LIMITS AS $p \to \infty$ OF p-LAPLACIAN CONCAVE-CONVEX PROBLEMS

FERNANDO CHARRO AND IRENEO PERAL

ABSTRACT. We study the behavior as $p \to \infty$ of the sequel of positive weak solutions of the concave-convex problem

(P)
$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda\, u^{q(p)} + u^{r(p)} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\lambda > 0$ and the exponents q, r satisfy

$$\lim_{p\to\infty} \frac{q(p)}{p-1} = Q, \qquad \lim_{p\to\infty} \frac{r(p)}{p-1} = R, \qquad \text{with} \quad 0 < Q < 1 < R.$$

We characterize any positive uniform limit of a sequence of weak solutions of (P) as a viscosity solution of

$$\min\left\{ |\nabla u_\Lambda| - \max\{\Lambda u_\Lambda^Q, u_\Lambda^R\}, -\Delta_\infty u_\Lambda \right\} = 0 \quad \text{in } \Omega.$$

Notice that the limit process decouples the nonlinearity. We obtain existence, non-existence and global multiplicity of positive viscosity solutions of the limit problem in terms of the parameter Λ.

1. INTRODUCTION

Our goal is to study the behavior as $p \to \infty$ of the sequence of positive solutions of the concave-convex problem

(1.1)
$$\begin{cases} -\Delta_p u = \lambda\, u^{q(p)} + u^{r(p)} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplacian, $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\lambda > 0$ and the exponents q, r are assumed to satisfy

(1.2) $$\lim_{p\to\infty} \frac{q(p)}{p-1} = Q, \qquad \lim_{p\to\infty} \frac{r(p)}{p-1} = R, \qquad \text{with} \quad 0 < Q < 1 < R.$$

We refer to problem (1.1) as concave-convex in the sense that $0 < q(p) < p-1 < r(p)$ for p large enough. In particular, we shall always assume $p > n$.

The asymptotics of problems having a power-type right-hand side, namely,

$$-\Delta_p u = \lambda\, u^{\alpha(p)} \quad \text{in } \Omega$$

1

have been already studied in the literature in the eigenvalues case, where $\alpha(p) = p - 1$ (see [13, 19, 20]), the concave power case $\alpha(p) = q(p)$ (see [9]), and the case with a convex power $\alpha(p) = r(p)$ (see [8]).

Hence, it seems natural to consider the combined effect of a concave and a convex power as $p \to \infty$.

Problem (1.1) is studied in [4, 7, 15, 14]. In [7], it is proved the existence of a threshold value $\lambda_{\max,p}$ such that there exist a minimal positive solution of (1.1) for $\lambda < \lambda_{\max,p}$ and no positive solution exists for $\lambda > \lambda_{\max,p}$. It is important to point out that this results hold with no restriction in the size of the convex exponent r even when $p < n$. We provide in Section 6 a *quantitative construction* of the branch of minimal positive solutions of (1.1) as well as the proof of non existence beyond the threshold $\lambda_{\max,p}$.

In [4, 14] it is proved that there exists a second positive solution for every $\lambda \in (0, \lambda_{\max,p})$ whenever $r < p^* - 1$, with p^* the Sobolev critical exponent. Notice that this means no restriction in the case of our interest since, since the critical exponent is $p^* = \infty$ when $p > n$.

The paper is organized as follows. In Section 2 we provide some necessary preliminaries. Then, in Section 3 we introduce the limit concave-convex problem in a formal way. We prove that, whenever $\Lambda = \lim_{p\to\infty} \lambda_p^{1/p}$ any positive uniform limit $u_\Lambda = \lim_{p\to\infty} u_{\lambda_p,p}$ is a viscosity solution of

$$
(1.3) \qquad
\begin{cases}
\min\left\{|\nabla u_\Lambda| - \max\{\Lambda u_\Lambda^Q, u_\Lambda^R\}, -\Delta_\infty u_\Lambda\right\} = 0 & \text{in } \Omega, \\
u_\Lambda > 0 \text{ in } \Omega \\
u_\Lambda = 0 \text{ on } \partial\Omega.
\end{cases}
$$

Then, once the limit problem is presented, we study the existence non-existence and multiplicity of positive solutions of the limit problem (1.3) in terms of the parameter Λ.

More precisely, in Section 4 we prove some non-existence results. We prove that there is no positive solution of the limit problem (1.3) for

$$
\Lambda > \hat\Lambda := \Lambda_1(\Omega)^{\frac{R-Q}{R-1}} = \lim_{p\to\infty} \lambda_{\max,p}^{1/p},
$$

for $\Lambda_1(\Omega) := \left(\max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)\right)^{-1}$.

Moreover, we also prove that every solution u_Λ of (1.3) verifies $w_\Lambda \leq u_\Lambda$, where w_Λ is the unique positive solution of the limit concave problem

$$
(1.4) \qquad
\begin{cases}
\min\left\{|\nabla w_\Lambda| - \Lambda\, w_\Lambda^Q, -\Delta_\infty w_\Lambda\right\} = 0 & \text{in } \Omega, \\
w_\Lambda = 0 & \text{on } \partial\Omega.
\end{cases}
$$

The decoupling of the nonlinearity in (1.3) under the limit process is a key feature for the analysis.

Next, in Section 5 we study the existence of a curve of positive minimal solutions of problem (1.3). A remarkable fact is that, for every $\Lambda \leq \hat\Lambda$, the unique solution of the limit concave problem (1.4) is also a solution of the limit concave-convex problem (1.3). The reason is that there is a critical size for solutions under which the convex power has no influence in problem (1.3) since

$$
\max\{\Lambda t^Q, t^R\} = \Lambda\, t^Q \quad \Leftrightarrow \quad t \leq \Lambda^{\frac{1}{R-Q}}.
$$

In fact, we prove that for every $\Lambda \le \hat{\Lambda}$ the unique positive solution of (1.3) with $\|u_\Lambda\|_\infty \le \Lambda^{\frac{1}{R-Q}}$ (the critical size) is given by the solution of (1.4).

We would like to stress that this makes a significant difference with the case $p < \infty$, where the concave and convex power always have a mutual influence, no matter the size of the solutions.

Feeding back this new information on the size of solutions, we extend the non-existence result and prove that in fact there is no positive solution of (1.3) with

$$\Lambda^{\frac{1}{R-Q}} < \|u_\Lambda\|_\infty < \Lambda_1(\Omega)^{\frac{1}{R-1}}.$$

Then, in Section 6, we turn our attention back to justify uniform convergence as $p \to \infty$ of solutions of (1.1) to solutions of (1.3). Assuming the natural normalization

$$\lambda_p^{1/p} \to \Lambda \text{ as } p \to \infty,$$

we justify the uniform convergence of the sequence of minimal solutions of (1.1) corresponding to λ_p to the minimal positive solution of (1.3) corresponding to Λ. We get the necessary estimates from the construction of the minimal branch at level p.

In Section 7, we pass to the limit as $p \to \infty$ on subsequences of mountain pass solutions of the problems at level p. The main feature is a global multiplicity result in Λ for the limit problem (1.3), that is, the existence of at least two positive viscosity solutions for all $\Lambda \in (0, \hat{\Lambda})$.

Finally, in Section 8 we find explicit solutions whenever the domain Ω satisfy a certain geometric condition. These solutions make apparent the results on existence and multiplicity in previous sections.

2. Preliminaries

We recall here some facts that will be needed later. First we recall the well-known Morrey estimates, with an explicit expression of the constants that will be crucial in the sequel. See [8] for the proof.

Lemma 2.1. *Assume $n < p < \infty$ and $u \in W_0^{1,p}(\Omega)$. Then $u \in \mathcal{C}^\gamma(\Omega)$, where $\gamma = 1 - \frac{n}{p}$, and the following hold:*

   i) *L^∞-estimate:*

$$\|u\|_{L^\infty(\Omega)} \le C_p \cdot \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}},$$

   with constant

(2.1) $\qquad C_p = p \, |B_1(0)|^{-\frac{1}{p}} \, n^{-\frac{n(p+1)}{p^2}} \, (p-1)^{\frac{n(p-1)}{p^2}} \, (p-n)^{\frac{n}{p^2}-1} \lambda_1(p;\Omega)^{\frac{n-p}{p^2}}.$

   ii) *Hölder estimate:*

$$\frac{|u(x) - u(y)|}{|x - y|^\gamma} \le \tilde{C}_p \cdot \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}},$$

   where

$$\tilde{C}_p = \frac{2C}{|\partial B_1(0)|^{\frac{1}{p}}} \left(\frac{p-1}{p-n} \right)^{1-\frac{1}{p}}$$

   and C is a constant depending only on n.

Remark 2.2. It can be checked that $\lim_{p\to\infty} C_p = \max_{x\in\Omega} \text{dist}(x, \partial\Omega)$.

In the following lemma we state that weak solutions of our problem are also viscosity solutions. The proof, which we omit here, follows analogously to [20, Lemma 1.8] (see also [6]).

Lemma 2.3. *If u is a continuous weak solution of (1.1), then it is a viscosity solution of the same problem, rewritten as*

$$(2.2) \qquad \begin{cases} F_p(\nabla u, D^2 u) = \lambda\, u^{q(p)} + u^{r(p)} & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

where

$$F_p(\xi, X) = -|\xi|^{p-2} \cdot \text{trace}\left(\left(Id + (p-2)\frac{\xi \otimes \xi}{|\xi|^2}\right)X\right).$$

The divergence form is more useful from the variational point of view, while the expanded form (2.2) is preferred in the viscosity framework. In the sequel we shall always consider the more suitable form of our problem between (1.1) and (2.2) without any further reference.

Next, we present some background on certain auxiliary problems that will be profusely used in the sequel.

2.1. **The problem with right-hand side 1.** In [21] the problem

$$\begin{cases} -\Delta_p v = 1 & in\ \Omega \\ v \in W_0^{1,p}(\Omega) \end{cases}$$

is studied in connection with torsional creep problems when Ω is a general bounded domain. In the case of our interest, $p > n$ every function in $v \in W_0^{1,p}(\Omega)$ can be considered continuous in $\overline{\Omega}$ and 0 on the boundary in the classical sense.

However, it is interesting to mention that in general the boundary datum is not satisfied in the classical sense, that is, given $x_0 \in \partial\Omega$, it is not necessarily true that $\lim_{x\to x_0} v(x) = 0$ when $1 < p \le n$. Nevertheless, the points where $\lim_{x\to x_0} v(x) = 0$ can be characterized by means of a version of the Wiener Criterion stated by Maz'ja [29] in the nonlinear framework (see [16, 22, 23] and also [12, 28]).

The existence result we shall need below is the following. See [21] and [18, Theorem 3.11] for the proof.

Proposition 2.4. *Let Ω be a bounded domain and $n < p < \infty$. Then, there exists a unique solution $v_{1,p} \in W_0^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ of the auxiliary problem*

$$(2.3) \qquad \begin{cases} -\Delta_p v_{1,p} = 1 & in\ \Omega \\ v_{1,p} = 0 & on\ \partial\Omega, \end{cases}$$

and $v_{1,p}$ converge uniformly as $p \to \infty$ to the unique viscosity solution to

$$\begin{cases} \min\{|\nabla v| - 1, -\Delta_\infty v\} = 0 & in\ \Omega, \\ v = 0 & on\ \partial\Omega. \end{cases}$$

Moreover, $v(x) = dist(x, \partial\Omega)$.

2.2. The p-eigenvalue problem.

We shall also need some facts about first eigenvalues and eigenfunctions. Let us recall that the first eigenvalue $\lambda_1(p;\Omega)$ is characterized by the nonlinear Rayleigh quotient

$$\lambda_1(p;\Omega) = \inf_{\phi \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla \phi|^p \, dx}{\int_\Omega |\phi|^p \, dx}.$$

In [24] (see also [25]) it is proved that the first eigenvalue of the p-Laplacian is simple (that is, that the first eigenfunction is unique up to multiplication by constants) when Ω is a mere bounded domain. Previous results ([3, 15, 30] and the references in [24]) require further regularity of $\partial\Omega$. Moreover, it is also proved in [24] that in a general bounded domain Ω only the first eigenfunction is positive and that the first eigenvalue is isolated (there exists $\epsilon > 0$ such that there are no eigenvalues in $(\lambda_1, \lambda_1 + \epsilon]$).

Proposition 2.5 ([24]). *Let Ω be a bounded domain and $n < p < \infty$. Then, there exists a nontrivial positive solution $\phi_{1,p} \in W_0^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ of*

$$\begin{cases} -\Delta_p \phi_{1,p} = \lambda_1(p;\Omega) \, |\phi_{1,p}|^{p-2}\phi_{1,p} & \text{in } \Omega \\ \phi_{1,p} = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $\lambda_1(p;\Omega)$ is simple and isolated.

The result above is also true in the range $1 < p \leq n$, but then the boundary datum is not necessarily realized in the classical sense and must be interpreted in the trace sense, that is, $\phi_{1,p} \in W_0^{1,p}(\Omega)$. Also in this case, the points $x_0 \in \partial\Omega$ where $\lim_{x \to x_0} \phi_{1,p}(x) = 0$ are characterized by the Wiener's criterion mentioned above, see [22].

