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ZERO ORDER PERTURBATIONS TO FULLY NONLINEAR EQUATIONS: COMPARISON, EXISTENCE AND UNIQUENESS

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Abstract. We study existence of solutions to

$$\begin{cases} F(\nabla u, D^2u) = f(\lambda, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where F is elliptic and homogeneous of degree m, and either $f(\lambda, u) = \lambda u^q$ or $f(\lambda, u) = \lambda u^q + u^r$, for $0 < q < m < r$, and $\lambda > 0$. Furthermore, in the first case we obtain that the solution is unique as a consequence of a comparison principle up to the boundary. Several examples, including uniformly elliptic operators and the infinity laplacian are considered.

1. Introduction

We study elliptic problems of the type

$$\begin{cases} F(\nabla u, D^2u) = f(\lambda, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

under suitable hypotheses of ellipticity and structure on F. In general we will consider a function $F : \mathbb{R}^n \times S^n \to \mathbb{R}$, verifying:

- $(F1)$ *D*egenerate Ellipticity: For every $p \in \mathbb{R}^n$, $F(p, X) \le F(p, Y)$ whenever $Y \le X$, with $X, Y \in S^n$.
- $(F2)$ *H*omogeneity of degree m: $F(tp, tX) = t^m \cdot F(p, X)$ for all $t > 0$. We further assume $F(0, 0) = 0$.

Our main concern is with the existence of nontrivial solutions (in the viscosity sense) for right hand sides

$$f_1(\lambda, u) = \lambda\, u^q \quad \text{and} \quad f_2(\lambda, u) = \lambda u^q + u^r,$$

for $0 < q < m < r$, with m the degree of homogeneity of F.

Our main result, Theorem 2.1, is a comparison result up to the boundary for problems with a right hand side satisfying

$$\frac{f(t)}{t^q} > 0 \quad \text{is non-increasing for all } t > 0 \text{ and some } 0 < q < m, \tag{1.2}$$

which can be interpreted as a viscosity counterpart for fully nonlinear equations of the comparison result by Brezis-Oswald in the variational setting (see [9] and also [8]). Notice that f_1 verifies (1.2). In particular the comparison result up to the boundary yields uniqueness of positive viscosity solutions of (1.1) if f fulfills (1.2).

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Since $f_1^{\frac{1}{m}}$ is a concave power, we will refer to this problem as a *concave problem.* To avoid any ambiguity, it is worth to emphasize that we are not assuming any concavity property on F.

Similarly, the case with f_2 will be referred to as a *concave-convex problem.* The main result in this case shows that there exists $\Lambda > 0$ such that (1.1) has at least one nontrivial solution for $\lambda \in (0, \Lambda)$ and does not have any nontrivial viscosity solution for $\lambda > \Lambda$.

The uniqueness of solutions in the concave case has been studied in some particular cases, such as the p-laplacian in the variational framework (see for instance [1] and the references therein) and, in the viscosity framework, fully nonlinear equations arising as a limit of p-laplacian type equations (see [14]) and k-hessian equations (see [20]). Our main goal is to show that this is a general fact for equations given by a homogeneous F. For the proof of existence in both cases, with f_1 and f_2, we follow the elementary ideas in [7].

The paper is organized as follows. In Section 2, the main comparison result (Theorem 2.1) is established. As an application, if F satisfies $(F1)$ and $(F2)$ above and f is under hypothesis (1.2), we obtain uniqueness of solutions for problem (1.1).

In Section 3, we prove the existence of solutions for both f_1 and f_2 when F is uniformly elliptic, which implies $m = 1$. In the case with f_1, our result is included in Theorem 6 in [6]. However, we provide an alternative proof based on the comparison result in Theorem 2.1 and the simple construction in [7].

Again by using the monotonicity argument in [7] we are able to prove the existence for a concave-convex right hand side f_2. We can find a related existence result for Pucci's operators in [10], where it is considered a right hand side of the type $f(u) = \lambda u + g(\lambda, u)$ with g continuous and $g(\lambda, s) = o(|s|)$ as $s \to 0$, and bifurcation (from the eigenvalues) techniques are used. However, in our case the solutions branch off from $(0, 0)$ since the concave part is not differentiable at 0. In the p-Laplacian case this kind of behavior was studied in [2].

Finally, in Section 4, we extend the simple techniques in Section 3 to some non-uniformly elliptic problems which have degree of homogeneity greater than one such as the infinity laplacian (with both normalizations), a context in which these results are new, and Monge-Ampere equations, for which we obtain some extension of known results. See for instance [20] where different methods are used. Other examples considered range from the linear problem with variable coefficients to the p-laplacian, or the equation

$$\min\{|\nabla u| - \lambda u^q, -\Delta_\infty u\} = 0,$$

already considered in [14].

2. Some comparison results

The following comparison principle is the main result of this section.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and consider a viscosity subsolution u and a supersolution v of*

$$F(\nabla w, D^2 w) = f(w) \quad \text{in } \Omega, \tag{2.1}$$

where $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ satisfies $(F1)$, $(F2)$ for some m, and $f(\cdot)$ satisfy

$$\frac{f(t)}{t^q} > 0 \quad \text{is non-increasing for all } t > 0 \text{ and some } 0 < q < m. \tag{2.2}$$

Suppose that both, u and v are strictly positive in Ω, continuous up to the boundary and satisfy $u \le v$ on $\partial\Omega$. Then, $u \le v$ in $\overline{\Omega}$.

As a consequence, we have the following uniqueness result for the Dirichlet problem.

Corollary 2.2. *The problem*

$$\begin{cases} F(\nabla u, D^2u) = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

with F verifying $(F1)$ and $(F2)$ for some m, and f verifying (2.2), has at most one positive viscosity solution.

Notice that our equation, written in the form $G(w, \nabla w, D^2w) = 0$ with

$$\begin{array}{rcl} G : \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n & \longrightarrow & \mathbb{R} \\ (r, p, X) & \longrightarrow & F(p, X) - f(r), \end{array}$$

is not *proper* (in the sense of [16]) since f may fail to be non-increasing in r, and it is consequently out of the scope of the general theory in [16].

Let us point out that a logarithmic change of variables $\tilde{w} = \log(w)$, can be used to transform the equation into

$$F\big(\nabla\tilde{w}, D^2\tilde{w} + \nabla\tilde{w} \otimes \nabla\tilde{w}\big) = e^{-m\,\tilde{w}} \cdot f\big(e^{\tilde{w}}\big) \quad \text{in } \Omega,$$

which is proper, either under the hypothesis

$$\frac{f(t)}{t^q} \quad \text{is strictly decreasing } \forall t > 0 \text{ and some } 0 < q < m,$$

or

$$\frac{f(t)}{t^m} \quad \text{is strictly decreasing } \forall t > 0,$$

the precise assumption in [9].

Then it is possible to show comparison using standard arguments (see [16], [21], and [23]). However, strict positivity of the supersolution is needed in the whole $\overline{\Omega}$ in order to carry out the proof. Hence the above argument is not sufficient to conclude uniqueness of nontrivial solutions of (2.3).

Thus, we instead follow the change of variables and subsequent ideas in [14] to prove comparison up to the boundary even under zero boundary data.

Lemma 2.3. *Let $w > 0$ be a supersolution (subsolution) of problem (2.1) in Ω and consider some $q < m$ for which (2.2) holds. Then,*

$$\tilde{w}(x) = \frac{1}{1 - \frac{q}{m}} \cdot w^{1-\frac{q}{m}}(x)$$

is a viscosity supersolution (subsolution) of

$$F\Big(\nabla\tilde{w}, D^2\tilde{w} + \frac{q}{m-q}\frac{\nabla\tilde{w} \otimes \nabla\tilde{w}}{\tilde{w}}\Big) = \frac{f\left(\left[\left(1 - \frac{q}{m}\right)\tilde{w}(x)\right]^{\frac{1}{1-\frac{q}{m}}}\right)}{\left[\left(1 - \frac{q}{m}\right)\tilde{w}(x)\right]^{\frac{q}{1-\frac{q}{m}}}}, \tag{2.4}$$

in every Ω^* such that $\overline{\Omega^*} \subset \Omega$.

Proof. Let us denote $\tilde{q} = q/m$ for simplicity. Consider $\tilde{\phi} \in \mathcal{C}^2(\Omega)$, a function touching $\tilde{w}(x)$ from below at $x_0 \in \Omega$ and define $\phi(x) = \Big[(1 - \tilde{q})\,\tilde{\phi}(x)\Big]^{\frac{1}{1-\tilde{q}}}$, which touches $w(x)$ from below at x_0. Notice that $\phi(x)$ is $\mathcal{C}^2$ in a neighborhood of x_0 since $w > 0$

in Ω implies $\tilde\phi(x)>0$ near x_0. We can compute the derivatives of $\phi(x)$ in terms of those of $\tilde\phi(x)$

$$\nabla\phi(x_0)=\Big[(1-\tilde q)\,\tilde\phi(x_0)\Big]^{\frac{\tilde q}{1-\tilde q}}\,\nabla\tilde\phi(x_0),$$
$$D^2\phi(x_0)=\Big[(1-\tilde q)\,\tilde\phi(x_0)\Big]^{\frac{\tilde q}{1-\tilde q}}\Big(D^2\tilde\phi(x_0)+\frac{\tilde q}{1-\tilde q}\,\frac{\nabla\tilde\phi(x_0)\otimes\nabla\tilde\phi(x_0)}{\tilde\phi(x_0)}\Big).$$

As w is a viscosity solution of (2.1), we have that

$$\begin{aligned}f\Big(\big[(1-\tilde q)\tilde w(x_0)\big]^{\frac{1}{1-\tilde q}}\Big)&=f\big(w(x_0)\big)\le F\big(\nabla\phi(x_0),D^2\phi(x_0)\big)\\&=\big[(1-\tilde q)\,\tilde\phi(x_0)\big]^{\frac{\tilde q m}{1-\tilde q}}\cdot F\Big(\nabla\tilde\phi(x_0),D^2\tilde\phi(x_0)+\frac{\tilde q}{1-\tilde q}\,\frac{\nabla\tilde\phi(x_0)\otimes\nabla\tilde\phi(x_0)}{\tilde\phi(x_0)}\Big),\end{aligned}$$

by homogeneity. Since $\tilde w(x_0)=\tilde\phi(x_0)$, we conclude that $\tilde u$ is a viscosity supersolution of (2.4). The subsolution case is analogous. □

The function giving rise to equation (2.4), namely,

$$\begin{aligned}\tilde G:\mathbb{R}^+\times\mathbb{R}^n\times\mathcal{S}^n\quad&\longrightarrow\quad\mathbb{R}\\(r,p,X)\quad&\longrightarrow\quad F\left(p,X+\frac{q}{m-q}\frac{p\otimes p}{r}\right)-\frac{f\left(\Big[\big(1-\frac{q}{m}\big)\,r\Big]^{\frac{1}{1-\frac{q}{m}}}\right)}{\Big[\big(1-\frac{q}{m}\big)\,r\Big]^{\frac{q}{1-\frac{q}{m}}}},\end{aligned}$$

is both degenerate elliptic and proper. In the following lemma, we show that it is possible to construct strict supersolutions starting from any positive supersolution.

