_1,p}\}_p$. Then, $\|u_{\Lambda_1}\|_\infty = 1$. Moreover, $u_{\Lambda_1} > 0$ in Ω.*

Proof. From Lemma 4.1 and Lemma 5.1 we get

$$\left[C_p \cdot \lambda_1(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}} \right]^{-\frac{1}{\frac{q}{p}-1}} \leq \|u_{\lambda_1,p}\|_\infty \leq C_p \cdot \lambda_1(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}}.$$

Letting $p \to \infty$, we arrive at

$$1 = \left[\Lambda_1(\Omega)^{-1} \cdot \Lambda_1(\Omega) \right]^{-\frac{1}{Q-1}} \leq \|u_{\Lambda_1}\|_\infty \leq \Lambda_1(\Omega)^{-1} \cdot \Lambda_1(\Omega) = 1.$$

For the proof of the positivity of the limit, notice that u_{Λ_1} is ∞−superharmonic in the sense of [15]. Then, the Harnack inequality for ∞−superharmonic functions (see [14] and [15]) implies $u_{\Lambda_1} > 0$ inside Ω. $\qquad \square$

A similar result holds for the family of least-energy nodal solutions.

Theorem 5.5. *Suppose $Q > 1$, and let v_{Λ_2} be a uniform limit of the sequence $\{v_{\lambda_2,p}\}_p$. Then, $1 \leq \|v_{\Lambda_2}\|_\infty \leq \Lambda_2(\Omega) \cdot \Lambda_1(\Omega)^{-1}$. Moreover, v_{Λ_2} is sign-changing.*

Proof. From Lemma 4.1 and Lemma 5.2 we get

$$\left[\widehat{C}_p \cdot \lambda_2(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}} \right]^{-\frac{1}{\frac{q}{p}-1}} \leq \|v_{\lambda_2,p}\|_\infty \leq 2^{\frac{1}{p}} \cdot C_p \cdot \lambda_2(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}}.$$

Letting $p \to \infty$, we arrive at

$$1 = \left[\Lambda_2(\Omega)^{-1} \cdot \Lambda_2(\Omega) \right]^{-\frac{1}{Q-1}} \leq \|v_{\Lambda_2}\|_\infty \leq \Lambda_1(\Omega)^{-1} \cdot \Lambda_2(\Omega).$$

To prove that v_{Λ_2} is sign-changing, one can proceed as in Lemma 5.1 in order to obtain

$$\min\left\{ \|v_{\lambda_2,p}^+\|_\infty, \|v_{\lambda_2,p}^-\|_\infty \right\} \geq \left[C_p \cdot \lambda_2(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}} \right]^{-\frac{1}{\frac{q}{p}-1}} > 0$$

Letting $p \to \infty$ we obtain the claim. $\qquad\square$

Remark 5.6. Recall that $\Lambda_2(\Omega) \cdot \Lambda_1(\Omega)^{-1} \leq 2$ (see [11], Theorem 6.4) with an equality if and only if Ω is a ball. Hence, in general bounded domains one has $1 \leq \|v_{\Lambda_2}\|_\infty \leq 2$. On the other hand, it is easy to produce examples of domains for which $\Lambda_1(\Omega) = \Lambda_2(\Omega)$ and consequently $\|v_{\Lambda_2}\|_\infty = 1$; annuli and long enough stadiums (convex hulls of two balls with the same radius) belong to this category.

Due to the homogeneity of problem (1), we have

$$u_{\lambda_p,p} = \left(\lambda_p^{-1} \cdot \lambda_1(p;\Omega) \right)^{\frac{1}{q-p}} \cdot u_{\lambda_1,p}. \tag{19}$$

As a consequence, assuming $Q > 1$ and $\lim_{p\to\infty} \lambda^{1/p} = \Lambda > 0$, whenever we have convergence for the sequence $\{u_{\lambda_1,p}\}_p$ so we will for the sequence $\{u_{\lambda_p,p}\}_p$. Hence, from Corollary 4.2 and Theorem 5.4, we get the following consequence.