In the following lemma, we recall the behavior as $p \to \infty$ of the first eigenvalue of the p-Laplacian (see [20] for the proof).

Lemma 2.6. $\lim_{p \to \infty} \lambda_1(p,\Omega)^{\frac{1}{p}} = \Lambda_1(\Omega) = \left(\max_{x \in \Omega} \operatorname{dist}(x, \partial\Omega) \right)^{-1}.$

2.3. The concave problem.

The last tool we need in the construction below is the solution to the concave problem.

Proposition 2.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p > n$ and $0 < q < p - 1$. Then, for every $\lambda > 0$, there exists a unique $u_\lambda \in W_0^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ solution of the problem*

$$(2.4) \qquad \begin{cases} -\Delta_p u_\lambda = \lambda \, u_\lambda^q, & \text{in } \Omega, \\ u_\lambda > 0 & \text{in } \Omega, \\ u_\lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, by homogeneity,

$$u_\lambda(x) = \lambda^{\frac{1}{(p-1)-q}} \, u_1(x),$$

where u_1 is the solution of the problem with $\lambda = 1$.

Uniqueness of positive solutions is a known result and can be found for instance in [1] in the variational setting (see also [10] for a proof in the viscosity framework). For the proof of existence, one can construct a sub- and supersolution using appropriate rescalings of $\phi_{1,p}$ and $v_{1,p}$ respectively (see for instance [10]). Then, we can iterate between this sub- and supersolution and construct the solution.

We conclude this preliminary section collecting some facts about the concave limit problem

$$(2.5) \quad \begin{cases} \min\left\{|\nabla w_\Lambda| - \Lambda w_\Lambda^Q, -\Delta_\infty w_\Lambda\right\} = 0 & \text{in } \Omega, \\ w_\Lambda > 0 & \text{in } \Omega \\ w_\Lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

The proofs of the following results can be found in [9].

The main result is the following Comparison Principle, which turns out to be a valuable tool in the study of the limit concave-convex problem.

Proposition 2.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and consider a viscosity subsolution $u \in \mathcal{C}(\overline{\Omega})$ and a viscosity supersolution $v \in \mathcal{C}(\overline{\Omega})$ of*

$$(2.6) \quad \min\left\{|\nabla w(x)| - w^Q(x), -\Delta_\infty w(x)\right\} = 0 \quad \text{in } \Omega.$$

Assume that both $u, v > 0$ in Ω and $u \leq v$ on $\partial\Omega$. Then, $u \leq v$ in $\overline{\Omega}$.

The following result deals with existence of positive solutions to (2.5). The uniqueness assertion is a consequence of the Comparison Principle.

Proposition 2.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\Lambda > 0$ and a sequence $\{\lambda_p\}_p$ such that $\lambda_p^{1/p} \to \Lambda$ as $p \to \infty$. Then, the sequence $\{u_{\lambda_p,p}\}_{p>n}$ of weak solutions of*

$$\begin{cases} -\Delta_p u_{\lambda_p,p} = \lambda_p\, u_{\lambda_p,p}^{q(p)} & \text{in } \Omega \\ u_{\lambda_p,p} > 0 & \text{in } \Omega \\ u_{\lambda_p,p} = 0 & \text{on } \partial\Omega, \end{cases}$$

converge uniformly to w_Λ, the unique positive solution of the limit problem (2.5) corresponding to Λ.

Finally, we have the following estimates for w_Λ.

Proposition 2.10. *Let $\Omega \subset \mathbb{R}^n$ a bounded domain and $0 < Q < 1$. Consider $\Lambda > 0$ and w_Λ the positive solution of (2.5). Then, we have*

$$(2.7) \quad \begin{aligned} \left(\Lambda\, \|dist(\cdot, \partial\Omega)\|_\infty\right)^{\frac{1}{1-Q}} v(x) &\leq w_\Lambda(x) \\ &\leq \left(\Lambda\, \|dist(\cdot, \partial\Omega)\|_\infty\right)^{\frac{1}{1-Q}} \frac{dist(x, \partial\Omega)}{\|dist(\cdot, \partial\Omega)\|_\infty}, \end{aligned}$$

for every $x \in \Omega$, where $v(x)$ is the first (maximal) ∞−eigenfunction (see [20]) normalized to $\|v\|_{L^\infty} = 1$. Moreover

$$(2.8) \quad \|w_\Lambda\|_{L^\infty(\Omega)} = \left(\Lambda \cdot \|dist(\cdot, \partial\Omega)\|_\infty\right)^{\frac{1}{1-Q}}.$$

3. The limit problem.

In the present section, we characterize uniform limits of solutions of (1.1) as solutions of a PDE. For the moment we shall assume the uniform convergence which will be proved in Sections 6 and 7.

Proposition 3.1. *Assume that $\{\lambda_p\}_p$ verifies $\lim_{p\to\infty} \lambda_p^{1/p} = \Lambda$ and let $q(p), r(p)$ and $0 < Q < 1 < R$ as in (1.2). Suppose that $u_{\lambda_p,p} \in W_0^{1,p}(\Omega)$ is a weak solution of*

$$(3.1) \qquad \begin{cases} -\Delta_p u_{\lambda_p,p} = \lambda_p u_{\lambda_p,p}^{q(p)} + u_{\lambda_p,p}^{r(p)} & in \ \Omega \\ u_{\lambda_p,p} > 0 & in \ \Omega \\ u_{\lambda_p,p} = 0 & on \ \partial\Omega, \end{cases}$$

and that $u_{\lambda_p,p} \to u_\Lambda > 0$ uniformly as $p \to \infty$. Then, u_Λ is a viscosity solution of the limit concave-convex problem

$$(3.2) \qquad \begin{cases} \min\{|\nabla u_\Lambda| - \max\{\Lambda u_\Lambda^Q, u_\Lambda^R\}, -\Delta_\infty u_\Lambda\} = 0 & in \ \Omega, \\ u_\Lambda > 0 & in \ \Omega \\ u_\Lambda = 0 & on \ \partial\Omega. \end{cases}$$

Proof. Let $x_0 \in \Omega$ and $\phi \in \mathcal{C}^2(\Omega)$ such that $u_\Lambda - \phi$ has a strict local minimum at x_0. By hypotesis, u_Λ is the uniform limit of $u_{\lambda_p,p}$, so there exists a sequence of points $x_p \to x_0$ such that $(u_{\lambda_p,p} - \phi)(x_p)$ is a local minimum for each p. As $u_{\lambda_p,p}$ is a continuous weak solution of (3.1), it is also a viscosity solution and so a supersolution. Then, we get

$$-(p-2)|\nabla\phi(x_p)|^{p-4}\left\{\frac{|\nabla\phi(x_p)|^2}{p-2}\Delta\phi(x_p) + \langle D^2\phi(x_p)\nabla\phi(x_p), \nabla\phi(x_p)\rangle\right\}$$
$$= -\Delta_p\phi(x_p) \geq \lambda_p u_{\lambda_p,p}^{q(p)}(x_p) + u_{\lambda_p,p}^{r(p)}(x_p).$$

Rearranging terms, we obtain

$$(3.3) \qquad -(p-2)\left[\frac{|\nabla\phi(x_p)|}{\left(\lambda_p u_{\lambda_p,p}^{q(p)}(x_p) + u_{\lambda_p,p}^{r(p)}(x_p)\right)^{\frac{1}{p-4}}}\right]^{p-4} \left\{\frac{|\nabla\phi(x_p)|^2}{p-2}\Delta\phi(x_p) \right.$$
$$\left. + \langle D^2\phi(x_p)\nabla\phi(x_p), \nabla\phi(x_p)\rangle\right\} \geq 1.$$

Notice that

$$\max\left\{\lambda_p^{\frac{1}{p-4}} u_{\lambda_p,p}^{\frac{q(p)}{p-4}}(x_p), u_{\lambda_p,p}^{\frac{r(p)}{p-4}}(x_p)\right\} \leq \left(\lambda_p u_{\lambda_p,p}^{q(p)}(x_p) + u_{\lambda_p,p}^{r(p)}(x_p)\right)^{\frac{1}{p-4}}$$
$$\leq 2^{\frac{1}{p-4}} \max\left\{\lambda_p^{\frac{1}{p-4}} u_{\lambda_p,p}^{\frac{q(p)}{p-4}}(x_p), u_{\lambda_p,p}^{\frac{r(p)}{p-4}}(x_p)\right\},$$

and hence,

$$\lim_{p\to\infty}\left(\lambda_p u_{\lambda_p,p}^{q(p)}(x_p) + u_{\lambda_p,p}^{r(p)}(x_p)\right)^{\frac{1}{p-4}} = \max\{\Lambda u_\Lambda^Q(x_0), u_\Lambda^R(x_0)\}.$$

We point out that the latter quantity is positive since $u_\Lambda > 0$ by hypothesis. Then, if we suppose that

$$|\nabla\phi(x_0)| < \max\{\Lambda u_\Lambda^Q(x_0), u_\Lambda^R(x_0)\}$$

we obtain a contradiction letting $p \to \infty$ in (3.3). Thus, it must be

$$(3.4) \qquad |\nabla\phi(x_0)| - \max\{\Lambda u_\Lambda^Q(x_0), u_\Lambda^R(x_0)\} \geq 0.$$

We also have that

$$(3.5) \qquad -\Delta_\infty\phi(x_0) = -\langle D^2\phi(x_0)\nabla\phi(x_0), \nabla\phi(x_0)\rangle \geq 0,$$

because we would get a contradiction with (3.3) otherwise.

We can put together (3.4) and (3.5) writing

$$\min\left\{|\nabla\phi(x_0)| - \max\{\Lambda u_\Lambda^Q(x_0), u_\Lambda^R(x_0)\}, -\Delta_\infty\phi(x_0)\right\} \geq 0$$

and conclude that u_Λ is a viscosity supersolution of equation (3.2).

It remains to be shown that u_Λ is a viscosity subsolution of the limit equation (3.2), i.e. we have to show that, for each $x_0 \in \Omega$ and $\phi \in \mathcal{C}^2(\Omega)$ such that $u_\Lambda - \phi$ attains a strict local maximum at x_0 (note that x_0 and ϕ are not the same than before) we have

$$\min\left\{|\nabla\phi(x_0)| - \max\{\Lambda u_\Lambda^Q(x_0), u_\Lambda^R(x_0)\}, -\Delta_\infty\phi(x_0)\right\} \leq 0.$$

We can suppose that

$$|\nabla\phi(x_0)| > \max\{\Lambda u_\Lambda^Q(x_0), u_\Lambda^R(x_0)\},$$

because we are done otherwise. As we did before, the uniform convergence of $u_{\lambda_p, p}$ to u_Λ provides a sequence of points $x_p \to x_0$ which are local maxima of $u_{\lambda_p, p} - \phi$. Recalling the definition of viscosity subsolution we have

$$-(p-2)\left[\frac{|\nabla\phi(x_p)|}{\left(\lambda u_{\lambda_p, p}^{q(p)}(x_p) + u_{\lambda_p, p}^{r(p)}(x_p)\right)^{\frac{1}{p-4}}}\right]^{p-4}\left\{\frac{|\nabla\phi(x_p)|^2}{p-2}\Delta\phi(x_p)\right.$$
$$\left. + \langle D^2\phi(x_p)\nabla\phi(x_p), \nabla\phi(x_p)\rangle\right\} \leq 1,$$

for each fixed p. Letting $p \to \infty$ we obtain $-\Delta_\infty\phi(x_0) \leq 0$ because in other case we get a contradiction. $\qquad\square$

4. A PRIORI NON-EXISTENCE RESULTS FOR THE LIMIT PROBLEM

Here, we present some results on non-existence of positive viscosity solutions of the limit concave-convex problem

$$(4.1) \quad \begin{cases} \min\left\{|\nabla u_\Lambda| - \max\{\Lambda u_\Lambda^Q, u_\Lambda^R\}, -\Delta_\infty u_\Lambda\right\} = 0 & \text{in } \Omega, \\ u_\Lambda > 0 & \text{in } \Omega, \\ u_\Lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

We have summarized the results in this section in Figure 1.

4.1. **Non-existence below the curve of positive solutions of the limit concave problem.** We are going to prove next that there are no positive solutions of the limit concave-convex problem (4.1) under the curve of positive solutions of the concave limit problem

$$(4.2) \quad \begin{cases} \min\left\{|\nabla w_\Lambda| - \Lambda\, w_\Lambda^Q, -\Delta_\infty w_\Lambda\right\} = 0 & \text{in } \Omega, \\ w_\Lambda > 0 & \text{in } \Omega, \\ w_\Lambda = 0 & \text{on } \partial\Omega. \end{cases}$$
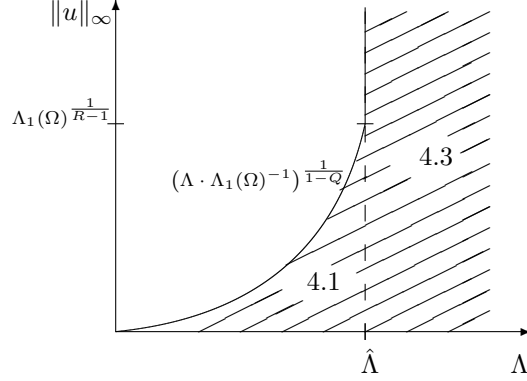
We have the following result.