Lemma 2.4. *Let $\tilde v(x)>0$ be a viscosity supersolution of (2.4) in Ω^* such that $\overline{\Omega^*}\subset\Omega$. Then, for any $\epsilon>0$,*

$$\tilde v_\epsilon(x)=(1+\epsilon)\cdot\big(\tilde v(x)+\epsilon\big),\tag{2.5}$$

is a strict supersolution of the same equation; indeed,

$$F\Big(\nabla\tilde v_\epsilon,D^2\tilde v_\epsilon+\frac{q}{m-q}\frac{\nabla\tilde v_\epsilon\otimes\nabla\tilde v_\epsilon}{\tilde v_\epsilon}\Big)\ge(1+\epsilon)^m\cdot\frac{f\Big(\Big[\big(1-\frac{q}{m}\big)\,\tilde v_\epsilon(x)\Big]^{\frac{1}{1-\frac{q}{m}}}\Big)}{\Big[\big(1-\frac{q}{m}\big)\,\tilde v_\epsilon(x)\Big]^{\frac{q}{1-\frac{q}{m}}}}.\tag{2.6}$$

Moreover, $\tilde v_\epsilon\to v$ uniformly in $\overline{\Omega^*}$ as $\epsilon\to0$.

Proof. Let $\phi\in\mathcal{C}^2$ be a function touching $\tilde v_\epsilon(x)$ from below in some $x_0\in\Omega^*$. We define

$$\Phi(x)=\frac{1}{1+\epsilon}\,\phi(x)-\epsilon,$$

which clearly touches $\tilde v(x)$ from below in x_0. We can compute the derivatives of $\Phi(x)$ in terms of those of $\phi(x)$, this is

$$\nabla\Phi(x_0)=(1+\epsilon)^{-1}\,\nabla\phi(x_0)\quad\text{and}\quad D^2\Phi(x_0)=(1+\epsilon)^{-1}\,D^2\phi(x_0).\tag{2.7}$$

Since $\tilde v(x)$ is a viscosity supersolution of (2.4) in Ω^*, we deduce

$$\begin{aligned}\frac{f\Big(\Big[\big(1-\frac{q}{m}\big)\,\tilde v(x_0)\Big]^{\frac{1}{1-\frac{q}{m}}}\Big)}{\Big[\big(1-\frac{q}{m}\big)\,\tilde v(x_0)\Big]^{\frac{q}{1-\frac{q}{m}}}}&\le F\Big(\nabla\Phi(x_0),D^2\Phi(x_0)+\frac{q}{m-q}\,\frac{\nabla\Phi(x_0)\otimes\nabla\Phi(x_0)}{\tilde v(x_0)}\Big)\\&=\frac{1}{(1+\epsilon)^m}\,F\Big(\nabla\phi(x_0),D^2\phi(x_0)+\frac{q}{m-q}\,\frac{\nabla\phi(x_0)\otimes\nabla\phi(x_0)}{\big(\tilde v_\epsilon(x_0)-\epsilon(1+\epsilon)\big)}\Big).\end{aligned}$$

Notice that our hypothesis (2.2) implies

$$\frac{f\Big(\Big[\left(1-\frac{q}{m}\right)\tilde{v}(x_0)\Big]^{\frac{1}{1-\frac{q}{m}}}\Big)}{\Big[\left(1-\frac{q}{m}\right)\tilde{v}(x_0)\Big]^{\frac{q}{1-\frac{q}{m}}}} \geq \frac{f\Big(\Big[\left(1-\frac{q}{m}\right)\tilde{v}_\epsilon(x_0)\Big]^{\frac{1}{1-\frac{q}{m}}}\Big)}{\Big[\left(1-\frac{q}{m}\right)\tilde{v}_\epsilon(x_0)\Big]^{\frac{q}{1-\frac{q}{m}}}}.$$

Since

$$D^2\phi(x_0)+\frac{q}{m-q}\frac{\nabla\phi(x_0)\otimes\nabla\phi(x_0)}{\big(\tilde{v}_\epsilon(x_0)-\epsilon(1+\epsilon)\big)} \geq D^2\phi(x_0)+\frac{q}{m-q}\frac{\nabla\phi(x_0)\otimes\nabla\phi(x_0)}{\tilde{v}_\epsilon(x_0)}$$

in the matrix sense, we get (2.6) by degenerate ellipticity. For the second statement, notice that

$$\|\tilde{v}_\epsilon - v\|_{L^\infty(\overline{\Omega^*})} \leq \epsilon\,\|\tilde{v}\|_{L^\infty(\Omega)} + \epsilon\,(1+\epsilon). \qquad \square$$

Proof of Theorem 2.1. Since $u-v\in\mathcal{C}(\overline{\Omega})$ and $\overline{\Omega}$ is compact, $u-v$ attains a maximum at $\overline{\Omega}$. In order to arrive at a contradiction, we suppose that $\max_{\overline{\Omega}}(u-v)>0$. Consider

$$\tilde{u}(x)=\frac{u(x)^{1-\frac{q}{m}}}{1-\frac{q}{m}} \qquad \text{and} \qquad \tilde{v}(x)=\frac{v(x)^{1-\frac{q}{m}}}{1-\frac{q}{m}}, \tag{2.8}$$

and define

$$\tilde{v}_\epsilon(x)=(1+\epsilon)\cdot\big(\tilde{v}(x)+\epsilon\big). \tag{2.9}$$

Notice that, since $(u-v)|_{\partial\Omega}\leq 0$, we have

$$\tilde{u}-\tilde{v}_\epsilon=\tilde{u}-(1+\epsilon)\,\tilde{v}-(1+\epsilon)\,\epsilon<0 \qquad \text{on } \partial\Omega.$$

Furthermore, by uniform convergence, we have $\max_{\overline{\Omega}}(\tilde{u}-\tilde{v}_\epsilon)>0$ for ϵ small enough, and hence, we can fix $\epsilon>0$ and suppose that there exists Ω^* with $\overline{\Omega^*}\subset\Omega$ containing all the maximum points of $\tilde{u}-\tilde{v}_\epsilon$. We have proved in Lemmas 2.3 and 2.4 that $\tilde{u}$ and $\tilde{v}_\epsilon$ are respectively a subsolution and a strict supersolution of (2.4) in Ω^*.

Now, for each $\tau>0$, let (x_τ,y_τ) be a maximum point of $\tilde{u}(x)-\tilde{v}_\epsilon(y)-\frac{\tau}{2}|x-y|^2$ in $\overline{\Omega}\times\overline{\Omega}$. By the compactness of $\overline{\Omega}$, we can suppose that $x_\tau\to\hat{x}$ as $\tau\to\infty$ for some $\hat{x}\in\overline{\Omega}$ (notice that also $y_\tau\to\hat{x}$). Proposition 3.7 in [16] implies that $\hat{x}$ is a maximum point of $\tilde{u}-\tilde{v}_\epsilon$ and, consequently, it is an interior point of Ω^*. We also have

$$\lim_{\tau\to\infty}\Big(\tilde{u}(x_\tau)-\tilde{v}_\epsilon(y_\tau)-\frac{\tau}{2}|x_\tau-y_\tau|^2\Big)=\tilde{u}(\hat{x})-\tilde{v}_\epsilon(\hat{x})>0,$$

and then, for τ large enough, both x_τ and y_τ are interior points of Ω^* and

$$\tilde{u}(x_\tau)-\tilde{v}_\epsilon(y_\tau)-\frac{\tau}{2}|x_\tau-y_\tau|^2>0. \tag{2.10}$$

Applying the Maximum Principle for semicontinuous functions (see for instance, [15] and [16]), there exist two symmetric matrices X_τ, Y_τ such that

$$\big(\tau(x_\tau-y_\tau),X_\tau\big)\in\overline{J}^{2,+}\tilde{u}(x_\tau), \quad \text{and} \quad \big(\tau(x_\tau-y_\tau),Y_\tau\big)\in\overline{J}^{2,-}\tilde{v}_\epsilon(y_\tau),$$

and

$$\langle X_\tau\xi,\xi\rangle-\langle Y_\tau\eta,\eta\rangle\leq 3\tau\,|\xi-\eta|^2 \quad \forall\xi,\eta\in\mathbb{R}^n. \tag{2.11}$$

Where, following [16],

$$\begin{aligned} J^{2,+}w(\hat{x}) = \big\{(p,X)\in\mathbb{R}^n\times S^n : \phi(x)&=w(\hat{x})+\langle p,(x-\hat{x})\rangle \\ &+\frac{1}{2}\big\langle X(x-\hat{x}),(x-\hat{x})\big\rangle \text{ touches } w \text{ from above in } \hat{x}\big\}, \\ J^{2,-}w(\hat{x}) = \big\{(p,X)\in\mathbb{R}^n\times S^n : \psi(x)&=w(\hat{x})+\langle p,(x-\hat{x})\rangle \\ &+\frac{1}{2}\big\langle X(x-\hat{x}),(x-\hat{x})\big\rangle \text{ touches } w \text{ from below in } \hat{x}\big\}, \end{aligned}$$

and their closures,

$$\begin{aligned}\overline{J}^{2,+}w(\hat{x}) = \big\{(p,X)\in\mathbb{R}^n\times S^n : \exists\, x_n\in B_r(\hat{x}),\ (p_n,X_n)\in J^{2,+}w(x_n)\\ \text{s.t. } (x_n,p_n,X_n)\to(\hat{x},p,X) \text{ as } n\to\infty\big\},\\ \overline{J}^{2,-}w(\hat{x}) = \big\{(p,X)\in\mathbb{R}^n\times S^n : \exists\, x_n\in B_r(\hat{x}),\ (p_n,X_n)\in J^{2,-}w(x_n)\\ \text{s.t. } (x_n,p_n,X_n)\to(\hat{x},p,X) \text{ as } n\to\infty\big\}.\end{aligned}$$

By definition of viscosity sub- and supersolution (see [16]), we have

$$\begin{aligned}&F\left(\tau(x_\tau-y_\tau), X_\tau+\frac{q}{m-q}\frac{\tau^2(x_\tau-y_\tau)\otimes(x_\tau-y_\tau)}{\tilde{u}(x_\tau)}\right)\\ &\qquad\le \frac{f\left(\left[\left(1-\frac{q}{m}\right)\tilde{u}(x_\tau)\right]^{\frac{1}{1-\frac{q}{m}}}\right)}{\left[\left(1-\frac{q}{m}\right)\tilde{u}(x_\tau)\right]^{\frac{q}{1-\frac{q}{m}}}},\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}&F\left(\tau(x_\tau-y_\tau), Y_\tau+\frac{q}{m-q}\frac{\tau^2(x_\tau-y_\tau)\otimes(x_\tau-y_\tau)}{\tilde{v}_\epsilon(y_\tau)}\right)\\ &\qquad\ge (1+\epsilon)^m\cdot\frac{f\left(\left[\left(1-\frac{q}{m}\right)\tilde{v}_\epsilon(y_\tau)\right]^{\frac{1}{1-\frac{q}{m}}}\right)}{\left[\left(1-\frac{q}{m}\right)\tilde{v}_\epsilon(y_\tau)\right]^{\frac{q}{1-\frac{q}{m}}}}.\end{aligned}\tag{2.13}$$