Corollary 5.7. *Let $\Lambda > 0$ and $Q > 1$. For every sequence $\{p_i\}_i$ such that $\lim_{i\to\infty} \lambda_{p_i}^{1/p_i} = \Lambda$ and the sequence $\{u_{\lambda_1,p_i}\}_i$ converges to u_{Λ_1}, then the sequence of positive solutions $u_{\lambda_{p_i},p_i}$ converges uniformly in Ω to a function u_Λ, such that*

$$u_\Lambda = \left(\Lambda^{-1} \cdot \Lambda_1(\Omega) \right)^{\frac{1}{Q-1}} \cdot u_{\Lambda_1}. \tag{20}$$

Moreover,

$$\|u_\Lambda\|_{L^\infty(\Omega)} = \left(\Lambda^{-1} \cdot \Lambda_1(\Omega) \right)^{\frac{1}{Q-1}}. \tag{21}$$

An analogous result holds for the sequence of least-energy nodal solutions $v_{\lambda_p,p}$.

6. THE CASE $Q = 1$.

In this section we consider the limit as $p \to \infty$ of positive solutions and least-energy nodal solutions of (1) when

$$Q = \lim_{p\to\infty} \frac{q(p)}{p} = 1.$$

In this case things change considerably; namely, the asymptotic behaviour of λ_p is decisive in order to determine convergence or blow-up of the sequences of solutions.

We will make use of the estimates for solutions of (1) already found in the previous sections. Notice that the fact that

$$\lim_{p\to\infty} C_p = \Lambda_1(\Omega)^{-1} \quad \text{and} \quad \lim_{p\to\infty} \widehat{C}_p = \Lambda_2(\Omega)^{-1}$$

where C_p is the Morrey constant in Proposition 3.1, and $\widehat{C}_p$ is the constant in Proposition 5.2, is a key point in these arguments.

We have the following results:

Theorem 6.1. *Let $\{u_{\lambda_p,p}\}_p$ be a sequence of positive solutions of* (1). *Suppose that*

$$Q = \lim_{p\to\infty} \frac{q(p)}{p} = 1 \qquad and \qquad \Lambda = \lim_{p\to\infty} \lambda_p^{1/p}. \qquad (22)$$

Then:

(i) *If* $\Lambda > \Lambda_1(\Omega)$, *then* $\lim_{p\to\infty} ||u_{\lambda_p,p}||_\infty = 0$.

(ii) *If* $\Lambda < \Lambda_1(\Omega)$, *then* $\lim_{p\to\infty} ||u_{\lambda_p,p}||_\infty = \infty$.

We give the proof of the theorem only in the case of least-energy nodal solutions since the case of positive solutions is virtually identical.

Theorem 6.2. *Let $\{v_{\lambda_p,p}\}_p$ be a sequence of least-energy nodal solutions of* (1) *and suppose that* (22) *holds. Then:*

(i) *If* $\Lambda > \Lambda_2(\Omega)$, *then* $\lim_{p\to\infty} ||v_{\lambda_p,p}||_\infty = 0$.

(ii) *If* $\Lambda < \Lambda_2(\Omega)$, *then* $\lim_{p\to\infty} ||v_{\lambda_p,p}||_\infty = \infty$.

Proof. (i) In a similar way to relation (19) we have

$$v_{\lambda_p,p} = \left(\lambda_p^{-1} \cdot \lambda_2(p;\Omega)\right)^{\frac{1}{q-p}} \cdot v_{\lambda_2,p} = \left(\lambda_p^{-\frac{1}{p}} \cdot \lambda_2(p;\Omega)^{\frac{1}{p}}\right)^{\frac{1}{\frac{q}{p}-1}} \cdot v_{\lambda_2,p}.$$

By Lemma 4.1 we have

$$||v_{\lambda_2,p}||_\infty \le 2^{\frac{1}{p}} \cdot C_p \cdot \lambda_2(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{-\frac{1}{p}},$$

where C_p is the Morrey constant in Proposition 3.1. Combining the two expressions and letting $p \to \infty$ we get the result.