FIGURE 1. Regions of non-existence for the limit concave-convex problem (4.1) given by Propositions 4.1 and 4.3.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^n$ a bounded domain and w_Λ the unique positive viscosity solution of the concave problem (4.2) for each $\Lambda > 0$. Then, any positive viscosity solution u_Λ of (4.1) satisfies $u_\Lambda \geq w_\Lambda$. In particular,*

$$(4.3) \qquad \|u_\Lambda\|_{L^\infty(\Omega)} \geq \left(\Lambda \cdot \|dist(\cdot, \partial\Omega)\|_\infty\right)^{\frac{1}{1-Q}} = \left(\Lambda \cdot \Lambda_1(\Omega)^{-1}\right)^{\frac{1}{1-Q}}.$$

Remark 4.2. Recall that $\Lambda_1(\Omega) = \|dist(\cdot, \partial\Omega)\|_\infty$ is the first ∞-eigenvalue (see [20]).

Proof. Consider u_Λ, a nontrivial solution of (4.1). We shall prove that

$$(4.4) \qquad \min\{|\nabla u_\Lambda| - \Lambda\, u_\Lambda^Q, -\Delta_\infty u_\Lambda\} \geq 0 \quad \text{in } \Omega,$$

since then, given the fact that w_Λ is a solution of the concave problem (4.2), we have $w_\Lambda \leq u_\Lambda$ by comparison (Proposition 2.8). Estimate (4.3) is a consequence of this fact and Proposition 2.10.

In order to prove (4.4), consider $x_0 \in \Omega$ and $\phi \in \mathcal{C}^2$ such that $u_\Lambda - \phi$ has a minimum at x_0. As $u_\Lambda(x)$ is a solution of problem (4.1) we have

$$\min\left\{|\nabla\phi(x_0)| - \max\{\Lambda u_\Lambda^Q(x_0), u_\Lambda^R(x_0)\}, -\Delta_\infty\phi(x_0)\right\} \geq 0$$

It follows that $-\Delta_\infty\phi(x_0) \geq 0$ and $|\nabla\phi(x_0)| \geq \max\{\Lambda u_\Lambda^Q(x_0), u_\Lambda^R(x_0)\}$. Hence

$$|\nabla\phi(x_0)| - \Lambda u_\Lambda^Q(x_0) \geq \max\{\Lambda u_\Lambda^Q(x_0), u_\Lambda^R(x_0)\} - \Lambda u_\Lambda^Q(x_0) \geq 0.$$

We get

$$\min\left\{|\nabla\phi(x_0)| - \Lambda u_\Lambda^Q(x_0), -\Delta_\infty\phi(x_0)\right\} \geq 0$$

and consequently (4.4) holds in the viscosity sense. $\qquad\square$

4.2. **Non-existence of positive solutions for large Λ.** We show here the existence of a threshold $\hat{\Lambda}$ beyond which problem (4.1) has no positive solutions.

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then, problem (4.1) has no positive viscosity solution for $\Lambda > \hat{\Lambda}$ with*

$$(4.5) \qquad \hat{\Lambda} = \Lambda_1(\Omega)^{\frac{R-Q}{R-1}}.$$

Proof. Define $\mu = \Lambda_1(\Omega) + \epsilon$ with $\epsilon > 0$. Suppose for the sake of contradiction that problem (4.1) has a solution u_Λ for some $\Lambda > \mu^{\frac{R-Q}{R-1}}$.

First, we are going to show that this u_Λ is a supersolution of the eigenvalue problem with parameter μ. More precisely, we are going to show that

$$(4.6) \qquad \min\big\{|\nabla u_\Lambda| - \mu\, u_\Lambda, -\Delta_\infty u_\Lambda\big\} > 0 \quad \text{in } \Omega$$

in the viscosity sense. To this aim, let $x_0 \in \Omega$ and $\phi \in \mathcal{C}^2$ such that $u_\Lambda - \phi$ has a minimum in x_0. Since $u_\Lambda(x)$ is a solution of problem (4.1) we have

$$\min\Big\{|\nabla\phi(x_0)| - \max\{\Lambda u_\Lambda^Q(x_0), u_\Lambda^R(x_0)\}, -\Delta_\infty\phi(x_0)\Big\} \geq 0.$$

We deduce that $-\Delta_\infty\phi(x_0) \geq 0$ and $|\nabla\phi(x_0)| \geq \max\{\Lambda u_\Lambda^Q(x_0), u_\Lambda^R(x_0)\}$. Hence,

$$|\nabla\phi(x_0)| - \mu\, u_\Lambda(x_0) \geq \max\{\Lambda u_\Lambda^Q(x_0), u_\Lambda^R(x_0)\} - \mu\, u_\Lambda(x_0).$$

To deduce (4.6) it is enough to show that

$$\min_{t>0} \Phi_\Lambda(t) > \mu \quad \text{where} \quad \Phi_\Lambda(t) = \max\{\Lambda t^{Q-1}, t^{R-1}\}.$$

It is elementary to check that the function Φ_Λ is convex and has a unique minimum point at $t_{\min} = \Lambda^{\frac{1}{R-Q}}$. Notice that $\lim_{t\to\infty}\Phi_\Lambda(t) = \lim_{t\to 0}\Phi_\Lambda(t) = \infty$, and hence $t_{\min}$ is a global minimum. Then, it is easy to check that $\Lambda > \mu^{\frac{R-Q}{R-1}}$ implies $\Phi_\Lambda(t_{\min}) > \mu$.

Next, we notice that any first ∞-eigenfunction is a subsolution of the eigenvalue problem with parameter μ. Specifically, let v be a first $\infty-$eigenfunction, that is, a solution of

$$(4.7) \qquad \begin{cases} \min\big\{|\nabla v| - \Lambda_1(\Omega)\, v, -\Delta_\infty v\big\} = 0 \quad \text{in } \Omega, \\ v > 0 \quad \text{in } \Omega \\ v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Then,

$$\min\big\{|\nabla v| - \mu\, v, -\Delta_\infty v\big\} \leq 0 \quad \text{in } \Omega.$$

For later purpose, we shall assume $\|v\|_\infty < \big(\Lambda\mu^{-1}\big)^{\frac{1}{1-Q}}$.

Now, we have to show that u_Λ and v are ordered, namely, that $0 < v \leq u_\Lambda$ in Ω. We are going to use the Comparison Principle for the concave problem (Proposition 2.8). Indeed, it is easy to see that

$$\min\big\{|\nabla u_\Lambda| - \Lambda\, u_\Lambda^Q, -\Delta_\infty u_\Lambda\big\} \geq 0 \quad \text{in } \Omega,$$

while, using the normalization of v, one can check that

$$(4.8) \qquad \min\big\{|\nabla v| - \Lambda\, v^Q, -\Delta_\infty v\big\} \leq 0 \quad \text{in } \Omega.$$

To see this, let $x_0 \in \Omega$ and $\phi \in \mathcal{C}^2$ such that $v - \phi$ has a maximum at x_0. Since v is a $\infty-$eigenfunction, it satisfies

$$\min\{|\nabla\phi(x_0)| - \Lambda_1(\Omega)\, v(x_0), -\Delta_\infty\phi(x_0)\} \leq 0 \quad \text{in } \Omega.$$

We can suppose $-\Delta_\infty\phi(x_0) > 0$ and $|\nabla\phi(x_0)| - \Lambda_1(\Omega)\, v(x_0) \leq 0$ since we are done otherwise. Clearly, as $\Lambda_1(\Omega) < \mu$, we have

$$|\nabla\phi(x_0)| - \Lambda\, v^Q(x_0) \leq \big(\Lambda_1(\Omega)\, \|v\|_\infty^{1-Q} - \Lambda\big)\, v(x_0)^Q \leq 0,$$

by the normalization of v. Consequently

$$\min\left\{|\nabla\phi(x_0)| - \Lambda\, v^Q(x_0), -\Delta_\infty\phi(x_0)\right\} \le 0,$$

that is, (4.8) holds in the viscosity sense. As $v = u_\Lambda = 0$ on $\partial\Omega$, we get $0 < v \le u_\Lambda$ by the Comparison Principle, Proposition 2.8.

So far, we have a subsolution v and a supersolution u_Λ of the eigenvalue problem

$$(4.9) \qquad\qquad \min\left\{|\nabla w| - \mu\, w, -\Delta_\infty w\right\} = 0 \quad \text{in } \Omega$$

which verify $0 < v \le u_\Lambda$. Next we claim that it is possible to construct a solution of (4.9) iterating between v and u_Λ. Then, the argument concludes since we have constructed a positive $\infty-$eigenfunction associated to $\mu = \Lambda_1 + \epsilon$, a contradiction with the fact that $\Lambda_1(\Omega)$ is isolated (see [19, Theorem 8.1] and [20, Theorem 3.1]).

Since the argument above runs for every $\epsilon > 0$, we deduce that there is no nontrivial solution of (4.1) for $\Lambda > \hat{\Lambda}$.

Finally, we conclude proving the claim. First, define $w_1(x)$, viscosity solution of

$$\begin{cases} \min\left\{|\nabla w_1| - \mu\, v, -\Delta_\infty w_1\right\} = 0 \quad \text{in } \Omega \\ w_1 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

To prove that such a w_1 exists, notice that v is a subsolution of the problem and that u_Λ is a supersolution since, from (4.6) and $v \le u_\Lambda$ we deduce

$$\min\left\{|\nabla u_\Lambda| - \mu\, v, -\Delta_\infty u_\Lambda\right\} \ge 0.$$

We point out that the Comparison Principle holds for this equation, see for instance [18, Theorem 4.18 and Remark 4.23] (see also [17]). For the mentioned Comparison Principle, notice that every ∞-superharmonic function is Lipschitz continuous, see [27]. Hence, we can apply the Perron method ([11, Theorem 4.1]), to get a unique w_1 such that

$$v \le w_1 \le u_\Lambda \quad \text{in } \Omega.$$

Then, we define w_2, the solution of

$$\begin{cases} \min\left\{|\nabla w_2| - \mu\, w_1, -\Delta_\infty w_2\right\} = 0 \quad \text{in } \Omega \\ w_2 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

In this case, w_1 is a subsolution and u_Λ is a supersolution, since

$$\min\left\{|\nabla w_1| - \mu\, v, -\Delta_\infty w_1\right\} = 0 \quad \Rightarrow \quad \min\left\{|\nabla w_1| - \mu\, w_1, -\Delta_\infty w_1\right\} \le 0,$$

while

$$\min\left\{|\nabla u_\Lambda| - \mu\, u_\Lambda, -\Delta_\infty u_\Lambda\right\} \ge 0 \quad \Rightarrow \quad \min\left\{|\nabla u_\Lambda| - \mu\, w_1, -\Delta_\infty u_\Lambda\right\} \ge 0.$$

As $w_1 = u_\Lambda = 0$ on $\partial\Omega$, by comparison and the Perron method, we obtain that there exists a unique w_2 satisfying

$$v \le w_1 \le w_2 \le u_\Lambda \quad \text{in } \Omega.$$

Iterating this procedure, we construct an increasing sequence

$$v \le w_1 \le w_2 \le \ldots \le w_{k-1} \le w_k \le u_\Lambda$$

of solutions of

$$(4.10) \qquad\qquad \begin{cases} \min\left\{|\nabla w_k| - \mu\, w_{k-1}, -\Delta_\infty w_k\right\} = 0 \quad \text{in } \Omega \\ w_k = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Notice that $\|w_k\|_\infty$ is uniformly bounded by construction. On the other hand, as $-\Delta_\infty w_k \geq 0$ in Ω, we have (see [26, 27] and also [18, Section 6] for a related construction) that

$$|\nabla w_k(x)| \leq \frac{w_k(x)}{\operatorname{dist}(x, \partial\Omega)} \leq \frac{u_\Lambda(x)}{\operatorname{dist}(x, \partial\Omega)} \qquad \text{a.e. } x \in \Omega,$$

for all $k > 1$. From there, both $\|w_k\|_\infty$ and $\|\nabla w_k\|_\infty$ are uniformly bounded in compact subsets of Ω. We observe that v, u_Λ are barriers on $\partial\Omega$ for each w_k. Hence by the Ascoli-Arzela theorem and the monotonicity of the sequence $\{w_k\}$, the whole sequence converges uniformly in Ω to some $w \in \mathcal{C}(\overline{\Omega})$ which verifies $w = 0$ on $\partial\Omega$. Then, we can take limits in the viscosity sense in (4.10) and obtain that the limit w is a viscosity solution of (4.9), which proves the claim. □

5. Existence of a curve of minimal solutions for the limit problem

In this section we prove that problem (4.1) has a minimal positive solution for each $\Lambda \leq \hat{\Lambda}$ such that

$$\|u_\Lambda\|_{L^\infty(\Omega)} = \left(\Lambda \cdot \|\operatorname{dist}(\cdot, \partial\Omega)\|_\infty\right)^{\frac{1}{1-Q}}.$$

Hence, the bounds obtained in Section 4 (Propositions 4.1 and 4.3) are sharp.