Since $\tilde{v}_\epsilon(y_\tau)<\tilde{u}(x_\tau)$ and $X_\tau\le Y_\tau$, from (2.10) and (2.11) respectively, we have

$$X_\tau+\frac{q}{m-q}\frac{\tau^2(x_\tau-y_\tau)\otimes(x_\tau-y_\tau)}{\tilde{u}(x_\tau)}\le Y_\tau+\frac{q}{m-q}\frac{\tau^2(x_\tau-y_\tau)\otimes(x_\tau-y_\tau)}{\tilde{v}_\epsilon(y_\tau)}$$

in the sense of matrices. Subtracting (2.12) from (2.13), by hypothesis (2.2) and degenerate ellipticity, we have

$$\begin{aligned}0&<\left[(1+\epsilon)^m-1\right]\cdot\frac{f\left(\left[\left(1-\frac{q}{m}\right)\tilde{v}_\epsilon(y_\tau)\right]^{\frac{1}{1-\frac{q}{m}}}\right)}{\left[\left(1-\frac{q}{m}\right)\tilde{v}_\epsilon(y_\tau)\right]^{\frac{q}{1-\frac{q}{m}}}}\\ &\le(1+\epsilon)^m\cdot\frac{f\left(\left[\left(1-\frac{q}{m}\right)\tilde{v}_\epsilon(y_\tau)\right]^{\frac{1}{1-\frac{q}{m}}}\right)}{\left[\left(1-\frac{q}{m}\right)\tilde{v}_\epsilon(y_\tau)\right]^{\frac{q}{1-\frac{q}{m}}}}-\frac{f\left(\left[\left(1-\frac{q}{m}\right)\tilde{u}(x_\tau)\right]^{\frac{1}{1-\frac{q}{m}}}\right)}{\left[\left(1-\frac{q}{m}\right)\tilde{u}(x_\tau)\right]^{\frac{q}{1-\frac{q}{m}}}}\\ &\le F\left(\tau(x_\tau-y_\tau), Y_\tau+\frac{q}{m-q}\frac{\tau^2(x_\tau-y_\tau)\otimes(x_\tau-y_\tau)}{\tilde{v}_\epsilon(y_\tau)}\right)\\ &\quad-F\left(\tau(x_\tau-y_\tau), X_\tau+\frac{q}{m-q}\frac{\tau^2(x_\tau-y_\tau)\otimes(x_\tau-y_\tau)}{\tilde{u}(x_\tau)}\right)\le 0,\end{aligned}$$

a contradiction. □

We also provide the following simple result, which turns out to be very useful when proving existence of solutions.

Theorem 2.5. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain and consider $f\in\mathcal{C}(\overline{\Omega})$ with $f>0$ in Ω, and $F:\mathbb{R}^n\times S^n\to\mathbb{R}$ satisfying (F1) and (F2) for some m. Let $u,v\in\mathcal{C}(\overline{\Omega})$, respectively a viscosity sub- and supersolution of*

$$F(\nabla w, D^2w)=f(x)\quad\text{in } \Omega.$$

Assume $u\le v$ on $\partial\Omega$, then $u\le v$ in $\overline{\Omega}$.

Proof. Without loss of generality, we can assume $v \geq 0$ in $\overline{\Omega}$, since adding a constant to both u and v does not affect the problem. Now, let $v_\epsilon(x) = (1+\epsilon)\, v(x)$. By homogeneity, v_ϵ is a strict supersolution, indeed,

$$F(\nabla v_\epsilon, D^2 v_\epsilon) \geq (1+\epsilon)^m f(x) > f(x) \quad \text{and } u - v_\epsilon \leq 0 \text{ on } \partial\Omega. \tag{2.14}$$

Now, we argue by contradiction. Suppose that there exists $x_0 \in \overline{\Omega}$ such that

$$(u - v_\epsilon)(x_0) = \max_{\overline{\Omega}} (u - v_\epsilon) > 0.$$

Then, (2.14) implies $x_0 \notin \partial\Omega$. As in the proof of Theorem 2.1, one finds $x_\tau, y_\tau \to x_0$ and symmetric matrices X_τ, Y_τ such that

$$\big(\tau(x_\tau - y_\tau), X_\tau\big) \in \overline{J}^{2,+} u(x_\tau) \quad \text{and} \quad \big(\tau(x_\tau - y_\tau), Y_\tau\big) \in \overline{J}^{2,-} v_\epsilon(y_\tau),$$

with $X_\tau \leq Y_\tau$. Then, by definition of viscosity sub- and supersolution, we have

$$F\big(\tau(x_\tau - y_\tau), X_\tau\big) \leq f(x_\tau),$$

and

$$F\big(\tau(x_\tau - y_\tau), Y_\tau\big) \geq (1+\epsilon)^m f(y_\tau).$$

Then, by degenerate ellipticity, we have

$$(1+\epsilon)^m f(y_\tau) - f(x_\tau) \leq F\big(\tau(x_\tau - y_\tau), Y_\tau\big) - F\big(\tau(x_\tau - y_\tau), X_\tau\big) \leq 0.$$

Letting $\tau \to \infty$, we have, by continuity,

$$0 < \big((1+\epsilon)^m - 1\big)\, f(x_0) \leq 0,$$

a contradiction. Thus, $u \leq v_\epsilon$ in Ω, and, letting $\epsilon \to 0$, we get $u \leq v$ in Ω. □

3. Existence of solutions to uniformly elliptic equations

For simplicity, let us assume that Ω is a smooth domain. In some results we will indicate a relaxation of such regularity hypothesis without looking for the optimal hypotheses.

Assume that $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ satisfies $(F2)$ for $m = 1$ and upgrade $(F1)$ to the following uniform ellipticity condition:

$(F1')$ There exist constants $0 < \theta \leq \Theta$ such that for all $X, Y \in S^n$ with $Y \geq 0$,

$$-\Theta\, \mathrm{trace}(Y) \leq F(p, X+Y) - F(p, X) \leq -\theta\, \mathrm{trace}(Y)$$

for every $p \in \mathbb{R}^n$.

Notice that the uniform ellipticity forces $m = 1$.

We further require the following structure condition:

$(F3)$ There exists $\gamma > 0$ such that, for all $X, Y \in S^n$, and $p, q \in \mathbb{R}^n$,

$$\mathcal{P}^-_{\theta,\Theta}(X - Y) - \gamma\, |p - q| \leq F(p, X) - F(q, Y) \leq \mathcal{P}^+_{\theta,\Theta}(X - Y) + \gamma\, |p - q|,$$

where $\mathcal{P}^\pm_{\theta,\Theta}$ are the extremal Pucci's operators defined as

$$\mathcal{P}^+_{\theta,\Theta}(M) = -\theta \sum_{\lambda_i > 0} \lambda_i(M) - \Theta \sum_{\lambda_i < 0} \lambda_i(M),$$
$$\mathcal{P}^-_{\theta,\Theta}(M) = -\Theta \sum_{\lambda_i > 0} \lambda_i(M) - \theta \sum_{\lambda_i < 0} \lambda_i(M),$$

with $\lambda_i(M)$, $i = 1, \ldots n$, the eigenvalues of M. Indeed,

$$\mathcal{P}^-_{\theta,\Theta}(M) = \inf_{A \in \mathcal{A}_{\theta,\Theta}} \{-\mathrm{trace}\,(AM)\}, \qquad \mathcal{P}^+_{\theta,\Theta}(M) = \sup_{A \in \mathcal{A}_{\theta,\Theta}} \{-\mathrm{trace}\,(AM)\}$$

for

$$\mathcal{A}_{\theta,\Theta} = \big\{A \in S^n :\ \theta|\xi|^2 \leq \langle A\xi, \xi\rangle \leq \Theta|\xi|^2 \quad \forall \xi \in \mathbb{R}^n\big\}.$$

We will quote several results from [11] along this work. Notice that in [11], Pucci's operators are defined with a different sign convention. Both definitions are related through the following expressions

$$\mathcal{M}^-(M,\theta,\Theta) = -\mathcal{P}^+_{\theta,\Theta}(M), \qquad \mathcal{M}^+(M,\theta,\Theta) = -\mathcal{P}^-_{\theta,\Theta}(M),$$

where $\mathcal{M}^\pm(M,\theta,\Theta)$ is the notation used in [11].

Remark 3.1. As it is pointed out in [13], the structure condition (F3) when $p = q$ is nothing but uniform ellipticity.

We state the existence and uniqueness of nontrivial solutions for every $\lambda > 0$ of

$$\begin{cases} F(\nabla u, D^2 u) = \lambda\, u^q & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < q < 1$.

Then, we consider $0 < q < 1 < r$ and the problem

$$\begin{cases} F(\nabla u, D^2 u) = \lambda\, u^q + u^r & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We prove the existence of a value $\Lambda > 0$ such that there exists a nontrivial solution for $\lambda \in (0,\Lambda)$, and any nontrivial solution otherwise.

The mentioned results rely on a monotone iteration method and the analysis of simpler auxiliary problems for which showing the existence of solutions is easier. We point out that our results can be extended to m-homogeneous degenerate elliptic equations, merely satisfying $(F1)$ and $(F2)$. We will give examples of this issue in the next section.

3.1. **Existence of solutions for the problem with a concave power.**

Assume that $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ satisfies the above hypotheses. We are interested in studying the existence of nontrivial solutions to

$$\begin{cases} F(\nabla u_\lambda, D^2 u_\lambda) = \lambda\, u_\lambda^q & \text{in } \Omega,\\ u_\lambda > 0 & \text{in } \Omega,\\ u_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $0 < q < 1$ and $\lambda > 0$. The following is our main result in this section.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ satisfies $(F1')$, $(F2)$ for $m = 1$ and $(F3)$ with $0 < q < 1$. Then, there exists a unique solution to* (3.1) *for every $\lambda > 0$ given by*

$$u_\lambda(x) = \lambda^{\frac{1}{1-q}}\, u_1(x), \tag{3.2}$$

where u_1 is the solution with $\lambda = 1$.

Notice that (3.2) is a smooth curve of solutions with respect to λ. By homogeneity, it is enough to prove the result for $\lambda = 1$, namely,

$$\begin{cases} F(\nabla u_1, D^2 u_1) = u_1^q & \text{in } \Omega,\\ u_1 > 0 & \text{in } \Omega,\\ u_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

In order to obtain positive sub and supersolutions to problem (3.3), we introduce the following auxiliary problems,

$$
\begin{cases} F(\nabla v, D^2 v) = 1 \quad \text{in } \Omega, \\ v > 0 \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} F(\nabla w, D^2 w) = d(x) \quad \text{in } \Omega, \\ w > 0 \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.4}
$$

where $d(x)$ is the normalized distance to the boundary, namely,

$$
d(x) = \frac{\operatorname{dist}(x, \partial\Omega)}{\|\operatorname{dist}(\cdot, \partial\Omega)\|_\infty}.
$$

Our construction rely on the following fact.

Proposition 3.3. *Both auxiliary problems in* (3.4) *have a unique viscosity solution.*

Proof. We show the existence of w, since the proof for v is similar. First, notice that 0 is a subsolution to our problem. Next, we are going to construct a supersolution following Proposition 3.2 in [17]. Let U be the unique viscosity solution to

$$
\begin{cases} \mathcal{P}^-_{\theta,\Theta}(D^2 U) - \gamma\,|\nabla U| = 1 \quad \text{in } \Omega, \\ U = 0 \quad \text{on } \partial\Omega. \end{cases}
$$

We point out that here it is enough to have an exterior cone condition on Ω. The structure condition $(F3)$ implies

$$
F(\nabla U, D^2 U) \geq \mathcal{P}^-_{\theta,\Theta}(D^2 U) - \gamma\,|\nabla U| = 1 \geq d(x)
$$

in the viscosity sense. Hence $0 \leq w \leq U$ by comparison, (Theorem 2.5) and, we can invoke the Perron method (see for instance, Theorem 4.1 in [16]) to get that there exists a unique w such that

$$
\begin{cases} F(\nabla w, D^2 w) = d(x) \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega. \end{cases}
$$

For the strict positivity of w, notice that $w \geq 0$ by construction and $w \neq 0$ (we get a contradiction with the equation otherwise). Since

$$
\mathcal{P}^+_{\theta,\Theta}(D^2 w) + \gamma\,|\nabla w| \geq F(\nabla w, D^2 w) = d(x) > 0
$$

in the viscosity sense, the weak Harnack inequality (an adaptation of Theorem 4.8 in [11] using the ABP estimate in Proposition 2.12 in [13]) implies $w > 0$ in Ω. $\square$

Next, we use v, w, the solutions to the auxiliary problems, to construct a subsolution and a supersolution to (3.3).