(ii) From Lemma 5.2 one has

$$||v_{\lambda_2,p}||_\infty \ge \left[\widehat{C}_p \cdot \lambda_2(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}}\right]^{-\frac{1}{\frac{q}{p}-1}},$$

where $\widehat{C}_p$ is the constant in Proposition 5.2. Using the scaling property (19) as in the proof of (i), we get

$$||v_{\lambda_p,p}||_\infty \ge \left[\widehat{C}_p \cdot \lambda_p^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p}}\right]^{-\frac{1}{\frac{q}{p}-1}}.$$

Letting $p \to \infty$ and recalling that $\widehat{C}_p \to \Lambda_2(\Omega)^{-1} < \Lambda^{-1}$ we obtain the claim.   □

Remark 6.3. If $Q = 1$ and $\Lambda = \Lambda_1(\Omega)$ (resp. $\Lambda = \Lambda_2(\Omega)$), the estimates we found are not enough in order to establish convergence or blow-up of the sequence $\{u_{\lambda_p,p}\}_p$ (resp. $\{v_{\lambda_p,p}\}_p$). In the particular case $\lambda_p = \lambda_1(p;\Omega)$ (resp. $\lambda_p = \lambda_2(p;\Omega)$) we can only state that $\{u_{\lambda_1,p}\}_p$ (resp. $\{v_{\lambda_2,p}\}_p$) converge to a function u_{Λ_1} (resp. v_{Λ_2}). In the general case, the asymptotic behaviour is determined by the particular sequence of $q(p)$ and λ_p, as we can see in the following example. Set $\lambda_p = 2\lambda_1(p;\Omega)$ and $q(p) = p + \frac{1}{p}$; in this case $\Lambda = \Lambda_1(\Omega)$ and $Q = 1$. From (19) we have

$$u_{\lambda_p,p} = 2^{-\frac{1}{q-p}} \cdot u_{\lambda_1,p} = 2^{-p} \cdot u_{\lambda_1,p}$$

so that $||u_{\lambda_p,p}||_\infty \to 0$ as $p \to \infty$. If we now set $\lambda_p = \frac{1}{2}\lambda_1(p;\Omega)$ (so that again $\Lambda = \Lambda_1(\Omega)$ and $Q = 1$), we have

$$u_{\lambda_p,p} = 2^{\frac{1}{q-p}} \cdot u_{\lambda_1,p} = 2^p \cdot u_{\lambda_1,p}$$

and hence $||u_{\lambda_p,p}||_\infty \to \infty$ as $p \to \infty$ if u_{Λ_1} is nontrivial.

7. THE LIMIT PROBLEM

In the present section, we characterize uniform limits of solutions of (1) as solutions of a PDE. See [7] and [12] for related results in the eigenvalue case and [5] for the case $Q < 1$.

Proposition 7.1. *Let $\{u_p\}_p$ be a sequence of solutions of* (1). *Set*

$$Q = \lim_{p \to \infty} \frac{q(p)}{p} \quad \text{and} \quad \Lambda = \lim_{p \to \infty} \lambda_p^{1/p}.$$

If $\{u_p\}_p$ converge uniformly to a function u_Λ as $p \to \infty$, then u_Λ is a viscosity solution of the equation

$$\begin{cases} F_\Lambda(u, \nabla u, D^2 u) &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{23}$$

where

$$F_\Lambda(s, \xi, X) = \begin{cases} \min\{|\xi| - \Lambda s^Q, -X\xi \cdot \xi\} & \text{if } s > 0 \\ -X\xi \cdot \xi & \text{if } s = 0 \\ \max\{-\Lambda|s|^{Q-1}s - |\xi|, -X\xi \cdot \xi\} & \text{if } s < 0. \end{cases}$$