Indeed, we prove that for every $\Lambda \in (0, \hat{\Lambda}]$, the minimal positive solution of the limit concave-convex problem coincides with the unique positive solution of the concave problem (4.2). As we have mentioned in the introduction, the reason is that there is a critical size of solutions (depending on Λ) under which the convex power has no effect in the limit concave-convex problem.

From this fact we shall deduce that the minimal solution is the unique positive solution of (4.1) with $\|u_\Lambda\|_\infty \leq \Lambda^{\frac{1}{R-Q}}$.

This new information on the size of solutions allows to extend the result and prove that, actually, there is no positive solution with

$$\Lambda^{\frac{1}{R-Q}} < \|u_\Lambda\|_\infty < \Lambda_1^{\frac{1}{R-1}}.$$

We would like to stress the major role that the Comparison Principle for the concave problem (Proposition 2.8) plays in the arguments in this section.

Finally, just mention that in Section 6 we shall prove that the minimal solutions of the limit problem found in this Section are uniform limits of minimal solutions of (1.1).

In Figure 2 we have collected all the information obtained in this section.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\Lambda \in (0, \hat{\Lambda}]$. Consider w_Λ, the unique positive solution of the concave problem (4.2). Then, if u_Λ is a positive solution of (4.1) such that $\|u_\Lambda\|_\infty \leq \Lambda^{\frac{1}{R-Q}}$, necessarily $u_\Lambda \equiv w_\Lambda$.*

Remark 5.2. Notice that, since $\Lambda < \hat{\Lambda} = \Lambda_1(\Omega)^{\frac{R-Q}{R-1}}$, Proposition 2.10 implies

$$\|w_\Lambda\|_\infty = (\Lambda \cdot \Lambda_1(\Omega)^{-1})^{\frac{1}{1-Q}} \leq \Lambda^{\frac{1}{R-Q}}.$$

Moreover, the homogeneity properties of the concave equation (4.2) imply that $w_\Lambda(x) = \Lambda^{\frac{1}{1-Q}} \cdot w_1(x)$, where w_1 is the unique solution of (4.2) with $\Lambda = 1$. Consequently, the set of minimal solutions of (3.2) is a differentiable curve in Λ for $\Lambda < \hat{\Lambda}$.
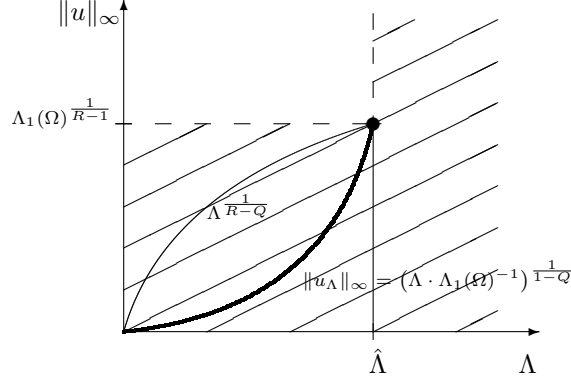
FIGURE 2. Curve of minimal solutions of problem (4.1) and regions of non-existence given by Propositions 4.1, 4.3, 5.1 and 5.3.

Proof. We are going to check that if $\|u_\Lambda\|_\infty \leq \Lambda^{\frac{1}{R-Q}}$, then

$$\text{(5.1)} \qquad \min\{|\nabla u_\Lambda| - \Lambda\, u_\Lambda^Q, -\Delta_\infty u_\Lambda\} = 0 \quad \text{in } \Omega,$$

from which, by uniqueness (Proposition 2.8), we conclude $u_\Lambda \equiv w_\Lambda$.

First, we prove that u_Λ is a subsolution of the concave problem (5.1). Consider $\phi \in \mathcal{C}^2$ such that $u_\Lambda - \phi$ has a local maximum at $x_0 \in \Omega$. Then, since u_Λ is a viscosity solution of (4.1), we have that either

$$-\Delta_\infty \phi(x_0) \leq 0,$$

or

$$|\nabla\phi(x_0)| - \max\{\Lambda\, u_\Lambda^Q(x_0), u_\Lambda^R(x_0)\} \leq 0.$$

In the first case there is nothing to prove so we can assume that $-\Delta_\infty \phi(x_0) > 0$ and the second alternative holds. Then,

$$|\nabla\phi(x_0)| - \Lambda\, u_\Lambda^Q(x_0) \leq \max\{\Lambda\, u_\Lambda^Q(x_0), u_\Lambda^R(x_0)\} - \Lambda\, u_\Lambda^Q(x_0).$$

We claim that $\|u_\Lambda\|_\infty \leq \Lambda^{\frac{1}{R-Q}}$ implies

$$\text{(5.2)} \qquad \max\{\Lambda\, u_\Lambda^Q(x), u_\Lambda^R(x)\} = \Lambda\, u_\Lambda^Q(x) \qquad \forall x \in \Omega,$$

and consequently

$$\min\left\{|\nabla\phi(x_0)| - \Lambda\, u_\Lambda^Q(x_0), -\Delta_\infty \phi(x_0)\right\} \leq 0,$$

which is what we aim for. To see (5.2), suppose for contradiction that there exists $z \in \Omega$ such that

$$\max\{\Lambda\, u_\Lambda^Q(z), u_\Lambda^R(z)\} = u_\Lambda^R(z).$$

Then, $\Lambda\, u_\Lambda^Q(z) \leq u_\Lambda^R(z)$ and we obtain a contradiction unless $z \in \Omega$ is such that $u_\Lambda(z) = \|u_\Lambda\|_\infty = \Lambda^{\frac{1}{R-Q}}$. In the latter case, we have

$$\max\{\Lambda\, u_\Lambda^Q(z), u_\Lambda^R(z)\} = u_\Lambda^R(z) = \Lambda\, u_\Lambda^Q(z),$$

and (5.2) follows.

Finally, arguing as in Proposition 4.1, it is easy to check that u_Λ is a supersolution of (5.1) and we conclude. $\qquad\square$

Proposition 5.3. *Let Ω be a bounded domain and let $\Lambda \in (0, \hat{\Lambda})$ fixed. Then, every nontrivial solution u_Λ of* (4.1) *with $\|u_\Lambda\|_\infty > \Lambda^{\frac{1}{R-Q}}$, verifies*

$$\|u_\Lambda\|_\infty \geq \Lambda_1(\Omega)^{\frac{1}{R-1}}.$$

Proof. If $\|u_\Lambda\|_\infty > \Lambda^{\frac{1}{R-Q}}$, then

(5.3) $$\max\{\Lambda u_\Lambda^Q(x), u_\Lambda^R(x)\} \leq \max\{\Lambda \|u_\Lambda\|_\infty^Q, \|u_\Lambda\|_\infty^R\} = \|u_\Lambda\|_\infty^R.$$

Then, u_Λ is a viscosity subsolution of

(5.4) $$\begin{cases} \min\{|\nabla v(x)| - \|u_\Lambda\|_\infty^R, -\Delta_\infty v(x)\} = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since

$$C(x) = \|u_\Lambda\|_\infty^R \cdot \text{dist}(x, \partial\Omega)$$

is the unique solution of (5.4) (see for instance [17, 18]), we have, by comparison, that

$$u_\Lambda(x) \leq C(x) = \|u_\Lambda\|_\infty^R \cdot \text{dist}(x, \partial\Omega) \qquad \forall x \in \Omega,$$

from which we deduce $\|u_\Lambda\|_\infty \geq \Lambda_1(\Omega)^{\frac{1}{R-1}}$. $\square$

6. Limits of positive minimal solutions as $p \to \infty$

In this section we show uniform convergence as $p \to \infty$ of minimal solutions of (1.1) to minimal solution of the limit problem (3.2). Notice that we have already proved existence of minimal solutions to (3.2) in the previous section.

To this aim, we provide the explicit construction of the branch of minimal positive solutions of (1.1) as well as the proof of non-existence beyond the threshold $\lambda_{\max,p}$.

Despite this construction is known from [7], we track down the precise dependence on p of the parameters involved, necessary to provide estimates of $\lambda_{\max,p}$ in terms of p, q, r, n from which the asymptotic behavior of the threshold can be deduced. Also, we put some effort in proving the results for mere bounded domains.

Let $\Omega \subset \mathbb{R}^n$ a bounded domain and $\lambda > 0$. We can assume without loss of generality that $0 < q < p - 1 < r$ in the concave-convex problem (1.1).

Next, we shall show the existence of a threshold $\lambda_{\max,p}$ such that no positive solution of (1.1) exists for $\lambda > \lambda_{\max,p}$ and then we shall construct a branch of positive minimal solutions of (1.1) for $\lambda < \lambda_{\max,p}$. Finally, we shall use this construction to pass to the limit as $p \to \infty$.

6.1. **Non-existence of positive solutions for large λ.** We show here that there exists a value $\hat{\lambda}_p > 0$ such that no positive weak solution of (1.1) exists for $\lambda > \hat{\lambda}_p$. This $\hat{\lambda}_p$ is not necessarily sharp; as we shall see, $\hat{\lambda}_p \geq \lambda_{\max,p}$, where $\lambda_{\max,p}$ is the threshold value. However, $\hat{\lambda}_p$ has an explicit expression in terms of p.

Proposition 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $n < p < \infty$. Then, problem* (1.1) *has no positive solution in $W_0^{1,p}(\Omega)$ whenever $\lambda > \hat{\lambda}_p$, where*

(6.1) $$\hat{\lambda}_p = \lambda_1(p; \Omega)^{\frac{r-q}{r-(p-1)}} \left(r - (p-1)\right) \left(\frac{((p-1)-q)^{(p-1)-q}}{(r-q)^{r-q}}\right)^{\frac{1}{r-(p-1)}}.$$

Proof. Let $\mu = \lambda_1(p; \Omega) + \epsilon$ with $\epsilon > 0$. Suppose for the sake of contradiction that problem (1.1) has a positive solution $u_\lambda \in W_0^{1,p}(\Omega)$ for some

$$\lambda > \mu^{\frac{r-q}{r-(p-1)}} (r - (p-1)) \left(\frac{((p-1) - q)^{(p-1)-q}}{(r-q)^{r-q}} \right)^{\frac{1}{r-(p-1)}}.$$

We are going to show that then

(6.2) $$-\Delta_p u_\lambda = \lambda\, u_\lambda^q + u_\lambda^r > \mu u_\lambda^{p-1} \quad \text{in } \Omega,$$

in the weak sense. In fact, in order to have the former inequality, it is enough to see that

$$\min_{t>0} \Phi_\lambda(t) > \mu \quad \text{where} \quad \Phi_\lambda(t) = \lambda\, t^{q-(p-1)} + t^{r-(p-1)}.$$

It is elementary to check that

$$\frac{d}{dt} \Phi_\lambda(t) = 0 \quad \Leftrightarrow \quad t_\lambda = \left(\frac{\lambda\big((p-1) - q\big)}{\big(r - (p-1)\big)} \right)^{\frac{1}{r-q}},$$

which is a minimum. As $\Phi_\lambda(t) \to \infty$ when $t \to 0$ and $t \to \infty$, it is a global minimum. Then,

$$\min_{t>0} \Phi_\lambda(t) = \Phi_\lambda(t_\lambda) = \frac{\lambda^{\frac{r-(p-1)}{r-q}} (r-q)}{((p-1)-q)^{\frac{(p-1)-q}{r-q}} (r-(p-1))^{\frac{r-(p-1)}{r-q}}} > \mu$$

because of our election of λ.

Next, we notice that any first eigenfunction is a subsolution of the eigenvalue problem with parameter μ. Specifically, let $\phi_{1,p}$ be a first eigenfunction, that is, a solution of

$$\begin{cases} -\Delta_p \phi_{1,p} = \lambda_1(p; \Omega) |\phi_{1,p}|^{p-2} \phi_{1,p} & \text{in } \Omega, \\ \phi_{1,p} > 0 & \text{in } \Omega \\ \phi_{1,p} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, obviously,

$$-\Delta_p \phi_{1,p} < \mu\, \phi_{1,p}^{p-1} \quad \text{in } \Omega.$$

For later purpose, we shall assume $\|\phi_{1,p}\|_\infty < \left(\lambda \mu^{-1} \right)^{\frac{1}{(p-1)-q}}$.

Now, we have to show that u_Λ and $\phi_{1,p}$ are ordered, namely, that $0 < \phi_{1,p} \leq u_\Lambda$ in Ω. We are going to make use of the Comparison Principle for the concave problem (see [1, 10]). Indeed,

$$-\Delta_p u_\lambda = \lambda\, u_\lambda^q + u_\lambda^r \geq \lambda\, u_\lambda^q \quad \text{in } \Omega,$$

while, using the normalization of $\phi_{1,p}$, one can check that

$$-\Delta_p \phi_{1,p} \leq \lambda_1(p; \Omega) \|\phi_{1,p}\|_\infty^{(p-1)-q} \phi_{1,p} < \lambda\, \phi_{1,p}^q \quad \text{in } \Omega.$$

Then, by comparison, $\phi_{1,p} \leq u_\lambda$.