Lemma 3.4. $\overline{u}(x) = \|v\|_\infty^{\frac{q}{1-q}} \cdot v(x)$ *is a viscosity supersolution of* (3.3).

Proof. We denote $T = \|v\|_\infty^{\frac{q}{1-q}}$ for brevity. By homogeneity and the definition of v, we have

$$
F(\nabla\overline{u}, D^2\overline{u}) = T \cdot F(\nabla v, D^2 v) = T,
$$

in the viscosity sense while, by definition of T, we have $\overline{u}^q \leq T^q \|v\|_\infty^q = T$. $\square$

Now, we are going to construct a subsolution to (3.3).

Lemma 3.5. *There exists $\delta > 0$ small enough, such that*

$$
\underline{u}(x) = t \cdot w(x)
$$

is a viscosity subsolution of (3.3) *for every $t \in \big(0, \delta\big)$.*

Proof. By definition, we have

$$F(\nabla \underline{u}, D^2\underline{u}) = t \cdot F(\nabla w, D^2 w) = t\, d(x).$$

Indeed, $t\, d(x) \leq \underline{u}^q(x)$ since this is equivalent to $t^{1-q}\, d(x) \leq w^q(x)$, which holds for all $t < \delta$ small enough by the Hopf boundary lemma (Proposition A.1 in the Appendix below). $\square$

At this stage, the comparison principle (Theorem 2.1), and Lemmas 3.4 and 3.5 allow us to invoke the Perron method (Theorem 4.1 in [16]), which completes the proof of Theorem 3.2.

Remark 3.6. Theorem 3.2 remains true for domains for which Hopf's Lemma holds and it is possible to construct a barrier in each point of the boundary in problems (3.4) (for instance if Ω satisfies both an interior sphere and an uniform exterior cone condition).

3.2. **Study of a concave-convex problem.** Consider $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ as in the previous section, namely, satisfying $(F1'), (F2)$ and the structure condition $(F3)$. Now, our goal is to study the existence and non-existence of viscosity solutions of the problem,

$$\begin{cases} F(\nabla u, D^2 u) = \lambda\, u^q + u^r & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where $0 < q < 1 < r$ and $\lambda > 0$.

3.2.1. *Existence of solutions for small* λ. The present section is devoted to the proof of the following result, which extends the arguments in [7] to the viscosity framework.

Theorem 3.7. *Consider a smooth bounded domain $\Omega \subset \mathbb{R}^n$ and let $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ satisfy $(F1'), (F2)$ with $m = 1$, and $(F3)$. Then, there exists a constant $\lambda_0 > 0$ such that, for every $\lambda \in (0, \lambda_0]$, problem* (3.5) *has at least one nontrivial viscosity solution.*

Since the scaling property (3.2) is not available for problem (3.5), let us consider for every $\lambda > 0$ the following variant of the auxiliary problems in Section 3

$$\begin{cases} F(\nabla v_\lambda, D^2 v_\lambda) = \lambda & \text{in } \Omega, \\ v_\lambda > 0 & \text{in } \Omega, \\ v_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} F(\nabla w_\lambda, D^2 w_\lambda) = \lambda\, d(x) & \text{in } \Omega, \\ w_\lambda > 0 & \text{in } \Omega, \\ w_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where $d(x)$ is the normalized distance to the boundary, namely,

$$d(x) = \frac{\operatorname{dist}(x, \partial\Omega)}{\|\operatorname{dist}(\cdot, \partial\Omega)\|_\infty}.$$

Notice that, by homogeneity,

$$v_\lambda(x) = \lambda \cdot v_1(x), \tag{3.7}$$

where v_1 is the solution to (3.4). A similar relation holds for w_λ. Then, Proposition 3.3 gives existence of a unique viscosity solution to both auxiliary problems in (3.6).

Now, we are able to construct a supersolution to (3.5).

Lemma 3.8. *There exists $\lambda_0 > 0$ such that $\forall \lambda \in (0, \lambda_0]$ we can find a constant $T(\lambda)$ such that*

$$\overline{u}_\lambda(x) = T(\lambda) \cdot v_\lambda(x)$$

is a viscosity supersolution of (3.5)*. In fact,*

$$\lambda_0 = (r-1) \left(\frac{(1-q)^{1-q}}{\big[\|v_1\|_\infty (r-q)\big]^{r-q}} \right)^{\frac{1}{r-1}}, \tag{3.8}$$

and

$$T(\lambda) = \frac{1}{\|v_1\|_\infty} \left(\frac{1-q}{r-1} \right)^{\frac{1}{r-q}} \lambda^{\frac{q-r+1}{(r-q)}}.$$

Proof. On the one hand, by homogeneity and the definition of v_λ, we have

$$F(\nabla \overline{u}_\lambda, D^2 \overline{u}_\lambda) = T(\lambda)\, F(\nabla v_\lambda, D^2 v_\lambda) = T(\lambda)\, \lambda.$$

On the other hand, using (3.7), one has

$$\lambda \big(\overline{u}_\lambda\big)^q + \big(\overline{u}_\lambda\big)^r \le \lambda^{1+q} T(\lambda)^q \|v_1\|_\infty^q + \lambda^r T(\lambda)^r \|v_1\|_\infty^r.$$

Thus, we will be done whenever

$$\lambda^{1+q} T(\lambda)^q \|v_1\|_\infty^q + \lambda^r T(\lambda)^r \|v_1\|_\infty^r \le T(\lambda)\, \lambda. \tag{3.9}$$

To prove (3.9) is equivalent to demonstrate that

$$\Phi_\lambda(T) \le 1, \quad \text{with} \quad \Phi_\lambda(T) = c^q \lambda^q T^{q-1} + c^r \lambda^{r-1} T^{r-1},$$

where $c = \|v_1\|_\infty$ for brevity. Indeed, we are going to show that, for λ small enough, the minimum of Φ_λ is smaller than 1. It is easy to check that

$$\frac{d}{dT} \Phi_\lambda(T) = 0 \quad \Leftrightarrow \quad T(\lambda) = \frac{1}{c} \left(\frac{1-q}{r-1} \right)^{\frac{1}{r-q}} \lambda^{\frac{q-r+1}{(r-q)}},$$

which, indeed, is a minimum of Φ_λ. Since we want

$$\Phi_\lambda(T(\lambda)) = c\, \lambda^{\frac{r-1}{r-q}} \left(\frac{r-q}{r-1} \right) \left(\frac{1-q}{r-1} \right)^{\frac{q-1}{r-q}} \le 1,$$

we get, after a simple calculation, that

$$\lambda \le \lambda_0,$$

where λ_0 is given by (3.8). □

Now, we are going to construct a subsolution of (3.5).

Lemma 3.9. *Let $0 < q < 1 < r$. Then, for every $\lambda > 0$, there exists $\delta(\lambda) > 0$ small enough such that*

$$\underline{u}_\lambda(x) = t\, w_\lambda(x)$$

is a viscosity subsolution of (3.5) *for every $t \in \big(0, \delta(\lambda)\big)$.*

Proof. By definition, we have

$$F(\nabla \underline{u}_\lambda, D^2 \underline{u}_\lambda) = t \cdot F(\nabla w_\lambda, D^2 w_\lambda) = t\, \lambda\, d(x)$$

where

$$d(x) = \frac{\operatorname{dist}(x, \partial\Omega)}{\|\operatorname{dist}(\cdot, \partial\Omega)\|_\infty}.$$

If it is true that

$$t\, \lambda\, d(x) \le \lambda \underline{u}_\lambda^q + \underline{u}_\lambda^r,$$

we are done. This is equivalent to

$$t^{1-q}\, d(x) \le w_\lambda^q + \frac{1}{\lambda} t^{r-q} w_\lambda^r.$$

Hopf's lemma (Proposition A.1) implies that, fixed $\lambda > 0$, we can find $\delta(\lambda)$ small enough such that the above is true for all $t < \delta$. $\square$

Finally, we have the following ordering result.

Lemma 3.10. *For all $\lambda \in (0, \lambda_0]$, and $\underline{u}_\lambda$, $\overline{u}_\lambda$ as in Lemmas 3.8 and 3.9, there is a small enough t as in Lemma 3.9 such that*

$$\underline{u}_\lambda \leq \overline{u}_\lambda \quad \text{in } \Omega.$$

Proof. On the boundary $\underline{u}_\lambda = \overline{u}_\lambda = 0$, while

$$F(\nabla \underline{u}_\lambda, D^2 \underline{u}_\lambda) = t\, \lambda\, d(x) \leq T(\lambda)\, \lambda \leq F(\nabla \overline{u}_\lambda, D^2 \overline{u}_\lambda) \quad \text{in } \Omega,$$

by definition of $\underline{u}_\lambda$, $\overline{u}_\lambda$. Then, Theorem 2.5 gives the desired result. $\square$

Now, we can finish the proof of the main Theorem.

Proof of Theorem 3.7. Let λ_0 be given by Lemma 3.8, and fix $\lambda \in (0, \lambda_0]$. Applying Lemmas 3.8, 3.9, and 3.10, we get that there exist constants $T(\lambda)$ and $t \in \big(0, \delta(\lambda)\big)$ such that $\overline{u}_\lambda(x) = T(\lambda)\, v_\lambda(x)$ and $\underline{u}_\lambda(x) = t \cdot w_\lambda(x)$ are, respectively, a viscosity super- and subsolution and moreover $\underline{u}_\lambda \leq \overline{u}_\lambda$ in Ω. For simplicity, we will drop the λ and write $\underline{u}$ and $\overline{u}$ hereafter. Now, define $w_1(x)$, the solution to

$$\begin{cases} F(\nabla w_1, D^2 w_1) = \lambda\, \overline{u}^q + \overline{u}^r & \text{in } \Omega, \\ w_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

in the viscosity sense. For brevity, all the equations and inequalities in the sequel will stand in the viscosity sense without any further reference. Hence, by definition,

$$F(\nabla \overline{u}, D^2 \overline{u}) \geq \lambda\, \overline{u}^q + \overline{u}^r \quad \text{and} \qquad F(\nabla w_1, D^2 w_1) = \lambda\, \overline{u}^q + \overline{u}^r,$$

with $w_1 = \overline{u} = 0$ on $\partial\Omega$. Theorem 2.5 implies $w_1 \leq \overline{u}$ in Ω. Moreover, since $\underline{u} \leq \overline{u}$, we get

$$F(\nabla \underline{u}, D^2 \underline{u}) \leq \lambda\, \underline{u}^q + \underline{u}^r \leq \lambda\, \overline{u}^q + \overline{u}^r \quad \text{and} \qquad F(\nabla w_1, D^2 w_1) = \lambda\, \overline{u}^q + \overline{u}^r,$$

with $\underline{u} = w_1 = 0$ on $\partial\Omega$. Hence, by comparison (Theorem 2.5), we get $\underline{u} \leq w_1$ in Ω and, combining both estimates,