Proof. Let $x_0 \in \Omega$; if $u(x_0) > 0$, we can proceed analogously as in [5], Proposition 8. Let us then suppose that $u(x_0) < 0$, and let $\varphi \in \mathscr{C}^2(\Omega)$ be a function such that $u_\Lambda - \varphi$ has a strict local minimum at x_0. As u_Λ is the uniform limit of u_p, there exists a sequence of points $x_i \to x_0$ such that $(u_{p_i} - \varphi)(x_i)$ is a local minimum for each i. Then, as u_{p_i} is a viscosity solution and so a supersolution, we get

$$-\Delta_{p_i}\varphi(x_i) = -(p_i - 2)|\nabla\varphi(x_i)|^{p_i-4}\left\{\frac{|\nabla\varphi(x_i)|^2}{p_i - 2}\Delta\varphi(x_i) + \langle D^2\varphi(x_i)\nabla\varphi(x_i), \nabla\varphi(x_i)\rangle\right\}$$

$$\geq \lambda_p |u_{p_i}(x_i)|^{q_i-2} u_{p_i}(x_i).$$

where we set $q_i = q(p_i)$. Since $u(x_0) < 0$, this relation can also be written as

$$(p_i - 2)|\nabla\varphi(x_i)|^{p_i-4}\left\{\frac{|\nabla\varphi(x_i)|^2}{p_i - 2}\Delta\varphi(x_i) + \langle D^2\varphi(x_i)\nabla\varphi(x_i), \nabla\varphi(x_i)\rangle\right\}$$

$$\leq \lambda_p |u_{p_i}(x_i)|^{q_i-1}.$$

Rearranging terms, we obtain

$$(p_i - 2)\left[\frac{|\nabla\varphi(x_i)|}{\lambda_p^{\frac{1}{p_i-4}}|u_{p_i}(x_i)|^{\frac{q_i-1}{p_i-4}}}\right]^{p_i-4}\left\{\frac{|\nabla\varphi(x_i)|^2}{p_i - 2}\Delta\varphi(x_i) + \langle D^2\varphi(x_i)\nabla\varphi(x_i), \nabla\varphi(x_i)\rangle\right\} \leq 1.$$

If $|\nabla\varphi(x_0)| > \Lambda|u_\Lambda(x_0)|^Q$, then, necessarily, $-\Delta_\infty\varphi(x_0) \geq 0$, since otherwise we obtain a contradiction letting $i \to \infty$ in the previous inequality. On the other hand, if

$$|\nabla\varphi(x_0)| - \Lambda|u_\Lambda(x_0)|^Q \leq 0. \tag{24}$$

then

$$-\Lambda|u_\Lambda(x_0)|^{Q-1}u_\Lambda(x_0) - |\nabla\varphi(x_0)| \geq 0$$

and

$$\max\left\{-\Lambda|u_\Lambda(x_0)|^{Q-1}u_\Lambda(x_0) - |\nabla\varphi(x_0)|, -\Delta_\infty\varphi(x_0)\right\} \geq 0.$$

Hence, we conclude that u_Λ is a viscosity supersolution of equation (23).

It remains to be shown that u_Λ is a viscosity subsolution of the limit equation (23), i.e. we have to show that, for each $x_0 \in \Omega$ and $\varphi \in \mathscr{C}^2(\Omega)$ such that $u_\Lambda - \varphi$ attains a strict local maximum at x_0, we have

$$\max\left\{-\Lambda|u_\Lambda(x_0)|^{Q-1}u_\Lambda(x_0) - |\nabla\varphi(x_0)|, -\Delta_\infty\varphi(x_0)\right\} \leq 0.$$

As we did before, the uniform convergence of u_{p_i} to u_Λ provides us a sequence of points $x_i \to x_0$ which are local maxima of $u_{1,p_i} - \varphi$. Recalling the definition of viscosity subsolution we get, analogously as before,

$$(p_i - 2) \left[\frac{|\nabla \varphi(x_i)|}{\lambda_p^{\frac{1}{p_i-4}} |u_{p_i}(x_i)|^{\frac{q_i-1}{p_i-4}}} \right]^{p_i-4} \left\{ \frac{|\nabla \varphi(x_i)|^2}{p_i - 2} \Delta \varphi(x_i) + \langle D^2 \varphi(x_i) \nabla \varphi(x_i), \nabla \varphi(x_i) \rangle \right\} \geq 1,$$

for each fixed p_i. We can suppose $|\nabla \varphi(x_0)| \geq \Lambda |u_\Lambda(x_0)|^Q$ because otherwise we get a contradiction letting $i \to \infty$. This implies

$$-\Lambda |u_\Lambda(x_0)|^{Q-1} u_\Lambda(x_0) - |\nabla \varphi(x_0)| \leq 0$$