Finally, we can apply an iteration method (see [7]) and obtain v such that $0 < \phi_{1,p} \leq v \leq u_\lambda$, a positive eigenfunction associated to μ, which is impossible since $\lambda_1(p, \Omega)$ is isolated (see Proposition 2.5).

Since the foregoing argument runs for every $\epsilon > 0$, we conclude that no positive solution of (1.1) exists for $\lambda > \hat\lambda$. $\qquad \square$

6.2. **Construction of the branch of minimal positive solutions of the concave-convex problem.** Now, we sketch the construction of a minimal positive solution for every $\lambda \in (0, \lambda_{\max,p})$ where $\lambda_{\max,p}$ will be properly defined below. The construction is divided into several partial results

Proposition 6.2 (Existence of positive solutions for small λ). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p > n$. Then, problem (1.1) has at least one positive solution $u \in W_0^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ for each $\lambda \in (0, \lambda_{0,p}]$ with*

$$(6.3) \qquad \lambda_{0,p} = \left(\|v_{1,p}\|_\infty^{1-p}\right)^{\frac{r-q}{r-(p-1)}} \left(r - (p-1)\right) \left(\frac{((p-1)-q)^{(p-1)-q}}{(r-q)^{r-q}}\right)^{\frac{1}{r-(p-1)}},$$

where $v_{1,p} \in W_0^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is the solution of (2.3).

For the proof, we construct a sub- and supersolution and apply an iteration method. First, we construct the supersolution.

Lemma 6.3. *Let Ω be a bounded domain, $p > n$ and $\lambda_{0,p}$ in (6.3). Take $v_{1,p} \in W_0^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$, the solution of (2.3). Then,*

$$\overline{u}_\lambda(x) = T_p(\lambda) \cdot v_{1,p}(x)$$

is a weak supersolution of (1.1) for every $\lambda \in (0, \lambda_{0,p}]$, where

$$T_p(\lambda) = \|v_{1,p}\|_\infty^{-1} \left(\frac{(p-1)-q}{r-(p-1)}\right)^{\frac{1}{r-q}} \lambda^{\frac{1}{r-q}}.$$

The proof follows by homogeneity with an argument somehow similar to the proof of (6.2) in the proof of Proposition 6.1. See [7, 10] for the details.

Next, we construct the subsolution.

Lemma 6.4. *Let Ω be a bounded domain and $p > n$. Then, for every $\lambda > 0$, the function $\underline{u}_\lambda = w_\lambda \in W_0^{1,p}(\Omega)$, solution of (2.4) is a weak subsolution of (1.1). Moreover, $\underline{u}_\lambda \leq \overline{u}_\lambda$ in Ω.*

For the proof, notice that, $\underline{u}_\lambda^q > 0$ obviously implies $-\Delta_p \underline{u}_\lambda = \lambda \underline{u}_\lambda^q < \lambda \underline{u}_\lambda^q + \underline{u}_\lambda^r$ in Ω. The second part follows by comparison for the concave problem (see [1, 10]).

Once we have constructed a weak sub- and supersolution $\underline{u}_\lambda$ and $\overline{u}_\lambda$ which are ordered on the boundary, we finish the proof of Proposition 6.2 by means of an iteration method as in [7].

From Propositions 6.1 and 6.2, we have the following result.

Proposition 6.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p > n$. There exists $\lambda_{\max,p} \in \mathbb{R}^+$ with $0 < \lambda_{\max,p} < \infty$ such that (1.1) has a weak positive solution for every $\lambda \in (0, \lambda_{\max,p})$ and no positive weak solution for $\lambda > \lambda_{max,p}$. Moreover,*

$$\lambda_{0,p} \leq \lambda_{\max,p} \leq \hat{\lambda}_p,$$

where $\hat{\lambda}_p$ and $\lambda_{0,p}$ are given by (6.1) and (6.3) respectively.

Proof. Define

$$\lambda_{\max,p} = \sup \left\{\lambda \in \mathbb{R}^+ : \ (1.1) \text{ has a positive solution}\right\}.$$

Proposition 6.2 implies $\lambda_{\max,p} > 0$, while Proposition 6.1 implies $\lambda_{\max,p} < \infty$. Indeed, we can take λ_M close to $\lambda_{\max,p}$, and u_M such that

$$\begin{cases} -\Delta_p u_M = \lambda_M \, u_M^q + u_M^r & \text{in } \Omega, \\ u_M > 0 & \text{in } \Omega, \\ u_M = 0 & \text{on } \partial\Omega. \end{cases}$$

Fixed $0 < \lambda < \lambda_M$, let $\underline{u}$ the unique positive solution of the concave problem

(6.4)
$$\begin{cases} -\Delta_p \underline{u} = \lambda \, \underline{u}^q & \text{in } \Omega, \\ \underline{u} > 0 & \text{in } \Omega, \\ \underline{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

Obviously, u_M is a weak supersolution of (6.4) and, by comparison $\underline{u} \le u_M$. Since,

$$-\Delta_p \underline{u} = \lambda \, \underline{u}^q < \lambda \, \underline{u}^q + \underline{u}^r,$$

and

$$-\Delta_p u_M = \lambda_M \, u_M^q + u_M^r > \lambda \, u_M^q + u_M^r,$$

we obtain the existence of $u_\lambda > 0$, solution of (1.1), by iteration. $\qquad\square$

We have the following consequence which yields the asymptotic behavior of $\lambda_{\max,p}$ as $p \to \infty$.

Corollary 6.6. *Consider the concave-convex problem* (1.1) *and assume that the exponents* $q = q(p)$ *and* $r = r(p)$ *satisfy condition* (1.2). *Then,*

$$\lim_{p\to\infty} \lambda_{\max,p}^{1/p} = \lim_{p\to\infty} \lambda_{0,p}^{1/p} = \lim_{p\to\infty} \hat{\lambda}_p^{1/p} = \hat{\Lambda} = \Lambda_1(\Omega)^{\frac{R-Q}{R-1}}.$$

The result follows from the explicit expressions for $\lambda_{0,p}$ and $\hat{\lambda}_p$, Proposition 2.4, and Lemma 2.6.

6.3. **The curve of minimal solutions obtained as a limit of minimal solutions with $p < \infty$.** Our aim in this subsection is to show that the curve of minimal positive solutions obtained in Section 5 for the limit problem (1.3) can be attained as a limit when $p \to \infty$ of minimal positive solutions of (1.1).

Theorem 6.7. *Fix* $\Lambda \in (0, \hat{\Lambda})$, *let* $\{\lambda_p\}_p$ *be a sequence such that* $\lim_{p\to\infty} \lambda_p^{1/p} = \Lambda$. *Consider* $\{u_{\lambda_p,p}\}_p$, *the corresponding sequence of minimal positive solutions of*

(6.5)
$$\begin{cases} -\Delta_p u_{\lambda_p,p} = \lambda_p \, u_{\lambda_p,p}^{q(p)} + u_{\lambda_p,p}^{r(p)} & \text{in } \Omega \\ u_{\lambda_p,p} > 0 & \text{in } \Omega \\ u_{\lambda_p,p} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then,

$$u_{\lambda_p,p} \to w_\Lambda \quad \text{uniformly as } p \to \infty,$$

with $w_\Lambda(x)$ *the unique positive solution of the limit concave problem* (4.2) *and minimal positive solution of problem* (4.1).

Before proving Theorem 6.7, we present some consequences of the Morrey estimates (Lemma 2.1) that will be needed in the proof.

Lemma 6.8. *Fix $p > n$. Then, for every $m \in (n, p)$, there exists a constant C independent of p such that every solution $u_{\lambda_p,p}$ of (6.5) satisfies*

$$(6.6) \quad \frac{|u_{\lambda_p,p}(x) - u_{\lambda_p,p}(y)|}{|x-y|^{1-\frac{n}{m}}} \leq C \max\left\{ \lambda^{\frac{1}{p}} \|u_{\lambda_p,p}\|_\infty^{\frac{q+1}{p}}, \|u_{\lambda_p,p}\|_\infty^{\frac{r+1}{p}} \right\} \quad \forall x, y \in \Omega.$$

Proof. Multiplying (6.5) by $u_{\lambda_p,p}$ and integrating by parts, we have

$$(6.7) \qquad \int_\Omega |\nabla u_{\lambda_p,p}|^p \, dx = \lambda \int_\Omega |u_{\lambda_p,p}|^{q+1} \, dx + \int_\Omega |u_{\lambda_p,p}|^{r+1} \, dx.$$

As $p > m$, combining (6.7), the Hölder inequality and the Morrey estimate (whose constant can be chosen independent of p) we get

$$\frac{|u_{\lambda_p,p}(x) - u_{\lambda_p,p}(y)|}{|x-y|^{1-\frac{n}{m}}} \leq C \left(\int_\Omega |\nabla u_{\lambda_p,p}|^m \, dx \right)^{1/m} \leq C |\Omega|^{\frac{1}{m}-\frac{1}{p}} \left(\int_\Omega |\nabla u_{\lambda_p,p}|^p \, dx \right)^{1/p}$$

$$= C |\Omega|^{\frac{1}{m}-\frac{1}{p}} \left(\lambda \int_\Omega |u_{\lambda_p,p}|^{q+1} \, dx + \int_\Omega |u_{\lambda_p,p}|^{r+1} \, dx \right)^{1/p}$$

$$\leq C|\Omega|^{\frac{1}{m}} \left(\lambda \|u_{\lambda_p,p}\|_\infty^{q+1} + \|u_{\lambda_p,p}\|_\infty^{r+1} \right)^{1/p} \leq C \max\left\{ \lambda^{\frac{1}{p}} \|u_{\lambda_p,p}\|_\infty^{\frac{q+1}{p}}, \|u_{\lambda_p,p}\|_\infty^{\frac{r+1}{p}} \right\},$$

where $C > 0$ is independent of m and p. $\square$

In the proof of Theorem 6.7 we shall make use of Lemma 6.8 and the construction of the branch of minimal positive solutions in the previous subsections.

Proof of Theorem 6.7. Let $\lambda_{0,p}$ as in (6.3). Since $\lambda_{0,p}^{1/p} \to \hat{\Lambda} > \Lambda$ as $p \to \infty$ (see Corollary 6.6), there exists p_0 large enough to ensure that $\lambda_{0,p}^{1/p} > \Lambda$ for all $p \geq p_0$.

Then, we know by Proposition 6.2 that $u_{\lambda_p,p}$ is constructed iterating between

$$\underline{u}_{\lambda_p,p}(x) = w_{\lambda_p,p}(x),$$

the positive solution to the concave problem (2.4) with parameter λ_p, and

$$\overline{u}_{\lambda_p,p}(x) = \left(\frac{(p-1)-q}{r-(p-1)} \right)^{\frac{1}{r-q}} \lambda_p^{\frac{1}{r-q}} \frac{v_{1,p}(x)}{\|v_{1,p}\|_\infty}.$$

Proposition 2.9 implies $\underline{u}_{\lambda_p,p} = w_{\lambda_p,p} \to w_\Lambda$ uniformly as $p \to \infty$. On the other hand, from Proposition 2.4 we have that $v_{1,p} \to \text{dist}(x, \partial\Omega)$, and consequently

$$\overline{u}_{\lambda_p,p} \to \Lambda^{\frac{1}{R-Q}} \frac{\text{dist}(x, \partial\Omega)}{\|\text{dist}(\cdot, \partial\Omega)\|_\infty} \quad \text{uniformly as } p \to \infty.$$

We deduce that we can find a constant C independent of p such that,

$$\|u_{\lambda_p,p}\|_\infty \leq \|\overline{u}_{\lambda_p,p}\|_\infty \leq C.$$

Then, from Lemma 6.8 and Ascoli-Arzela Theorem, we get that there exists a subsequence p' and a limit function $u(x)$ such that $u_{\lambda_{p'},p'} \to u$, uniformly as $p \to \infty$. Notice that, taking limits in

$$\underline{u}_{\lambda_{p'},p'} \leq u_{\lambda_{p'},p'} \leq \overline{u}_{\lambda_{p'},p'}$$

we arrive at

$$0 < w_\Lambda \leq u \leq \Lambda^{\frac{1}{R-Q}} \frac{\text{dist}(x, \partial\Omega)}{\|\text{dist}(\cdot, \partial\Omega)\|_\infty}.$$

Since $u > 0$, Proposition 3.1 imply that u is a solution of the concave-convex limit problem (4.1).

Finally, since $\|u\|_\infty \leq \Lambda^{\frac{1}{R-Q}}$, Proposition 5.1 implies $u = w_\Lambda$. Notice that, since the limit is unique, not only a subsequence but the whole sequence $u_{\lambda_p,p}$ converge.   □

7. Limits of Mountain Pass solutions as $p \to \infty$. Multiplicity of solutions for the limit problem

In this section, we provide the proof of existence of a second positive weak solution to problem (1.1). Then, we use this construction to prove existence of a second positive viscosity solution to the limit problem (1.3).

Throughout this section we shall always assume without loss of generality that $0 < q < p - 1 < r$.