$$\underline{u} \leq w_1 \leq \overline{u} \quad \text{in } \Omega.$$

Next, define w_2 as the solution to

$$\begin{cases} F(\nabla w_2, D^2 w_2) = \lambda\, w_1^q + w_1^r & \text{in } \Omega, \\ w_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Then one has

$$F(\nabla w_2, D^2 w_2) = \lambda\, w_1^q + w_1^r \leq \lambda\, \overline{u}^q + \overline{u}^r \quad \text{and} \qquad F(\nabla w_1, D^2 w_1) = \lambda\, \overline{u}^q + \overline{u}^r,$$

where $w_2 = w_1 = 0$ on $\partial\Omega$. By comparison (Theorem 2.5), $w_2 \leq w_1$ in Ω. On the other hand, $\underline{u} \leq w_1$ implies

$$F(\nabla \underline{u}, D^2 \underline{u}) \leq \lambda\, \underline{u}^q + \underline{u}^r \leq \lambda\, w_1^q + w_1^r \quad \text{and} \qquad F(\nabla w_2, D^2 w_2) = \lambda\, w_1^q + w_1^r,$$

and $\underline{u} = w_2 = 0$ on $\partial\Omega$. Hence, $\underline{u} \leq w_2$ in Ω and then

$$\underline{u} \leq w_2 \leq w_1 \leq \overline{u} \quad \text{in } \Omega.$$

We can iterate the above procedure and construct a sequence $\{w_k\}_{k\geq 1}$ of solutions to

$$\begin{cases} F(\nabla w_k, D^2 w_k) = \lambda\, w_{k-1}^q + w_{k-1}^r & \text{in } \Omega, \\ w_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

such that

$$\underline{u} \leq \ldots \leq w_k \leq w_{k-1} \leq \ldots \leq w_2 \leq w_1 \leq \overline{u} \quad \text{in } \Omega.$$

In particular, for every $x \in \Omega$, the sequence $\{w_k(x)\}_{k\geq 1}$ is bounded and nonincreasing and hence convergent. We denote $u(x)$ this pointwise limit of the w_k.

By the structure condition $(F3)$ and the definition of w_k on the boundary, we have

$$\begin{cases} \mathcal{P}^+_{\theta,\Theta}(D^2 w_k) + \gamma\,|\nabla w_k| \geq \lambda\, w^q_{k-1} + w^r_{k-1}, \\ w_k = 0 \quad \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \mathcal{P}^-_{\theta,\Theta}(D^2 w_k) - \gamma\,|\nabla w_k| \leq \lambda\, w^q_{k-1} + w^r_{k-1}, \\ w_k = 0 \quad \text{on } \partial\Omega. \end{cases}$$

By construction,

$$\|\lambda\, w^q_{k-1} + w^r_{k-1}\|_{L^n(\Omega)} \leq \|\lambda\, \overline{u}^q + \overline{u}^r\|_{L^n(\Omega)}, \qquad \forall k \geq 1.$$

Thus, we can adapt Proposition 4.14 and Remark 6 in [11] using the ABP estimate in [13] and then, deduce that there exists a modulus of continuity ρ^* independent of k such that

$$|w_k(x) - w_k(y)| \leq \rho^*(|x-y|), \qquad \forall x, y \in \overline{\Omega}.$$

Hence, the sequence $\{w_k\}$ is uniformly bounded and equicontinuous and, as a consequence of the Arzela-Ascoli compactness criterion, there exists a subsequence w_{k_j} such that w_{k_j} converges uniformly to u in $\overline{\Omega}$. Since the sequence is monotone, not only the subsequence but the whole sequence $\{w_k\}$ converges to u.

Finally, it is easy to prove that the uniform limit u is a viscosity solution of (3.1). First of all, notice that,

$$u(x) = \lim_{k\to\infty} w_k(x) = 0, \qquad \forall x \in \partial\Omega.$$

Then, consider $\phi \in \mathcal{C}^2$ and $x_0 \in \Omega$ such that $u - \phi$ has a strict local maximum at x_0, this is,

$$(u-\phi)(x) < (u-\phi)(x_0),$$

for all $x \neq x_0$ in a neighborhood of x_0. By uniform convergence, we deduce that $w_k - \phi$ has a local maximum at some x_k, this is,

$$(w_k - \phi)(x) \leq (w_k - \phi)(x_k),$$

for all $x \neq x_k$ near x_k. In addition, $x_k \to x_0$ as $k \to \infty$.

Since v_k is a viscosity solution of (3.10), we have

$$F\big(\nabla\phi(x_k), D^2\phi(x_k)\big) \leq \lambda\, w^q_{k-1}(x_k) + w^r_{k-1}(x_k).$$

Taking limits as $k \to \infty$, we get

$$F\big(\nabla\phi(x_0), D^2\phi(x_0)\big) \leq \lambda\, u^q(x_0) + u^r(x_0).$$

The supersolution case is analogous. □

3.2.2. *Non existence for large* λ. Under the hypotheses of this section, $(F1')$, $(F2)$ and $(F3)$, Theorem 8 in [6] holds. Hence, we know that there exists a principal eigenvalue λ_1 for F defined as

$$\lambda_1 = \sup\{\lambda \,|\, \exists v > 0 \text{ in } \Omega \text{ s.t. } F(\nabla v, D^2 v) \geq \lambda v\}.$$

in the sense that $\lambda_1 < \infty$ and there exists a nontrivial solution (eigenfunction) to

$$\begin{cases} F(\nabla v, D^2 v) = \lambda\, v \quad \text{in } \Omega, \\ v > 0 \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Moreover, by definition of λ_1, we know that for every $\lambda > \lambda_1$, problem (3.11) does not have strictly positive solutions.

Other references for the existence of eigenvalues in the fully nonlinear setting are [30], [10] and the references therein. On the other hand, in [21] the existence of a principal eigenvalue for the 1-homogeneous ∞-laplacian is studied. Notice that the 1-homogeneous ∞-laplacian is outside of the scope of the results in [6].

The existence of such principal eigenvalue and eigenfuntion is needed in the proof of the following result.

Theorem 3.11. *For λ large enough, problem* (3.5) *has no solution in the viscosity sense.*

Proof. Fix $\mu > \lambda_1$ and consider

$$\lambda_0 = \mu^{\frac{r-q}{r-1}}(r-1)\left(\frac{(1-q)^{1-q}}{(r-q)^{r-q}}\right)^{\frac{1}{r-1}}.$$

In order to reach a contradiction, suppose that there exists $\lambda > \lambda_0$ such that the problem

$$\begin{cases} F(\nabla u_\lambda, D^2 u_\lambda) = \lambda\, u_\lambda^q + u_\lambda^r & \text{in } \Omega,\\ u_\lambda > 0 & \text{in } \Omega,\\ u_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.12}$$

has a solution u_λ. Then, we have

$$F(\nabla u_\lambda, D^2 u_\lambda) = \lambda\, u_\lambda^q + u_\lambda^r > \mu u_\lambda \quad \text{in } \Omega, \tag{3.13}$$

in the viscosity sense. In fact, it is enough to demonstrate that

$$\min_{t\in\mathbb{R}^+} \Phi_\lambda(t) > \mu \quad \text{where} \quad \Phi_\lambda(t) = \lambda\, t^{q-1} + t^{r-1}.$$

It is easy to check that

$$\frac{d}{dt}\Phi_\lambda(t) = 0 \quad \Leftrightarrow \quad t_\lambda = \left(\frac{\lambda(1-q)}{(r-1)}\right)^{\frac{1}{r-q}},$$

which, indeed, is a minimum. Since $\Phi_\lambda(t) \to \infty$ both as $t \to 0$ and $t \to \infty$, it is a global minimum. Then,

$$\Phi_\lambda(t_\lambda) = \lambda^{\frac{r-1}{r-q}} \frac{(r-q)\,(1-q)^{\frac{q-1}{r-q}}}{(r-1)^{\frac{r-1}{r-q}}} > \mu$$

by our election of λ. On the other hand, define $\psi = \delta\varphi_1$, where φ_1 is a solution of (3.11). Hopf's Lemma (Proposition A.1) implies that there exists $\delta > 0$ such that $\psi \le u_\lambda$. Then,

$$F(\nabla\psi, D^2\psi) = \lambda_1\,\psi < \mu\, u_\lambda \qquad \text{in } \Omega, \tag{3.14}$$

by definition of μ.

By construction, we have $0 < \psi \le u_\lambda$, where ψ and u_λ satisfy (3.13) and (3.14). Hence, we can apply the iteration method as in the proof of existence to get v, satisfying $\psi \le v \le u$, a viscosity solution of

$$F(\nabla v, D^2 v) = \mu v.$$

Hence, v is a positive solution of (3.11), which is a contradiction with the definition of λ_1. $\square$

Corollary 3.12. *There exists $\Lambda \in \mathbb{R}^+$ with $0 < \Lambda < \infty$ such that* (3.5) *has a positive viscosity solution for every $\lambda \in (0, \Lambda)$.*

Proof. Let us define

$$\Lambda = \sup\big\{\lambda \in \mathbb{R}^+ : \text{ problem (3.5) has a solution}\big\}.$$

Theorem 3.7 implies $\Lambda > 0$, while Theorem 3.11 gives $\Lambda < \infty$. Inded, there exists λ_M near Λ and u_M, such that

$$\begin{cases} F(\nabla u_M, D^2 u_M) = \lambda_M \, u_M^q + u_M^r & \text{in } \Omega, \\ u_M > 0 & \text{in } \Omega, \\ u_M = 0 & \text{on } \partial\Omega. \end{cases}$$

Fix $0 < \mu < \lambda_M$ and take λ_m such that $0 < \lambda_m < \mu$. We have proved that there exists a unique u_m solution to

$$\begin{cases} F(\nabla u_m, D^2 u_m) = \lambda_m \, u_m^q & \text{in } \Omega, \\ u_m > 0 & \text{in } \Omega, \\ u_m = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.15}$$

Obviously, u_M is a supersolution of (3.15) and Theorem 2.1 implies $u_m \le u_M$. Since,

$$F(\nabla u_m, D^2 u_m) = \lambda_m \, u_m^q < \mu \, u_m^q + u_m^r,$$

and

$$F(\nabla u_M, D^2 u_M) = \lambda_M \, u_M^q + u_M^r > \mu \, u_M^q + u_M^r,$$

we can apply the iteration method described in the proof of Theorem 3.7 to get the existence of a solution $u_\mu > 0$. □

4. Examples and further results

In this section, we show several operators for which Theorem 3.2 still applies even though the hypotheses of Section 3 do not hold. We emphasize that the fundamental ingredients of the method are the Harnack inequality and the Hopf boundary Lemma; such properties are still available in the examples considered below.

4.1. **The linear problem with variable coefficients.** Consider the linear problem

$$\begin{cases} -\text{trace}\big(A(x) D^2 u_\lambda\big) + \langle b(x), \nabla u_\lambda \rangle = \lambda \, u_\lambda^q & \text{in } \Omega, \\ \quad u_\lambda > 0 & \text{in } \Omega, \\ \quad u_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

with $\lambda > 0$ and $0 < q < 1$. We assume that $A(x)$ is a symmetric and uniformly elliptic matrix, i.e., there exist constants $0 < \theta < \Theta$ such that

$$\theta|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \Theta|\xi|^2 \quad \forall \xi \in \mathbb{R}^n,$$

and that the coefficients $A(x), b(x)$ are, say, Lipschitz continuous. Then, it is possible to adapt the arguments in the proof of Theorem 2.1. Notice that the hypotheses are not optimal (see for instance Section 5.A in [16]); with this example, we intend to illustrate the technical difficulties arising in the proof of comparison, since the argument remains essentially the same. These technical difficulties are handled using the regularity of the coefficients.