Moreover, it must be $-\Delta_\infty \varphi(x_0) \leq 0$, otherwise we get another contradiction. We finally obtain that

$$\max \left\{ -\Lambda |u_\Lambda(x_0)|^{Q-1} u_\Lambda(x_0) - |\nabla \varphi(x_0)|, -\Delta_\infty \varphi(x_0) \right\} \leq 0.$$

which means that u_Λ is a subsolution of (23).
The proof in the case $u(x_0) = 0$ is almost identical to [11], Lemma 4.3. $\qquad\square$

8. SYMMETRY AND UNIQUENESS OF POSITIVE SOLUTIONS FOR THE LIMIT PROBLEM IN A BALL.

We devote this section to the proof of a symmetry result for positive limit solutions in a ball, which is related to a uniqueness property. Notice that Comparison Principles typically fail to hold in problems with a supperdiffusive power-type nonlinearity (in contrast to the subdiffusive case, see [5]), so we rely on comparison for infinity sub- and superharmonic functions as well as on our estimates of the L^∞ norm of positive limit solutions, which turn out to be crucial.

Recall that we say that u is a *limit solution* of (23) if it can be obtained as a limit of solutions of (1) for some sequences of $q(p)$ and λ_p in the sense of Proposition 7.1. In order to simplify the notation, in the following we will write $\Lambda_1 = \Lambda_1(B_r(0)) = r^{-1}$.

The following is the main result in this section.

Theorem 8.1. *Let $r > 0$ and $Q > 1$. Then, for every $\Lambda > 0$, the cone*

$$u_\Lambda(x) = \Lambda^{-\frac{1}{Q-1}} \cdot r^{-\frac{Q}{Q-1}} \cdot (r - |x|)$$

is the unique limit solution of the problem

$$\begin{cases} \min \left\{ |\nabla u_\Lambda(x)| - \Lambda u_\Lambda^Q(x), -\Delta_\infty u_\Lambda(x) \right\} = 0 & in\ B_r(0) \\ u_\Lambda = 0 & on\ \partial B_r(0). \end{cases} \tag{25}$$

We split the proof of Theorem 8.1 into several partial results. Without loss of generality, we can suppose $\Lambda = \Lambda_1$ in the argument due to the following scaling property of the limit problem. We will omit the proof since it is standard.

Lemma 8.2. *The solutions of the problem*

$$\begin{cases} \min \left\{ |\nabla u_\Lambda(x)| - \Lambda u_\Lambda^Q(x), -\Delta_\infty u_\Lambda(x) \right\} = 0 & in\ B_r(0) \\ u_\Lambda = 0 & on\ \partial B_r(0), \end{cases} \tag{26}$$

and those of the problem

$$\begin{cases} \min \left\{ |\nabla u_{\Lambda_1}(x)| - \Lambda_1 u_{\Lambda_1}^Q(x), -\Delta_\infty u_{\Lambda_1}(x) \right\} = 0 & in\ B_r(0) \\ u_{\Lambda_1} = 0 & on\ \partial B_r(0). \end{cases} \tag{27}$$

are related through the expression $u_\Lambda = \left(\Lambda^{-1} \Lambda_1 \right)^{\frac{1}{Q-1}} u_{\Lambda_1}$.

First, we show that there exists a unique cone which is a positive solution of the limit problem for $\Lambda = \Lambda_1$.

Lemma 8.3. *The normalized distance to the boundary,*

$$\delta(x) = \frac{dist(x, \partial B_r(0))}{\|dist(\cdot, \partial B_r(0))\|_\infty} = 1 - \frac{|x|}{r} \tag{28}$$

is the unique cone which is a viscosity solution of problem (27).