We introduce the functional associated to problem (1.1),

$$J_\lambda(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{\lambda}{q+1} \int_\Omega u_+^{q+1} \, dx - \frac{1}{r+1} \int_\Omega u_+^{r+1} \, dx.$$

Then, $J_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ is well defined and has the Mountain-Pass geometry. This is easily seen since $J_\lambda(0) = 0$ and for any fixed $\phi \in W_0^{1,p}(\Omega)$ and $t > 0$ we have that

$$J_\lambda(t\phi) = \frac{A}{p} t^p - \frac{\lambda B}{q+1} t^{q+1} - \frac{C}{r+1} t^{r+1}$$

with

$$A = \int_\Omega |\nabla \phi|^p \, dx, \qquad B = \int_\Omega |\phi|^{q+1} \, dx, \qquad C = \int_\Omega |\phi|^{r+1} \, dx.$$

As $r > p - 1$, we have that for λ small enough, $J_\lambda(t\phi) < 0$ for t sufficiently small, then $J_\lambda(t\phi) > 0$, and finally $J_\lambda(t\phi) < 0$ for every t large enough.

For convenience in the sequel, consider a first eigenfunction of the p-Laplacian $\phi_{1,p} \in W_0^{1,p}(\Omega)$ and t large so that $J_\lambda(t\phi_{1,p}) < 0$. Then, we can define

$$(7.1) \qquad \Gamma = \left\{ \gamma \in \mathcal{C}\big([0,1], W_0^{1,p}(\Omega)\big) : \ \gamma(0) = 0, \ \gamma(1) = t\phi_{1,p} \right\}$$

and the level of mountain pass

$$(7.2) \qquad \qquad c_p = \inf_{\gamma \in \Gamma} \sup_{s \in [0,1]} J_\lambda(\gamma(s)).$$

Since in the case $p > n$ the concave-convex problem (1.1) is subcritical for every r, we can apply the results in [14] and get that for every fixed p there exist at least two positive weak solutions for $\lambda < \lambda_{\max,p}$, one positive weak solution for $\lambda = \lambda_{\max,p}$ and no positive weak solution for $\lambda > \lambda_{\max,p}$. When $\lambda < \lambda_{\max,p}$ one solution corresponds to a negative local minimum of J_λ and the second one, at least for λ small enough, corresponds to a positive mountain pass level of J_λ. Moreover, it is also proved in [14] that the local minimizer coincides with the minimal solution obtained by the sub- and supersolution method in Section 6.

In this section we provide a quantitative version of this multiplicity result and prove the existence of a second positive solution to problem (1.1) by means of the Ambrosetti-Rabinowitz Mountain Pass Theorem (see [2]) up to a certain, possibly non-optimal, value of λ that can be estimated in terms of p, q, r and n. This limitation in λ is due to the fact that we need explicit estimates of the energy levels in terms of p; however, we shall see that our results are asymptotically sharp.

This results allow to pass to the limit as $p \to \infty$ and get a second positive viscosity solution of the limit problem (3.2) for every $\Lambda \in (0, \hat{\Lambda})$, a global multiplicity result. In the sequel we will denote,

$$(7.3) \qquad \mu_{1,p} = (2\,|\Omega|C_p^p)^{\frac{q-r}{r-(p-1)}} \left(\frac{q+1}{r+1} \right)$$

and

$$(7.4) \qquad \begin{aligned} \mu_{2,p} &= \mu_{1,p} \left(\frac{r+1}{p} \right)^{\frac{r-q}{r-(p-1)}} \\ &= (2\,|\Omega|\,C_p^p)^{\frac{q-r}{r-(p-1)}} \left(\frac{(q+1)\,(r+1)^{\frac{(p-1)-q}{r-(p-1)}}}{p^{\frac{r-q}{r-(p-1)}}} \right), \end{aligned}$$

where C_p is given by (2.1).

Remark 7.1. Notice that both $\mu_{1,p}^{1/p}$ and $\mu_{2,p}^{1/p}$ converge to $\hat{\Lambda} = \Lambda_1^{\frac{R-Q}{R-1}}$ as $p \to \infty$.

Theorem 7.2. *Whenever $\lambda \in [0, \mu_{2,p})$, for $\mu_{2,p}$ given by (7.4), there exists $u \in W_0^{1,p}(\Omega)$ such that*

$$J_\lambda(u) = c_p \qquad and \qquad J_\lambda'(u) = 0,$$

where c_p is the level of mountain pass given by (7.2).

In the proof of Theorem 7.2 we shall need the following calculus fact.

Lemma 7.3. *Let $0 < q < p - 1 < r$ and constants $\lambda, \alpha, \beta > 0$. Define,*

$$\mu_1 = \left(\frac{q+1}{r+1} \right) \alpha^{-1} \beta^{\frac{q-(p-1)}{r-(p-1)}}$$

and

$$\mu_2 = \mu_1 \left(\frac{r+1}{p} \right)^{\frac{r-q}{r-(p-1)}} = \left(\frac{(q+1)\,(r+1)^{\frac{(p-1)-q}{r-(p-1)}}}{p^{\frac{r-q}{r-(p-1)}}} \right) \alpha^{-1} \beta^{\frac{q-(p-1)}{r-(p-1)}}.$$

Then:

    i) *If $\lambda \in [0, \mu_1)$*

$$\sup_{t>0} \min \left\{ \frac{1}{p}t^p - \frac{\lambda\alpha}{q+1}t^{q+1}, \ \frac{1}{p}t^p - \frac{\beta}{r+1}t^{r+1} \right\}$$

$$= \left(\frac{1}{p} - \frac{1}{r+1} \right) \beta^{\frac{-p}{r-(p-1)}} > 0.$$

    ii) *If $\lambda \in [\mu_1, \mu_2)$*

$$\sup_{t>0} \min \left\{ \frac{1}{p}t^p - \frac{\lambda\alpha}{q+1}t^{q+1}, \ \frac{1}{p}t^p - \frac{\beta}{r+1}t^{r+1} \right\}$$

$$= \left(\frac{\lambda\alpha}{\beta} \right)^{\frac{p}{r-q}} \left(\frac{r+1}{q+1} \right)^{\frac{p}{r-q}} \left[\frac{1}{p} - \left(\frac{\lambda\alpha}{q+1} \right)^{\frac{r-(p-1)}{r-q}} \left(\frac{\beta}{r+1} \right)^{\frac{(p-1)-q}{r-q}} \right] > 0.$$

    iii) *If $\lambda \geq \mu_2$*

$$\sup_{t>0} \min \left\{ \frac{1}{p}t^p - \frac{\lambda\alpha}{q+1}t^{q+1}, \ \frac{1}{p}t^p - \frac{\beta}{r+1}t^{r+1} \right\} = 0.$$

Proof of Theorem 7.2. We only need to check that J_λ satisfies the geometry of the Mountain Pass theorem and that it satisfies the Palais-Smale condition. Then the result follows from the Ambrosetti-Rabinowitz Mountain Pass theorem (see [2]). As $p > n$, the Morrey estimate (see Lemma 2.1) implies

$$J_\lambda(u) \geq \Phi_\lambda \left(\|u\|_{W_0^{1,p}(\Omega)} \right)$$

for

$$\Phi_\lambda(t) = \min \left\{ \frac{1}{p}t^p - \frac{2\lambda|\Omega|C_p^{q+1}}{q+1}t^{q+1}, \frac{1}{p}t^p - \frac{2|\Omega|C_p^{r+1}}{r+1}t^{r+1} \right\}.$$

Lemma 7.3 with $\alpha = 2\,|\Omega|\,C_p^{q+1}$ and $\beta = 2\,|\Omega|\,C_p^{r+1}$ implies that

$$\max_{t>0} \Phi_\lambda(t) > 0 \qquad \text{if } \lambda < \mu_{2,p},$$

with $\mu_{2,p}$ given by (7.4). Hence J_λ has the geometry of the Mountain Pass theorem.

The proof of the fact that J_λ satisfies the Palais-Smale condition is standard; an estimate similar to the proof of Proposition 7.10, Morrey's estimates and the continuity of $(-\Delta_p)^{-1}$ give the necessary compactness. See for instance [30] and the references therein for details. □

Next, we provide estimates of the level of mountain pass (7.2).

Proposition 7.4. *For every fixed $p > n$, the level of mountain pass (7.2) satisfies,*

$$c_p \leq \left(\frac{r - (p-1)}{p\,(r+1)} \right) |\Omega|\,\lambda_1(p;\Omega)^{\frac{r+1}{r-(p-1)}}.$$

Remark 7.5. Notice that this result holds independently of $\lambda \geq 0$.

Proof. Consider the following radial path in $W_0^{1,p}(\Omega)$

$$\gamma : [0,1] \to W_0^{1,p}$$
$$s \mapsto sv$$

for v an appropriate rescaling of the first eigenfunction as in the definition of Γ in (7.1). Clearly, $\gamma \in \Gamma$ and by definition of c_p, we have that

$$c_p = \inf_{\gamma \in \Gamma} \sup_{s \in [0,1]} J_\lambda(\gamma(s)) \leq \sup_{s>0} J_\lambda(sv) = \sup_{s>0} \left\{ \frac{A}{p}s^p - \frac{\lambda B}{q+1}s^{q+1} - \frac{C}{r+1}s^{r+1} \right\}$$

for

$$A = \lambda_1(p;\Omega) \int_\Omega |v|^p \, dx, \qquad B = \int_\Omega |v|^{q+1} \, dx, \qquad C = \int_\Omega |v|^{r+1} \, dx.$$

We can estimate

$$\frac{A}{p}s^p - \frac{\lambda B}{q+1}s^{q+1} - \frac{C}{r+1}s^{r+1} \leq \frac{A}{p}s^p - \max \left\{ \frac{\lambda B}{q+1}s^{q+1}, \frac{C}{r+1}s^{r+1} \right\}$$

$$= \min \left\{ \frac{A}{p}s^p - \frac{\lambda B}{q+1}s^{q+1}, \frac{A}{p}s^p - \frac{C}{r+1}s^{r+1} \right\}.$$

Hence, we can apply Lemma 7.3 with $\alpha = B/A$ and $\beta = C/A$ and deduce that

$$c_p \leq \left(\frac{1}{p} - \frac{1}{r+1} \right) A^{\frac{r+1}{r-(p-1)}} C^{\frac{-p}{r-(p-1)}}.$$

Notice that this bound holds for every $\lambda > 0$. Then, as a consequence of the Hölder inequality, we have that

$$A \leq \lambda_1(p;\Omega) |\Omega|^{\frac{r-(p-1)}{r+1}} C^{\frac{p}{r+1}},$$

and hence,

$$c_p \leq \left(\frac{1}{p} - \frac{1}{r+1}\right) \lambda_1(p;\Omega)^{\frac{r+1}{r-(p-1)}} |\Omega|. \qquad \square$$

Proposition 7.6. *For every fixed $p > n$, the level of mountain pass (7.2) satisfies,*

$$c_p \geq \left(\frac{r-(p-1)}{p(r+1)}\right) \left(2|\Omega|C_p^{r+1}\right)^{\frac{-p}{r-(p-1)}} > 0$$

whenever $0 \leq \lambda \leq \mu_{1,p}$, for $\mu_{1,p}$ given by (7.3), and

$$c_p \geq \lambda^{\frac{p}{r-q}} \left(\frac{r+1}{q+1}\right)^{\frac{p}{r-q}} \left(\frac{1}{p \, C_p^p} - 2|\Omega|(r+1)^{\frac{(p-1)-q}{r-q}} \left(\frac{\lambda}{q+1}\right)^{\frac{r-(p-1)}{r-q}}\right) > 0$$

whenever $\mu_{1,p} < \lambda < \mu_{2,p}$, for $\mu_{2,p}$ given by (7.4).

Proof. As $p > n$, the Morrey estimate (see Lemma 2.1) implies

$$J_\lambda(u) \geq \Phi_\lambda \left(\|u\|_{W_0^{1,p}(\Omega)}\right)$$

for

$$\Phi_\lambda(t) = \min\left\{\frac{1}{p} t^p - \frac{2\lambda|\Omega|C_p^{q+1}}{q+1} t^{q+1}, \frac{1}{p} t^p - \frac{2|\Omega|C_p^{r+1}}{r+1} t^{r+1}\right\}.$$

Moreover, we know from Lemma 7.3 that $\sup_{t>0} \Phi_\lambda(t) > 0$ whenever $\lambda < \mu_{2,p}$ and

$$\Phi_\lambda \left(\|\gamma(1)\|_{W_0^{1,p}(\Omega)}\right) \leq J_\lambda(\gamma(1)) < 0$$

by definition of Γ. Consequently,

$$c_p = \inf_{\gamma \in \Gamma} \sup_{s \in [0,1]} J_\lambda(\gamma(s)) \geq \sup_{t>0} \Phi_\lambda(t).$$

Then, Lemma 7.3 with $\alpha = 2\,|\Omega|\,C_p^{q+1}$ and $\beta = 2\,|\Omega|\,C_p^{r+1}$ implies the result. $\qquad \square$

From Propositions 7.4 and 7.6 we get the following consequence.