Theorem 4.1. *Let $0 < q < 1$, $\lambda = 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Consider $u, v \in \mathcal{C}(\overline{\Omega})$ a viscosity sub- and supersolution of (4.1) with the coefficients $A(x), b(x)$ under the hypotheses above. Finally, suppose that both, u and v are strictly positive in Ω, that one of them is in $\mathcal{C}^\alpha(\Omega)$ for $\alpha > 1/2$, and $u \le v$ on $\partial\Omega$. Then, $u \le v$ in $\overline{\Omega}$.*

The proof is an adaptation of that of Theorem 2.1.

Proof. We argue by contradiction. As $u - v \in \mathcal{C}(\overline{\Omega})$ and $\overline{\Omega}$ is compact, $u - v$ attains a maximum at $\overline{\Omega}$. In order to arrive at a contradiction, let us suppose that $\max_{\overline{\Omega}}(u - v) > 0$. Consider the functions $\tilde{u}$ and $\tilde{v}_\epsilon$ as in the proof of Theorem 2.1. Arguing as before, we can fix $\epsilon > 0$ small enough and suppose that there exists Ω^* with $\overline{\Omega^*} \subset \Omega$ which contains all the maximum points of $\tilde{u} - \tilde{v}_\epsilon$ and

$$-\mathrm{trace}\big(A(x)D^2\tilde{u}\big) - \frac{q}{1-q}\,\frac{\langle A(x)\nabla\tilde{u}, \nabla\tilde{u}\rangle}{\tilde{u}} + \langle b(x), \nabla\tilde{u}\rangle \leq 1 \quad \text{in } \Omega^*,$$

and

$$-\mathrm{trace}\big(A(x)D^2\tilde{v}_\epsilon\big) - \frac{q}{1-q}\,\frac{\langle A(x)\nabla\tilde{v}_\epsilon, \nabla\tilde{v}_\epsilon\rangle}{\tilde{v}_\epsilon} + \langle b(x), \nabla\tilde{v}_\epsilon\rangle \geq (1+\epsilon) > 1 \quad \text{in } \Omega^*,$$

in the viscosity sense. Consider $w(x,y) = \tilde{u}(x) - \tilde{v}_\epsilon(y) - \frac{\tau}{2}|x-y|^2$ for each $\tau > 0$, land let (x_τ, y_τ) be a maximum point of w in $\overline{\Omega} \times \overline{\Omega}$. The arguments in the proof of Theorem 2.1 show that, for τ large enough, $x_\tau, y_\tau \in \Omega^*$ and $\tilde{u}(x_\tau) > \tilde{v}_\epsilon(y_\tau)$. Moreover, the Maximum Principle for semicontinuous functions implies the existence of two symmetric matrices X_τ, Y_τ such that

$$\big(\tau(x_\tau - y_\tau), X_\tau\big) \in \overline{J}^{2,+}\tilde{u}(x_\tau) \quad \text{and} \quad \big(\tau(x_\tau - y_\tau), Y_\tau\big) \in \overline{J}^{2,-}\tilde{v}_\epsilon(y_\tau),$$

and

$$-3\tau\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_\tau & 0 \\ 0 & -Y_\tau \end{pmatrix} \leq 3\tau\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{4.2}$$

Hence, we have

$$\begin{aligned}-\mathrm{trace}\,(A(x_\tau)X_\tau) - \frac{q}{1-q}\,&\frac{\tau^2\langle A(x_\tau)(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle}{\tilde{u}(x_\tau)}\\ &+ \tau\langle b(x_\tau), (x_\tau - y_\tau)\rangle \leq 1,\end{aligned}$$

and

$$\begin{aligned}-\mathrm{trace}\,(A(y_\tau)Y_\tau) - \frac{q}{1-q}\,&\frac{\tau^2\langle A(y_\tau)(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle}{\tilde{v}_\epsilon(y_\tau)}\\ &+ \tau\langle b(y_\tau), (x_\tau - y_\tau)\rangle \geq 1+\epsilon,\end{aligned}$$

and, subtracting the first equation from the second one, we have

$$\begin{aligned}0 < \epsilon \leq &- \mathrm{trace}\,(A(y_\tau)Y_\tau - A(x_\tau)X_\tau) - \frac{q}{1-q}\bigg\{\frac{\tau^2\langle A(y_\tau)(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle}{\tilde{v}_\epsilon(y_\tau)}\\ &+ \frac{\tau^2\langle A(x_\tau)(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle}{\tilde{u}(x_\tau)}\bigg\} - \tau\langle b(x_\tau) - b(y_\tau), (x_\tau - y_\tau)\rangle\\ \leq &- \mathrm{trace}\big(A(y_\tau)Y_\tau - A(x_\tau)X_\tau\big)\\ &+ \frac{q}{1-q}\,\frac{\tau^2\big\langle\big(A(x_\tau) - A(y_\tau)\big)(x_\tau - y_\tau), (x_\tau - y_\tau)\big\rangle}{\tilde{u}(x_\tau)}\\ &- \tau\big\langle b(x_\tau) - b(y_\tau), (x_\tau - y_\tau)\big\rangle,\end{aligned}$$

since $\tilde{v}_\epsilon(y_\tau) \leq \tilde{u}(x_\tau)$. Now, we estimate each term in the right hand side of the above expression separately. We follow Example 3.6 in [16]. For the first term, multiply the right part of (4.2) by the nonnegative, symmetric matrix

$$\begin{pmatrix} A(x_\tau)^{1/2}A(x_\tau)^{1/2} & A(y_\tau)^{1/2}A(x_\tau)^{1/2} \\ A(x_\tau)^{1/2}A(y_\tau)^{1/2} & A(y_\tau)^{1/2}A(y_\tau)^{1/2} \end{pmatrix},$$

where $A(x)^{1/2}$ is well defined since $A(x)$ is uniformly elliptic. Taking traces we arrive at

$$-\mathrm{trace}\,(A(y_\tau)Y_\tau - A(x_\tau)X_\tau) \leq 3\tau \cdot \mathrm{trace}\left[\left(A(y_\tau)^{1/2} - A(x_\tau)^{1/2}\right)^2\right].$$

As the matrix $A(x)$ is Lipschitz continuous and uniformly bounded away from 0, $A(x)^{1/2}$ is Lipschitz as well. Then,

$$-\mathrm{trace}\,(A(y_\tau)Y_\tau - A(x_\tau)X_\tau) \le C\,\tau\,|x_\tau - y_\tau|^2.$$

Next, for the second and third terms, we use that $A(\cdot), b(\cdot)$ are Lipschitz, indeed

$$\tau^2\,\langle (A(x_\tau)-A(y_\tau))\,(x_\tau-y_\tau),(x_\tau-y_\tau)\rangle \le C\,\tau^2|x_\tau-y_\tau|^3,$$

and

$$-\tau\big\langle b(x_\tau)-b(y_\tau), x_\tau-y_\tau\big\rangle \le c\,\tau\,|x_\tau-y_\tau|^2.$$

Collecting all the estimates above, we get

$$0<\epsilon \le C\cdot\left(\tau\,|x_\tau-y_\tau|^2+\tau^2\,|x_\tau-y_\tau|^3\right). \tag{4.3}$$

It remains to show that the right hand side of (4.3) tends to 0 as $\tau\to\infty$. We follow the observations in Section 5.A in [16]. By definition of x_τ, y_τ, we have

$$\tilde u(x)-\tilde v_\epsilon(y)-\frac{\tau}{2}|x-y|^2 \le \tilde u(x_\tau)-\tilde v_\epsilon(y_\tau)-\frac{\tau}{2}|x_\tau-y_\tau|^2,$$

for all (x,y) near (x_τ,y_τ). Since either u or v are Holder continuous, say $u\in\mathcal{C}^\alpha(\Omega^*)$, we have that $\tilde u\in\mathcal{C}^\alpha(\Omega^*)$ as well and then, setting $x=y=y_\tau$, we get

$$\frac{\tau}{2}|x_\tau-y_\tau|^2 \le \tilde u(x_\tau)-\tilde u(y_\tau)\le C\,|x_\tau-y_\tau|^\alpha.$$

Hence

$$\tau\,|x_\tau-y_\tau|^\sigma\to 0 \quad \text{as } \tau\to 0 \qquad \forall\sigma>2-\alpha.$$

Hence, since $\alpha>1/2$, the right hand side of (4.3) tends to 0 as $\tau\to\infty$, which is a contradiction. □

Remark 4.2. Notice the fundamental role that (4.2) plays in the final estimate of the above proof.

For the existence, notice that the operator is uniformly elliptic and hence both the Harnack inequality and the Hopf Lemma in Section 3 are still available. Hence, Theorem 3.2 applies in this context. Moreover once a viscosity solution is found, by elliptic regularity, it is $\mathcal{C}^{2,\alpha}$.

4.2. **The p-laplacian, $p<\infty$.**

The p-laplacian operator is given by

$$\Delta_p u = \mathrm{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = \mathrm{trace}\left(\Big(Id+(p-2)\frac{\nabla u\otimes\nabla u}{|\nabla u|^2}\Big)D^2u\right)\cdot|\nabla u|^{p-2}.$$

The divergence-form is useful for variational problems, while the expanded version, is preferred in the viscosity setting. It is immediate to check that it verifies hypotheses $(F1)$ and $(F2)$, it is degenerate elliptic and homogeneous of degree $p-1$.

Take $0<q<p-1<r$ and $\lambda>0$, and consider the problems

$$\begin{cases} -\Delta_p u_\lambda = \lambda\,u_\lambda^q, & \text{in } \Omega,\\ u_\lambda>0 & \text{in } \Omega,\\ u_\lambda=0 & \text{on } \partial\Omega,\end{cases} \tag{4.4}$$

and

$$\begin{cases} -\Delta_p u_\lambda = \lambda\,u_\lambda^q + u_\lambda^r & \text{in } \Omega,\\ u_\lambda>0 & \text{in } \Omega,\\ u_\lambda=0 & \text{on } \partial\Omega.\end{cases} \tag{4.5}$$

It is well-known that continuous variational weak solutions to this kind of problems are viscosity solutions, see [5] and [23].

The following already known result (see [1] and [7] and the references therein) can be achieved following the ideas in Section 3 as a consequence of Theorem 2.1, the Harnack inequality and the Hopf boundary lemma.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $0 < q < p-1 < r$, and $\lambda > 0$. Then,*

(1) There exists a unique solution to (4.4) for every $\lambda > 0$ satisfying

$$u_\lambda(x) = \lambda^{\frac{1}{p-q-1}}\, u_1(x),$$

where u_1 is the solution with $\lambda = 1$.

(2) Suppose in addition that Ω satisfies an interior sphere condition. Then exists $0 < \Lambda < \infty$ such that there is at least one positive viscosity solution of (4.5) for every $\lambda < \Lambda$, and no nontrivial solution for $\lambda > \Lambda$.

In this case the subsolution can be taken as a positive eigenfunction conveniently scaled. For variational eigenvalues of the p-laplacian see for instance [18] and [28].