Proof. Let $a > 0$, and define the cone

$$C_a(x) = a \cdot dist(x, \partial B_r(0)) = a \cdot (r - |x|). \tag{29}$$

We are going to prove that $C_a(x)$ is a viscosity solution of problem (26) if and only if $a = \Lambda_1 = r^{-1}$.

First of all, since $C_a(x)$ is smooth if $x \neq 0$, it can be checked by direct computation that

$$-\Delta_\infty C_a(x) = 0 \qquad \forall x \in B_r(0) \setminus \{0\},$$

in the classical sense. Hence, we need make sure that

$$|\nabla C_a(x)| - \Lambda_1 C_a^Q(x) \geq 0 \qquad \forall x \in B_r(0) \setminus \{0\}.$$

Indeed, plugging (29) into the latter expression (recall that $x \neq 0$ so the derivatives are classical), we find that

$$|\nabla C_a(x)| - \Lambda_1 C_a^Q(x) = a - \Lambda_1 a^Q (r - |x|)^Q \geq 0,$$

must hold for points with $|x|$ arbitrarily small. Hence, we find the following necessary condition for a,

$$a - \Lambda_1^{1-Q} a^Q \geq 0. \tag{30}$$

Next, let $\varphi \in \mathscr{C}^2$ such that $C_a - \varphi$ has a local maximum point at 0. We aim to prove that

$$\min\left\{|\nabla\varphi(0)| - \Lambda_1 C_a^Q(0), -\Delta_\infty\varphi(0)\right\} \leq 0. \tag{31}$$

It is well known that

$$\min\left\{|\nabla C_a(x)| - a, -\Delta_\infty C_a(x)\right\} = 0.$$

Hence, by definition of viscosity subsolution we have either $|\nabla\varphi(0)| \leq a$ or $-\Delta_\infty\varphi(0) \leq 0$. In the latter case, (31) holds and there is nothing to prove. Thus, we can suppose in the sequel that $-\Delta_\infty\varphi(0) > 0$ and $|\nabla\varphi(0)| \leq a$. We get $C_a(0) = a\Lambda_1^{-1}$ and

$$|\nabla\varphi(0)| - \Lambda_1 C_a^Q(0) \leq a - \Lambda_1^{1-Q} a^Q.$$

Recalling (30), we discover that we will be done only if

$$a - \Lambda_1^{1-Q} a^Q = 0.$$

Since $Q > 1$, the only nontrivial solution to this equation is $a = \Lambda_1 = r^{-1}$. $\qquad\square$

Next, we prove that any other possible limit solution associated to Λ_1 is not greater than the normalized distance to the boundary $\delta(x)$.

Lemma 8.4. *Let $u_{\Lambda_1}(x)$ be a limit viscosity solution of* (27). *Then, $u_{\Lambda_1} \leq \delta$ in $B_r(0)$.*

Proof. Fix $R \in (0, 1)$ and consider the auxiliary (subdiffusive) problem

$$\begin{cases} \min\left\{|\nabla w(x)| - \Lambda_1 w^R(x), -\Delta_\infty w(x)\right\} = 0 & \text{in } B_r(0) \\ w = 0 & \text{on } \partial B_r(0), \end{cases} \tag{32}$$

1. First, we seek to prove that u_{Λ_1} is a viscosity subsolution of (32).

To this aim, consider a point $x_0 \in B_r(0)$ and a function $\varphi \in \mathscr{C}^2$ such that $u_{\Lambda_1} - \varphi$ has a maximum at x_0. As u_{Λ_1} is a viscosity solution of (27), it satisfies

$$\min\left\{|\nabla\varphi(x_0)| - \Lambda_1 u_{\Lambda_1}^Q(x_0), -\Delta_\infty\varphi(x_0)\right\} \leq 0 \quad \text{in } B_r(0).$$