Corollary 7.7. *Whenever $0 \leq \lambda < \mu_{1,p}$, the level of mountain pass (7.2) satisfies,*

$$\lim_{p\to\infty} c_p^{1/p} = \Lambda_1(\Omega)^{\frac{R}{R-1}}.$$

Proposition 7.8. *The positive solution $u \in W_0^{1,p}(\Omega)$ obtained in Theorem 7.2 satisfies the following estimates,*

$$(7.5) \qquad \|u\|_\infty \geq \left(c_p \, |\Omega|^{-1} \frac{(q+1)(r+1)}{r-q}\right)^{\frac{1}{r+1}},$$

and,

$$\|\nabla u\|_{L^p(\Omega)} \geq C_p^{-1} \left(c_p \, |\Omega|^{-1} \frac{(q+1)(r+1)}{r-q}\right)^{\frac{1}{r+1}},$$

where c_p is the level of mountain pass in (7.2) and C_p is given by (2.1).

Proof. The second inequality follows easily from the first one using Morrey's inequality, Lemma 2.1. For the proof of the first inequality, let u be the positive solution obtained in Theorem 7.2. We have that,

$$c_p = J_\lambda(u) - \frac{1}{q+1}\langle J'_\lambda(u), u\rangle$$

implies,

$$c_p + \left(\frac{1}{q+1} - \frac{1}{p}\right)\int_\Omega |\nabla u|^p\, dx = \left(\frac{1}{q+1} - \frac{1}{r+1}\right)\int_\Omega u^{r+1}\, dx$$

Then, Morrey's inequality implies,

$$c_p + \left(\frac{1}{q+1} - \frac{1}{p}\right) C_p^{-p}\|u\|_\infty^p \le \left(\frac{1}{q+1} - \frac{1}{r+1}\right) |\Omega|\, \|u\|_\infty^{r+1}.$$

Using the following elementary inequality for non-negative numbers (see for instance [5, Section 14.7])

$$a^{\frac{1}{\alpha}} b^{\frac{1}{\beta}} \le \frac{a}{\alpha} + \frac{b}{\beta} \qquad \forall \beta > 1, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

we get

$$(\alpha c_p)^{\frac{1}{\alpha}} \left(\beta\left(\frac{1}{q+1} - \frac{1}{p}\right) C_p^{-p}\|u\|_\infty^p\right)^{\frac{1}{\beta}} \le \left(\frac{1}{q+1} - \frac{1}{r+1}\right) |\Omega|\, \|u\|_\infty^{r+1}.$$

Using that $\alpha = \beta/(\beta - 1)$ we get that

$$\|u\|_\infty^{r+1-\frac{p}{\beta}} \ge |\Omega|^{-1} \left(\frac{1}{q+1} - \frac{1}{r+1}\right)^{-1} \left(\frac{\beta c_p}{\beta - 1}\right)^{\frac{\beta - 1}{\beta}} \left(\beta\left(\frac{1}{q+1} - \frac{1}{p}\right) C_p^{-p}\right)^{\frac{1}{\beta}}$$

As this estimate holds for every $\beta > 1$, we can let $\beta \to \infty$ and get the estimate

$$\|u\|_\infty^{r+1} \ge c_p\, |\Omega|^{-1} \left(\frac{1}{q+1} - \frac{1}{r+1}\right)^{-1}. \qquad \square$$

Remark 7.9. In virtue of Corollary 7.7 we have that the right-hand side of (7.5) converges as $p \to \infty$ to $\Lambda_1(\Omega)^{\frac{1}{R-1}}$.

Proposition 7.10. *The positive solution $u \in W_0^{1,p}(\Omega)$ obtained in Theorem 7.2 satisifies*

$$(7.6) \qquad \|\nabla u\|_{L^p(\Omega)} \le \left[\frac{2\, p\, (r+1)}{r - (p-1)}\left(c_p + \frac{(p-1) - q}{p} A^{\frac{p}{(p-1)-q}}\right)\right]^{\frac{1}{p}}$$

where c_p is the level of mountain pass given by (7.2),

$$A = 2^{\frac{q+1}{p}} \lambda|\Omega|C_p^{q+1}\left(\frac{(r-q)\,((r+1)(q+1))^{\frac{q-(p-1)}{p}}}{(r - (p-1))^{\frac{q+1}{p}}}\right)$$

and C_p is given by (2.1). Moreover,

$$(7.7) \qquad \|u\|_\infty \le C_p\left[\frac{2\, p\, (r+1)}{r - (p-1)}\left(c_p + \frac{(p-1) - q}{p} A^{\frac{p}{(p-1)-q}}\right)\right]^{\frac{1}{p}}.$$

Proof. We only prove the first inequality as the second one follows from the first as a consequence of Morrey's estimates. Let u be the positive solution obtained in Theorem 7.2. From,

$$c_p = J_\lambda(u) - \frac{1}{r+1}\langle J'_\lambda(u), u\rangle$$

we get that,

$$\left(\frac{1}{p} - \frac{1}{r+1}\right)\int_\Omega |\nabla u|^p\, dx = c_p + \lambda\left(\frac{1}{q+1} - \frac{1}{r+1}\right)\int_\Omega u^{q+1}\, dx$$

$$\leq c_p + \lambda\,|\Omega|\, C_p^{q+1}\left(\frac{1}{q+1} - \frac{1}{r+1}\right)\left(\int_\Omega |\nabla u|^p\, dx\right)^{\frac{q+1}{p}}.$$

From the weighted Young's inequality with exponents $\frac{p}{(p-1)-q}$ and $\frac{p}{q+1}$, we get

$$\left(\frac{1}{p} - \frac{1}{r+1} - \epsilon^{\frac{p}{q+1}}\left(\frac{q+1}{p}\right)\right)\int_\Omega |\nabla u|^p\, dx$$

$$\leq c_p + \left(\frac{(p-1)-q}{p}\right)\left(\frac{\lambda}{\epsilon}\,|\Omega|\, C_p^{q+1}\left(\frac{1}{q+1} - \frac{1}{r+1}\right)\right)^{\frac{p}{(p-1)-q}}.$$

Finally, choosing

$$\epsilon = \left(\frac{r - (p-1)}{2(q+1)(r+1)}\right)^{\frac{q+1}{p}}$$

we obtain (7.6). $\qquad\square$

Remark 7.11. As a consequence of Corollary 7.7, the right-hand side of (7.7) converges as $p \to \infty$ to $\Lambda_1(\Omega)^{\frac{1}{R-1}}$ whenever $\Lambda < \hat\Lambda$.

From the previous results and Morrey's estimates (Lemma 2.1), we get the following result.

Theorem 7.12.

(1) $\displaystyle\lim_{p\to\infty} \mu_{1,p}^{1/p} = \lim_{p\to\infty} \mu_{2,p}^{1/p} = \hat\Lambda = \Lambda_1(\Omega)^{\frac{R-Q}{R-1}}.$

(2) *Fix* $\Lambda \in [0, \hat\Lambda)$ *and a sequence* $\{\lambda_p\}_p$ *such that* $\displaystyle\lim_{p\to\infty} \lambda_p^{1/p} = \Lambda$. *Then, there exists* p_0 *for which*

$$0 < \lambda_p < \mu_{1,p} \quad \text{for every } p \geq p_0.$$

(3) *Fix* $\Lambda \in [0, \hat\Lambda)$ *and let* $\{\lambda_p\}_p$ *be a sequence such that* $\displaystyle\lim_{p\to\infty} \lambda_p^{1/p} = \Lambda$. *For* $p \geq p_0$, *consider* $\{u_{\lambda_p,p}\}_p$, *the associated sequence of mountain pass positive solutions of*

$$\begin{cases} -\Delta_p u_{\lambda_p,p} = \lambda_p\, u_{\lambda_p,p}^{q(p)} + u_{\lambda_p,p}^{r(p)} & \text{in } \Omega \\ u_{\lambda_p,p} > 0 & \text{in } \Omega \\ u_{\lambda_p,p} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, up to a subsequence,

$$u_{\lambda_p,p} \to u_\Lambda \quad \text{uniformly as } p \to \infty,$$

with $u_\Lambda(x)$ *a viscosity solution to* (1.3) *with*

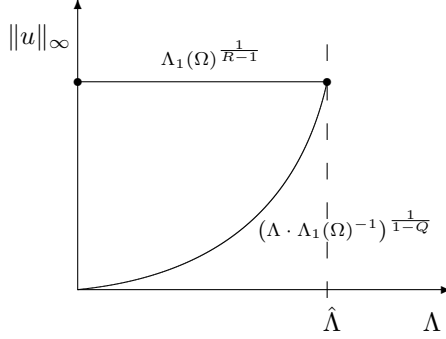$$\|u_\Lambda\|_\infty = \Lambda_1(\Omega)^{\frac{1}{R-1}}.$$

FIGURE 3. Diagram of solutions of problem (4.1).

Remark 7.13. Notice that the corresponding minimal solution for $\Lambda \in [0, \hat{\Lambda})$, w_Λ verifies

$$\|w_\Lambda\|_\infty = (\Lambda \cdot \Lambda_1(\Omega)^{-1})^{\frac{1}{1-Q}} < \Lambda_1(\Omega)^{\frac{1}{R-1}} = \|u_\Lambda\|_\infty.$$

Hence, problem (1.3) has at least two positive viscosity solutions for every $\Lambda \in (0, \hat{\Lambda})$, at least one for $\Lambda = 0, \hat{\Lambda}$ and no solution for $\Lambda > \hat{\Lambda}$, see Figure 3.

8. Multiplicity of solutions for the limit problem in special domains

To state the results in this section we need to introduce some notation. We define the *ridge set* of Ω,

$$\begin{aligned} \mathcal{R} &= \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \text{ is not differentiable at } x\} \\ &= \{x \in \Omega : \exists x_1, x_2 \in \partial\Omega, \ x_1 \neq x_2, \text{ s.t. } |x - x_1| = |x - x_2| = \operatorname{dist}(x, \partial\Omega)\} \end{aligned}$$

and its subset $\mathcal{M}$, the *set of maximal distance* to the boundary,

$$\mathcal{M} = \big\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) = \|\operatorname{dist}(\cdot, \partial\Omega)\|_\infty\big\}.$$

In the case of Ω a general bounded domain, we have proved in Proposition 5.1 the existence of a curve of minimal solutions for the limit problem (1.3), as well as several non-existence results in Propositions 4.1, 4.3 and 5.3 (see Figure 2).

Then, in Section 7 we have proved the existence of a second positive solution to the limit problem (1.3) as a limit of mountain pass solutions.

In this section we are going to show that in bounded domains satisfying the geometric condition $\mathcal{M} \equiv \mathcal{R}$ it is possible to find a curve of explicit positive solutions corresponding to the solutions already found in Sections 6 and 7. Some examples of domains satisfying the geometric condition are the ball, the annulus and the stadium (convex hull of two balls of the same radius). A square or an ellipse do not verify the condition.

Theorem 8.1. *Suppose that Ω is a bounded domain such that $\mathcal{M} \equiv \mathcal{R}$ and let $\Lambda > 0$, $0 < Q < 1 < R$ and $\hat{\Lambda} = \Lambda_1(\Omega)^{\frac{R-Q}{R-1}}$ as before. Consider the concave-convex*

problem,

$$
(8.1) \quad
\begin{cases}
\min\left\{|\nabla u_\Lambda(x)| - \max\{\Lambda u_\Lambda^Q(x), u_\Lambda^R(x)\}, -\Delta_\infty u_\Lambda(x)\right\} = 0 & in\ \Omega, \\
u_\Lambda > 0 & in\ \Omega \\
u_\Lambda = 0 & on\ \partial\Omega.
\end{cases}
$$

and solutions of the form

$$(8.2) \qquad\qquad u(x) = a \cdot dist(x, \partial\Omega), \qquad a > 0.$$

Then, problem (8.1)*,*

    i) *Has exactly two viscosity solutions of the form* (8.2) *for each* $\Lambda \in (0, \hat\Lambda)$*,*
       with $a_1(\Lambda) = \left(\Lambda\,\Lambda_1(\Omega)^{-Q}\right)^{\frac{1}{1-Q}}$*, and* $a_2(\Lambda) = \Lambda_1(\Omega)^{\frac{R}{R-1}}$*.*
    ii) *Has exactly one viscosity solution of the form* (8.2) *for* $\Lambda = 0$ *and* $\Lambda = \hat\Lambda$*,*
       both for $a = \Lambda_1(\Omega)^{\frac{R}{R-1}}$*.*
    iii) *Has no positive viscosity solution if* $\Lambda > \hat\Lambda$*.*

Proof. First of all, we are going to check that

$$-\Delta_\infty u(x) = 0 \qquad in\ \Omega \setminus \mathcal{R}$$

in the viscosity sense. Let $\phi \in \mathcal{C}^2$ and $x_0 \in \Omega \setminus \mathcal{R}$ such that $u - \phi$ has a local maximum at x_0. We can assume $u(x_0) = \phi(x_0)$ and $\nabla\phi(x_0) \neq 0$. A Taylor expansion, and the fact that ϕ touches u from above at x_0 yield

$$-\frac{\Delta_\infty \phi(x_0)}{|\nabla\phi(x_0)|^2} + o(1) \leq \frac{1}{\epsilon^2}\left(2u(x_0) - \max_{y \in B_\epsilon(x_0)} u(y) - \min_{y \in B_\epsilon(x_0)} u(y)\right)$$

as $\epsilon \to 0$. From (8.2) we have that

$$\max_{y \in B_\epsilon(x_0)} u(y) = u(x_0) + a\epsilon, \qquad \min_{y \in B_\epsilon(x_0)} u(y) = u(x_0) - a\epsilon$$

and we deduce that u is ∞-subharmonic in $\Omega \setminus \mathcal{R}$. The proof that it is also ∞-superharmonic is analogous.