4.3. **The 3-homogeneous infinity-laplacian.** Consider the infinity laplacian operator,

$$\Delta_\infty u = \langle D^2 u \nabla u, \nabla u\rangle = \sum_{i,j=1}^{n} u_{x_i,x_j} u_{x_i} u_{x_j}.$$

Clearly, it is homogeneous of degree 3 and degenerate elliptic and hence, not in the framework of Theorem 3.2. Nevertheless, the methods in Section 3 can be adapted to produce the following result.

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $0 < q < 3 < r$ and $\lambda > 0$. Then,*

(1) There exists a unique viscosity solution of

$$\begin{cases} -\Delta_\infty u_\lambda = \lambda\, u_\lambda^q & \text{in } \Omega, \\ u_\lambda > 0 & \text{in } \Omega, \\ u_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

Indeed, $u_\lambda(x) = \lambda^{\frac{1}{3-q}}\, u_1(x)$, where u_1 is the solution for $\lambda = 1$.

(2) There exists at least one solution of

$$\begin{cases} -\Delta_\infty u_\lambda = \lambda\, u_\lambda^q + u_\lambda^r & \text{in } \Omega, \\ u_\lambda > 0 & \text{in } \Omega, \\ u_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

for λ small enough.

The uniqueness assertion in the first part of Theorem 4.4 is a consequence of Theorem 2.1.

For the existence part, we follow the construction in Section 3. We start studying the auxiliary problems.

Proposition 4.5. *The auxiliary problem*

$$\begin{cases} -\Delta_\infty v_\lambda = \lambda & \text{in } \Omega, \\ v_\lambda > 0 & \text{in } \Omega, \\ v_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.8}$$

has a unique viscosity solution for every λ.

Proof. As 0 is a subsolution, we only have to construct a suitable supersolution and apply Perron's method. Consider $x_0 \notin \overline{\Omega}$ and let

$$\Phi(x) = \alpha - \frac{\beta}{2}|x-x_0|^2, \qquad \text{with} \quad \beta \geq \left(\frac{\lambda}{\mathrm{dist}(x_0,\Omega)^2}\right)^{1/3} \quad \text{and} \quad \alpha \geq \frac{\beta C^2}{2},$$

where $C = \max_{x\in\overline{\Omega}} |x-x_0|$. By direct computation,

$$-\Delta_\infty \Phi(x) = \beta^3\,|x-x_0|^2 \geq \lambda,$$

in Ω, while $w \leq \Phi$ on $\partial\Omega$, since

$$\Phi(x) = \alpha - \frac{\beta}{2}|x-x_0|^2 \geq 0,$$

by construction. In remains to construct barriers in order to force Φ to have the correct boundary value. Following [16] and [21], for every $z \in \partial\Omega$, we define

$$\Psi_z(x) = D\,|x-z|^{1/2}, \quad x \in \Omega$$

where

$$D^3 \geq 16\,\lambda\,\mathrm{diam}(\Omega)^{5/2}.$$

A straightforward computation shows

$$-\Delta_\infty \Psi_z(x) = \frac{D^3}{16}\,|x-z|^{-5/2} \geq \lambda,$$

therefore $v_\lambda(x) = 0$ on $\partial\Omega$.

Furthermore, $0 \leq v_\lambda \leq \Phi$ by comparison, (Theorem 2.5). Since $v_\lambda \neq 0$ (we get a contradiction with the equation otherwise), the Harnack inequality (see [26], [25] and [4]) implies $v_\lambda > 0$. □

Corollary 4.6. *The problem*

$$\begin{cases} -\Delta_\infty w_\lambda = \lambda\, d(x) & \text{in } \Omega \\ w_\lambda > 0 & \text{in } \Omega \\ w_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.9}$$

where $d(x)$ is the normalized distance to the boundary, namely,

$$d(x) = \frac{dist(x,\partial\Omega)}{\|dist(\cdot,\partial\Omega)\|_\infty},$$

has a viscosity solution.

Proof. It is sufficient to consider v_λ as a supersolution and proceed as in the previous proposition. □

Now, we can construct a supersolution of the concave problem (4.6) as in Lemma 3.4. For the construction of the subsolution we argue as in Lemma 3.5 using the Hopf Lemma in [4]. Then, the Comparison Principle (Theorem 2.1) yields existence (and uniqueness) of solutions to (4.6) via the Perron method for $\lambda = 1$. By homogeneity, we extend the result for every λ. In the concave-convex case, problem (4.7), we follow Lemmas 3.8 and 3.9. Hence, we construct a sub- and supersolution $\underline{u}_\lambda, \overline{u}_\lambda$ from v_λ, w_λ in (4.8) for $\lambda < \lambda_0$, the latter given by (3.8). Following the iteration procedure in Subsection 3.2 we get a sequence

$$\underline{u}_\lambda \leq \ldots \leq w_k \leq w_{k-1} \leq \ldots \leq w_2 \leq w_1 \leq \overline{u}_\lambda \quad \text{in } \Omega,$$

where

$$\begin{cases} -\Delta_\infty w_k = \lambda\, w_{k-1}^q + w_{k-1}^r & \text{in } \Omega, \\ w_k = 0 & \text{on } \partial\Omega. \end{cases}$$

for $k \geq 1$ (by convention $w_0 = \overline{u}_\lambda$). Notice that, since $-\Delta_\infty w_k > 0$ in Ω, we have (see [25] and [26]) that

$$|\nabla w_k(x)| \leq \frac{w_k(x)}{\operatorname{dist}(x, \partial\Omega)} \leq \frac{\overline{u}_\lambda(x)}{\operatorname{dist}(x, \partial\Omega)} \qquad \text{a.e. } x \in \Omega,$$

for every $k > 1$. Hence, both $\|w_k\|_\infty$ and $\|\nabla w_k\|_\infty$ are uniformly bounded on compact subsets of Ω. Since $w_k = 0$ for all k, by the Ascoli-Arzela compactness criterion and the monotonicity of the $\{w_k\}$, the whole sequence converges uniformly in Ω to some $u_\lambda \in \mathcal{C}(\Omega)$ which is a solution of (4.7) in the viscosity sense. We have now finished the proof of Theorem 4.4.

4.4. **The 1-homogeneous infinity-laplacian.** In the previous section we have considered the (classical) infinity laplacian with degree of homogeneity 1. Other definition can be considered, namely,

$$\tilde{\Delta}_\infty u = \left\langle D^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right\rangle.$$

This operator naturally arises in many equations obtained as a limit of p-laplacian type equations; it is easy to adapt the arguments in [23] and [14] to get an example of this issue. Furthermore, when studying evolution equations governed by the infinity laplacian (see [22] and the references therein), it is natural to consider the 1-homogeneous infinity laplacian rather than its 3-homogeneous version since then, the homogeneity of the parabolic part matches that of the infinity laplacian. Finally, the Poisson problem for this infinity laplacian has been recently studied in [29] using Tug-of-War games, and in [21], an eigenvalue-type equation is studied. Our aim here is to study the existence and uniqueness of solutions to

$$\begin{cases} -\tilde{\Delta}_\infty u_\lambda = \lambda\, u_\lambda^q \quad \text{in } \Omega, \\ \quad u_\lambda > 0 \quad \text{in } \Omega, \\ \quad u_\lambda = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.10}$$

and

$$\begin{cases} -\tilde{\Delta}_\infty u_\lambda = \lambda\, u_\lambda^q + u_\lambda^r \quad \text{in } \Omega, \\ \quad u_\lambda > 0 \quad \text{in } \Omega, \\ \quad u_\lambda = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.11}$$

with $\lambda > 0$ and $0 < q < 1 < r$. It would be necessary to make precise the definition of viscosity solution (respectively sub- and supersolution) in this context, since the operator is singular when $\nabla u = 0$. Our definition follows the one in [21] for the eigenvalue problem. First, given a matrix $A \in S^n$, we denote its largest and smallest eigenvalues by $M(A)$ and $m(A)$, respectively, this is,

$$M(A) = \max_{|\xi|=1} \langle A\xi, \xi \rangle, \qquad m(A) = \min_{|\xi|=1} \langle A\xi, \xi \rangle.$$

Then, we have the following definition.

Definition 4.7. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. An upper semicontinuous function $v : \Omega \to \mathbb{R}$ is a *viscosity subsolution of* (4.10) in Ω if, whenever $\hat{x} \in \Omega$ and $\phi \in \mathcal{C}^2$ are such that $(v - \phi)(x) < (v - \phi)(\hat{x}) = 0$ for all $x \neq \hat{x}$ in a neighborhood of $\hat{x}$, then

$$\begin{cases} -\tilde{\Delta}_\infty \phi(\hat{x}) \leq \lambda\, \phi^q(\hat{x}), & \text{if } \nabla\phi(\hat{x}) \neq 0, \\ -M(D^2\phi(\hat{x})) \leq \lambda\, \phi^q(\hat{x}), & \text{if } \nabla\phi(\hat{x}) = 0. \end{cases}$$

A lower semicontinuous function $w : \Omega \to \mathbb{R}$ is a *viscosity supersolution of* (4.10) in Ω if, whenever $\hat{x} \in \Omega$ and $\phi \in \mathcal{C}^2$ are such that $(w-\phi)(x) > (w-\phi)(\hat{x}) = 0$ for all $x \neq \hat{x}$ in a neighborhood of $\hat{x}$, then

$$\begin{cases} -\tilde{\Delta}_\infty \phi(\hat{x}) \geq \lambda\, \phi^q(\hat{x}), & \text{if } \nabla\phi(\hat{x}) \neq 0, \\ -m(D^2\phi(\hat{x})) \geq \lambda\, \phi^q(\hat{x}), & \text{if } \nabla\phi(\hat{x}) = 0. \end{cases}$$

Finally, a continuous function $u : \Omega \to \mathbb{R}$ is a *viscosity solution of* (4.10) in Ω if it is both a viscosity subsolution and viscosity supersolution.

Notice that this definition is slightly different to the one in [29]; however, it is easy to see that both definitions are equivalent.

If u satisfies $-\tilde{\Delta}_\infty u \geq 0$ in Ω in the viscosity sense, then $-\Delta_\infty u \geq 0$ in Ω as well. Hence the Harnack inequality and Hopf Lemma used in the previous section (see [4], [25] and [26]) also apply to the operator $\tilde{\Delta}_\infty$. Then, the results in [29] and the Harnack inequality imply the existence of a unique solution to the auxiliary problem

$$\begin{cases} -\tilde{\Delta}_\infty v = \lambda & \text{in } \Omega, \\ \quad v > 0 & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.12}$$

According to the results in [21], there exists a solution for the eigenvalue problem

$$\begin{cases} -\tilde{\Delta}_\infty w = \lambda_1\, w & \text{in } \Omega, \\ \quad w > 0 & \text{in } \Omega, \\ \quad w = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\|w\|_\infty = 1$ for general Ω. Then we can use as a subsolution $t\,w$, t small, and then avoid the use of the Hopf boundary lemma. Moreover, the existence of a first eigenvalue allows us to prove the non existence of solutions for large λ in the concave-convex case. Following the ideas in Section 3 and completing the details as in the case of the 3-homogeneous infinity laplacian, we get the following result.

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $0 < q < 1 < r$ and $\lambda > 0$. Then,*

(1) There exists a unique positive viscosity solution of (4.10) *for every $\lambda > 0$. In fact,*

$$u_\lambda(x) = \lambda^{\frac{1}{1-q}}\, u_1(x),$$

where u_1 is the solution for $\lambda = 1$.