If $-\Delta_\infty \varphi(x_0) \leq 0$ we are done. So we can suppose $-\Delta_\infty \varphi(x_0) > 0$ and $|\nabla \varphi(x_0)| - \Lambda_1 u_{\Lambda_1}^Q (x_0) \leq 0$. Since $\|u_{\Lambda_1}\|_\infty = 1$ (see Proposition 5.4) we clearly have

$$|\nabla \varphi(x_0)| - \Lambda_1 u_{\Lambda_1}^R (x_0) \leq \Lambda_1 \left(u_{\Lambda_1}^Q (x_0) - u_{\Lambda_1}^R (x_0) \right) \leq 0,$$

and then

$$\min \left\{ |\nabla \varphi(x_0)| - \Lambda_1 u_{\Lambda_1}^R (x_0), -\Delta_\infty \varphi(x_0) \right\} \leq 0 \quad \text{in } B_r(0).$$

2. Next, we prove that $\delta(x)$ in (28) is a viscosity supersolution of (32). It is well known that for any bounded domain, δ is the unique solution of

$$\begin{cases} \min \left\{ |\nabla \delta(x)| - \Lambda_1, -\Delta_\infty \delta(x) \right\} = 0 & \text{in } B_r(0) \\ \delta = 0 & \text{on } \partial B_r(0), \end{cases}$$

Consider a point $x_0 \in B_r(0)$ and a function $\varphi \in \mathscr{C}^2$ such that $\delta - \varphi$ has a minimum in x_0. By definition of viscosity supersolution, we have

$$|\nabla \varphi(x_0)| - \Lambda_1 \geq 0 \quad \text{and} \quad -\Delta_\infty \varphi(x_0) \geq 0.$$

We clearly get

$$|\nabla \varphi(x_0)| - \Lambda_1 \delta^R(x_0) \geq \Lambda_1 \left(1 - \delta^R(x_0) \right) \geq 0,$$

and then

$$\min \left\{ |\nabla \varphi(x_0)| - \Lambda_1 \delta^R(x_0), -\Delta_\infty \varphi(x_0) \right\} \geq 0 \quad \text{in } B_r(0).$$

3. Finally, since u_{Λ_1} and δ are respectively a sub- and supersolution of (32) both of them positive and satisfying $u_{\Lambda_1} = \delta = 0$ on $\partial B_r(0)$, the result follows by comparison (see [5]). $\qquad \square$

We are now able to finish the proof of Theorem 8.1.

Proof of Theorem 8.1. We observe that, since $u_{\Lambda_1} \leq \delta$ by Lemma 8.4, we have

$$\{x \in B_r(0) : u_{\Lambda_1}(x) = \|u_{\Lambda_1}\|_\infty = 1\} = \{0\}, \tag{33}$$

as the set on the left-hand side is nonempty (see Proposition 5.4). Moreover, $\delta(x)$ is the unique (see [10]) viscosity solution of the problem

$$\begin{cases} -\Delta_\infty \delta(x) = 0 & \text{in } B_r(0) \setminus \{0\} \\ \delta(x) = 0 & \text{on } \partial B_r(0) \\ \delta(0) = 1. \end{cases} \tag{34}$$

On the other hand, u_{Λ_1} is infinity superharmonic in $B_r(0)$ and hence a viscosity supersolution of (34). By comparison (see [10]), we get $u_{\Lambda_1} \geq \delta$. Then, from Lemma 8.4, we have $u_{\Lambda_1} \equiv \delta$, which is the claim. $\qquad \square$

Remark 8.5. The partial results in the proof of Theorem 8.1 hold in greater generality. Lemmas 8.2 and 8.4 are true in the case of a general bounded domain $\Omega \subseteq \mathbb{R}^n$. For Lemma 8.3, a sufficient condition (standard in the literature) is that the set of maximal distance to the boundary coincides with the set of points $x \in \Omega$ where $\text{dist}(x, \partial \Omega)$ is not of class $\mathscr{C}^1$. This assumption, a sort of symmetry condition on Ω, is satisfied by domains like balls, stadiums (convex hull of two identical balls) and annuli. Indeed, the crucial point where we use that the domain is a ball is (33).

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