Hence, we need to make sure that

$$|\nabla u(x)| - \max\left\{\Lambda u^Q(x), u^R(x)\right\} \geq 0 \qquad in\ \Omega \setminus \mathcal{R}$$

in the viscosity sense. Indeed, plugging (8.2) into the latter expression (recall that $x \notin \mathcal{R}$ so the derivatives are classical), we find that

$$|\nabla u(x)| - \max\left\{\Lambda u^Q(x), u^R(x)\right\} = a - \max\left\{\Lambda\, a^Q \mathrm{dist}(x, \partial\Omega)^Q,\ a^R \mathrm{dist}(x, \partial\Omega)^R\right\}.$$

Since we can choose points $x \notin \mathcal{R} \equiv \mathcal{M}$ arbitrarily close to $\mathcal{M}$, we find the following necessary condition for a,

$$(8.3) \qquad a - \max\left\{\Lambda\, a^Q \|\mathrm{dist}(\cdot, \partial\Omega)\|_\infty^Q,\ a^R \|\mathrm{dist}(\cdot, \partial\Omega)\|_\infty^R\right\} \geq 0.$$

Now, we turn our attention to the ridge set $\mathcal{R}$. First, observe that cones as in (8.2) are always supersolutions of (8.1) in the ridge set, since they cannot be touched from below with $\mathcal{C}^2$ functions at those points.

Hence, we only have to consider the subsolution case. So, let $x_0 \in \mathcal{R}$ and $\phi \in \mathcal{C}^2$ such that $u - \phi$ has a local maximum point at x_0. We aim to prove that

$$(8.4) \qquad \min\left\{|\nabla\phi(x_0)| - \max\{\Lambda u^Q(x_0), u^R(x_0)\}, -\Delta_\infty \phi(x_0)\right\} \leq 0.$$
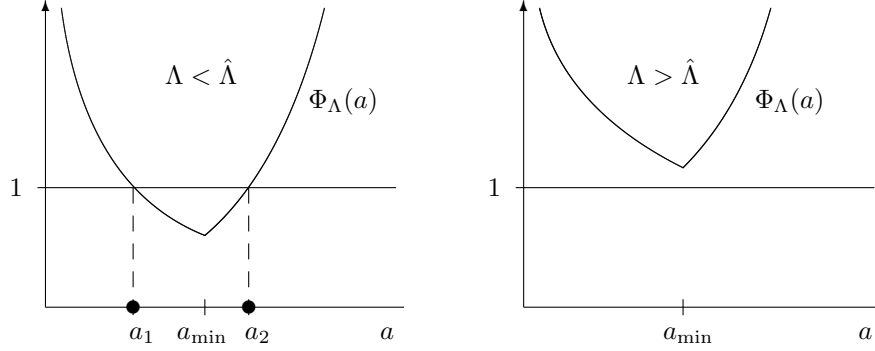
FIGURE 4. When $\Lambda < \hat{\Lambda}$ the equation $\Phi_\Lambda(t) = 1$ has exactly two solutions t_1, t_2, a unique solution if $\Lambda = \hat{\Lambda}$ and no solution if $\Lambda > \hat{\Lambda}$.

It is well known (see for instance [18, Lemma 6.10]) that u in (8.2) satisfies

$$\min\left\{|\nabla u(x)| - a, -\Delta_\infty u(x)\right\} = 0.$$

Thus, by definition of viscosity subsolution we have that either $|\nabla\phi(x_0)| \leq a$ or $-\Delta_\infty\phi(x_0) \leq 0$. In the latter case, (8.4) holds and there is nothing to prove. Thus, we can suppose in the sequel that $-\Delta_\infty\phi(x_0) > 0$ and $|\nabla\phi(x_0)| \leq a$.

Then, since $x_0 \in \mathcal{R} \equiv \mathcal{M}$, we have $u(x_0) = a\,\Lambda_1(\Omega)^{-1}$ and

$$|\nabla\phi(x_0)| - \max\{\Lambda u^Q(x_0), u^R(x_0)\} \leq a - \max\{\Lambda\,a^Q\Lambda_1(\Omega)^{-Q}, a^R\Lambda_1(\Omega)^{-R}\}.$$

Recalling (8.3), we discover that the only possibility is

(8.5) $$a - \max\{\Lambda\,a^Q\Lambda_1(\Omega)^{-Q}, a^R\Lambda_1(\Omega)^{-R}\} = 0.$$

The rest of the proof is devoted to study the number of positive solutions of this equation.

We rewrite (8.5) as $\Phi_\Lambda(a) = 1$ where $\Phi_\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is given by

$$\Phi_\Lambda(a) = \max\{\Lambda\,\Lambda_1^{-Q}a^{Q-1}, \Lambda_1^{-R}\,a^{R-1}\}.$$

It is elementary to check that the function Φ_Λ is positive, convex and has a minimum point at $a_{\min} = \Lambda_1\Lambda^{\frac{1}{R-Q}}$ (where the function is not differentiable). Notice that $\lim_{a\to\infty}\Phi_\Lambda(a) = \lim_{a\to 0}\Phi_\Lambda(a) = \infty$, so the minimum is a global minimum (see Figure 4). Such a minimum value is given by

$$\min_{a>0}\Phi_\Lambda(a) = \Phi(a_{\min}) = \Lambda_1^{-1}\Lambda^{\frac{R-1}{R-Q}}.$$

Given the geometry of Φ_Λ, the equation $\Phi_\Lambda(a) = 1$ will not have solutions if

$$\Phi(a_{\min}) = \Lambda_1^{-1}\Lambda^{\frac{R-1}{R-Q}} > 1 \quad \Leftrightarrow \quad \Lambda > \hat{\Lambda},$$

which is coherent with Proposition 4.3, where statement (iii) is proved. If $\Lambda = \hat{\Lambda}$, we have $\Phi(a_{\min}) = 1$ and the equation has a single solution $a = \Lambda_1^{\frac{R}{R-1}}$. In the case $0 < \Lambda \leq \hat{\Lambda}$ the equation $\Phi_\Lambda(a) = 1$ will have two different solutions, namely,

$$a_1 = \left(\Lambda \cdot \Lambda_1(\Omega)^{-Q}\right)^{\frac{1}{1-Q}}, \qquad a_2 = \Lambda_1(\Omega)^{\frac{R}{R-1}}.$$

Finally, if $\Lambda = 0$, it is easy to obtain the existence of a single solution which coincides with $a_2 = \Lambda_1(\Omega)^{\frac{R}{R-1}}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

References

[1] B. Abdellaoui, I. Peral; *Existence and nonexistence results for quasilinear elliptic equations involving the p-Laplacian with a critical potential*, Annali di Matematica Pura e Applicata, Vol 182, No 3 (2003), 247-270.

[2] A. Ambrosetti, P. Rabinowitz; *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14 (1973), 349-381.

[3] A. Anane; *Simplicitè et isolation de la premiere valeur propre du p-laplacien avec poids*, C. R. Acad. Sci. Paris Sér. I Math. 305 (1987), no. 16, 725–728.

[4] A. Ambrosetti, J. García Azorero, I. Peral; *Multiplicity results for some nonlinear elliptic equations*, J. Funct. Anal. 137 (1996), 29-242.

[5] E. Beckenbach, R. Bellman; *Inequalities*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Bd. 30 Springer-Verlag, Berlin-Gï¿½ttingen-Heidelberg 1961.

[6] T. Bhattacharya, E. DiBenedetto, J. Manfredi; *Limit as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems*, Rend. Sem. Mat. Univ. Politec. Torino (1989), pp. 15-68.

[7] L. Boccardo, M. Escobedo, I. Peral; *A Dirichlet problem involving crititical exponents*, Nonlinear Analalysis, Theory, methods & Applications 24 (1995), no.11, pp. 1639–1648.

[8] F. Charro, E. Parini; *Limits as $p \to \infty$ of p-laplacian problems with a superdiffusive power-type nonlinearity: positive and sign-changing solutions*, J. Math. Anal. Appl. 372 (2010), no. 2, 629–644.

[9] F. Charro, I. Peral; *Limit branch of solutions as $p \to \infty$ for a family of sub-diffusive problems related to the p-laplacian* Comm. Partial Differential Equations, vol. 32 (2007), no. 12, pp. 1965 − 1981.

[10] F. Charro, I. Peral; *Zero Order Perturbations to Fully Nonlinear equations: Comparison, existence and uniqueness*, Commun. Contemp. Math. 11 (1) (2009) 131–164.

[11] M. G. Crandall, H. Ishii, P. L. Lions; *User's Guide to Viscosity Solutions of Second Order Partial Differential Equations*, Bull. Amer. Math. Soc. 27 (1992), no. 1, pp. 1-67.

[12] L.C. Evans, R. F. Gariepy; *Measure theory and fine properties of functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992. viii+268 pp.

[13] N. Fukagai, M. Ito, K. Narukawa; *Limit as $p \to \infty$ of p-Laplace eigenvalue problems and L^∞-inequality of the Poincare type*. Differential Integral Equations 12 (1999), no. 2, pp. 183–206.

[14] J. P. García Azorero, J.J. Manfredi, I. Peral; *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations*. Commun. Contemp. Math. 2 (2000), no. 3, 385-404.

[15] J. García Azorero, I. Peral; *Existence and nonuniqueness for the p-laplacian: Nonlinear eigenvalues*, Comm. Partial Differential Equations, vol 12, no. 12 (1987) 1389-1430.

[16] R.Gariepy, W. Ziemer, P. William; *A regularity condition at the boundary for solutions of quasilinear elliptic equations*, Arch. Rational Mech. Anal. 67 (1977), no. 1, 25–39.

[17] R. Jensen; *Uniqueness of Lipschitz extensions: Minimizing the sup norm of the gradient*, Arch. Rational Mech. Anal. 123 (1993), pp. 51–74.

[18] P. Juutinen; *Minimization problems for Lipschitz functions via viscosity solutions*. Dissertation, University of Jyväskulä, Jyväskulä, 1998. Ann. Acad. Sci. Fenn. Math. Diss. No. 115 (1998), 53 pp.

[19] P. Juutinen, P. Lindqvist *On the higher eigenvalues for the ∞-eigenvalue problem*, Calc. Var. and Partial Differential Equations, 23:169–192, 2005.

[20] P. Juutinen, P. Lindqvist and J. Manfredi, *The ∞-eigenvalue problem*, Arch. Ration. Mech. Anal. 148 (1999), no. 2, pp. 89-105.

[21] B. Kawohl; *On a family of torsional creep problems*, J. Reine Angew. Math. 410 (1990), 1–22.

[22] T. Kilpeläinen, P. Lindqvist; *Nonlinear ground states in irregular domains*, Indiana Univ. Math. J. 49 (2000), no. 1, 325–331.

[23] T. Kilpeläinen, J. Malý; *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. 172 (1994), no. 1, 137–161.

[24] P. Lindqvist; *On the equation* div $(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc. 109 (1990), no. 1, 157–164.

[25] P. Lindqvist; *Addendum: "On the equation* div$(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$" *[Proc. Amer. Math. Soc. 109 (1990), no. 1, 157–164; MR1007505 (90h:35088)]*, Proc. Amer. Math. Soc. 116 (1992), no. 2, 583–584.

[26] P. Lindqvist, J. Manfredi; *The Harnack inequality for ∞-harmonic functions*, Elec. J. Diff. Eqs. 5 (1995), pp. 1-5.

[27] P. Lindqvist, J. Manfredi; *Note on ∞-superharmonic functions*, Revista Matemática de la Universidad Complutense de Madrid 10 (1997), pp. 1-9.

[28] J. Malý, W. P. Ziemer; *Fine regularity of solutions of elliptic partial differential equations*, Mathematical Surveys and Monographs, 51. American Mathematical Society, Providence, RI, 1997. xiv+291 pp.

[29] V. G.Maz'ja; *The continuity at a boundary point of the solutions of quasi-linear elliptic equations (Russian. English summary)*, Vestnik Leningrad. Univ. 25 1970 no. 13, 42–55.

[30] I. Peral *Some results on Quasilinear Elliptic Equations: Growth versus Shape*, 153-202, *Proceedings of the Second School of Nonlinear Functional Analysis and Applications to Differential Equations* I.C.T.P. Trieste, Italy, A. Ambrosetti and it alter editors. World Scientific, 1998.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS AT AUSTIN, 1 UNIVERSITY STATION C1200, AUSTIN, TX 78712

E-mail address: `fcharro@math.utexas.edu`

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID AND ICMAT, 28049 MADRID, SPAIN.

E-mail address: `ireneo.peral@uam.es`