(2) If Ω satisfies an interior sphere condition, then there exists $0 < \Lambda < \infty$ such that there is at least one positive viscosity solution of (4.11) *for every $\lambda < \Lambda$ and no nontrivial solution for $\lambda > \Lambda$.*

4.5. **Monge-Ampere equations.** The above theory also applies to equations of the Monge-Ampere type. It is well known (see for example [11] and [16]) that the Monge-Ampere operator

$$F : S^n \to \mathbb{R}$$
$$M \to \det(M),$$

is elliptic only for positive definite matrices. Hence, it is natural to look for strictly convex solutions. Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth, strictly convex set, $0 < q < n < r$ and $\lambda > 0$. Our model concave problem reads

$$\begin{cases} \det(D^2 u) = \lambda\, |u|^q & \text{in } \Omega, \\ u \quad \text{convex in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.13}$$

and, in the concave-convex case,

$$(4.14)\qquad \begin{cases} \det(D^2u) = \lambda\,|u|^q + |u|^r & \text{in } \Omega,\\ u \quad \text{convex in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Notice that $\partial\Omega$ is the zero level set of the strictly convex function u and hence it is natural to assume that Ω is strictly convex. Since $\det(\cdot)$ is increasing in the set of positive definite-matrices, the equation should be written in the form

$$-\det(D^2u) + \lambda\,|u|^q = 0 \qquad \text{in } \Omega$$

in order to follow the same convention as in the rest of the paper. As we are looking for negative solutions, we have to adapt the arguments in Theorem 2.1 which in this context reads as follows.

Theorem 4.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded strictly convex domain and consider a viscosity subsolution u and a supersolution v of*

$$(4.15)\qquad -\det(D^2w) + \lambda\,|w|^q = 0 \qquad \textit{in } \Omega,$$

where $0 < q < n$. Suppose that both, u and v are strictly negative in Ω, continuous up to the boundary and satisfy $u \le v$ on $\partial\Omega$. Then, $u \le v$ in $\overline{\Omega}$.

For the proof, as before, we can take $\lambda = 1$ by homogeneity. Replace (2.8) by

$$\tilde{u}(x) = \frac{-1}{1-\frac{q}{n}}\big(-u(x)\big)^{1-\frac{q}{n}} \qquad \text{and} \qquad \tilde{v}(x) = \frac{-1}{1-\frac{q}{n}}\big(-v(x)\big)^{1-\frac{q}{n}},$$

and (2.9) by

$$\tilde{v}_\epsilon(x) = (1-\epsilon)\cdot\big(\tilde{v}(x)+\epsilon\big).$$

Then, since $\tilde{u} \le \tilde{v}$ on $\partial\Omega$, we have

$$\tilde{u} - \tilde{v}_\epsilon = \tilde{u} - (1-\epsilon)\,\tilde{v} - (1-\epsilon)\epsilon < 0 \qquad \text{on } \partial\Omega.$$

Then, following the arguments in the proof of Theorem 2.1, we can suppose that there exists Ω^* such that $\overline{\Omega^*} \subset \Omega$ contains all the maximum points of $\tilde{u} - \tilde{v}_\epsilon$ for ϵ small enough. Then, $\tilde{u}, \tilde{v}_\epsilon$ satisfy

$$-\det\left(D^2\tilde{u} + \frac{q}{n-q}\,\frac{\nabla\tilde{u}\otimes\nabla\tilde{u}}{\tilde{u}}\right) + 1 \le 0 \qquad \text{in } \Omega^*$$

and

$$-\det\left(D^2\tilde{v}_\epsilon + \frac{q}{n-q}\,\frac{\nabla\tilde{v}_\epsilon\otimes\nabla\tilde{v}_\epsilon}{\tilde{v}_\epsilon}\right) + (1-\epsilon)^n \ge 0 \qquad \text{in } \Omega^*,$$

and we can complete the proof as in Theorem 2.1.

The Comparison Principle in Theorem 2.5 can also be adapted to the present setting in a similar way.

For the existence of solutions of (4.13) and (4.14), we have the following result.

Theorem 4.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, strictly convex, smooth domain, and consider $0 < q < n < r$ and $\lambda > 0$. Then,*

1. *(1) There exists a unique solution to (4.13) for every $\lambda > 0$ satisfying*

   $$u_\lambda(x) = \lambda^{\frac{1}{n-q}}\, u_1(x),$$

   where u_1 is the solution with $\lambda = 1$.
2. *(2) There exists $0 < \Lambda < \infty$ such that there is at least one positive viscosity solution of (4.14) for every $\lambda < \Lambda$, and no nontrivial solution for $\lambda > \Lambda$.*

The above result was already known, see [20] for example where the Leray-Schauder degree is used. Our proof is different and constructive.

For the proof of both statements, we adapt the monotone iteration scheme in Section 3. Hence, fix $\epsilon \in (0,1)$ and consider the auxiliary problem

$$\begin{cases} \det(D^2 v_\lambda) = \lambda \qquad \text{in } \Omega, \\ v_\lambda \text{ convex in } \overline{\Omega}, \\ v_\lambda = -\epsilon \quad \text{on } \partial\Omega. \end{cases}$$

Since $\lambda > 0$, it is known (see [12]) that there exists a unique strictly convex solution $v_\lambda \in \mathcal{C}^\infty(\overline{\Omega})$. Indeed, $v_\lambda < 0$ in $\overline{\Omega}$ by strict convexity. Furthermore, consider the Monge-Ampere eigenvalue problem

$$\begin{cases} \det(D^2 \psi_1) = \lambda_1 (-\psi_1)^n \qquad \text{in } \Omega, \\ \psi_1 \text{ convex in } \overline{\Omega}, \\ \psi_1 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

In [27], it is shown that there exists $\psi_1 \in \mathcal{C}^{1,1}(\overline{\Omega}) \cap \mathcal{C}^\infty(\Omega)$ such that $\psi_1 < 0$ in Ω and that $\lambda_1 > 0$ is isolated. Then, we can construct a subsolution and a supersolution of (4.13) (with $\lambda = 1$).

Lemma 4.11. *Let $T = \|v\|_\infty^{\frac{q}{n-q}}$ and $t \le \lambda_1^{\frac{-1}{n-q}}$. Then*

$$\underline{u}(x) = T\, v(x) \qquad \textit{and} \qquad \overline{u}(x) = t\, \psi_1(x)$$

are, respectively, classical sub- and supersolution of (4.13) *with* $\lambda = 1$.

The proof is similar to that of Lemmas 3.4 and 3.5 but, since now we are looking for negative solutions, the details are slightly different. In a similar fashion to Lemmas 3.8 and 3.9 we get strictly convex, classical subsolutions and supersolutions.

Lemma 4.12. *There exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$, we can find constants $T(\lambda), t > 0$ such that*

$$\underline{u}_\lambda(x) = T(\lambda)\, v_\lambda(x) \qquad \textit{and} \qquad \overline{u}_\lambda(x) = t\, \psi_1(x),$$

are respectively a classical sub- and supersolution of (4.14)*. Indeed, t is chosen such that $\underline{u}_\lambda \le \overline{u}_\lambda$ in Ω for all $\lambda \in (0, \lambda_0)$.*

Then, consider w_1, the solution of

$$\begin{cases} \det(D^2 w_1) = \lambda\, |\underline{u}_\lambda|^q + |\underline{u}_\lambda|^r \qquad \text{in } \Omega \\ w_1 \text{ convex in } \Omega \\ w_1 = -T(\lambda) \cdot \epsilon^2 \quad \text{on } \partial\Omega. \end{cases}$$

Theorem 1.1 in [12] imply that exists a unique $w_1 \in \mathcal{C}^\infty(\overline{\Omega})$ and it is strictly convex. By comparison,

$$\underline{u}_\lambda \le w_1 \le \overline{u}_\lambda \quad \text{in } \Omega.$$

Then, consider

$$\begin{cases} \det(D^2 w_2) = \lambda\, |w_1|^q + |w_1|^r \qquad \text{in } \Omega, \\ w_2 \text{ convex in } \Omega, \\ w_2 = -T(\lambda) \cdot \epsilon^3 \quad \text{on } \partial\Omega. \end{cases}$$

Again, Theorem 1.1 in [12] implies the existence of a unique $w_2 \in \mathcal{C}^\infty(\overline{\Omega})$, which is strictly convex. Then, by comparison,

$$\underline{u}_\lambda \le w_1 \le w_2 \le \overline{u}_\lambda \quad \text{in } \Omega.$$

We can iterate this procedure. In general, we get

$$\begin{cases} \det(D^2 w_k) = \lambda\, |w_{k-1}|^q + |w_{k-1}|^r & \text{in } \Omega, \\ w_k \text{ convex in } \Omega, \\ w_k = -T(\lambda)\cdot \epsilon^{k+1} \quad \text{on } \partial\Omega. \end{cases}$$

Each w_k is unique, strictly convex, and $\mathcal{C}^\infty(\overline{\Omega})$. Furthermore,

$$\underline{u}_\lambda \le w_1 \le w_2 \le \ldots \le w_{k-1} \le w_k \le \ldots \overline{u}_\lambda \qquad \text{in } \Omega.$$

Since every $w_k \in \mathcal{C}^\infty(\overline{\Omega})$, it is a weak solution in the sense of [32], and the Holder estimate in [32, Theorem 4.1] gives the necessary relative compactness. Hence, the Ascoli-Arzela compactness criterion and the monotonicity of the sequence, gives uniform convergence of the whole sequence to some $u \in \mathcal{C}(\Omega)$. Then, for every compact subset Ω^* of Ω, we can pass to the limit in the viscosity sense and get

$$\det(D^2 u) = \lambda\, |u|^q + |u|^r \qquad \text{in } \Omega^*.$$

Furthermore, u is convex (is an uniform limit of strictly convex functions) and verifies $u = 0$ on $\partial\Omega$ by construction.

4.6. **The equation** $\min\{|\nabla u| - \lambda u^q, -\Delta_\infty u\} = 0$**.** We finally consider the equation

$$\begin{cases} \min\{|\nabla u_\lambda| - \lambda u_\lambda^q, -\Delta_\infty u_\lambda\} = 0 & \text{on } \Omega, \\ u_\lambda > 0 & \text{on } \partial\Omega, \\ u_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.16}$$

for $\lambda > 0$ and $0 < q < 1$. The existence and uniqueness of positive solutions to (4.16) for every $\lambda > 0$ has already been studied in [14]. In fact, it is proven that there exists a smooth curve of solutions given by

$$u_\lambda(x) = \lambda^{\frac{1}{1-q}}\, u_1(x),$$

where u_1 is the solution to (4.16) with $\lambda = 1$.

Clearly, (4.16) does not fit the general shape considered in this work, namely (3.1). However, both the comparison principle in [14] and our Theorem 2.1 (for m=1) are particular cases of the following slightly more general result.

Theorem 4.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $F : \mathbb{R}\times\mathbb{R}^n \times S^n \to \mathbb{R}$ such that*

(1) $F(t\,s, t\,\xi, t\,X) = t\cdot F(s,\xi,X)$ for every $t > 0$, and $F(0,0,0) = 0$.
(2) $F(s,\xi,X) \le F(s,\xi,Y)$ whenever $Y \le X$.
(3) For fixed $\xi \in \mathbb{R}^n$ and $X \in S^n$, $F(\cdot,\xi,X)$ is strictly decreasing.

Let $u, v \in \mathcal{C}(\overline{\Omega})$ be a viscosity subsolution and a viscosity supersolution to

$$F\big(f(w), \nabla w, D^2 w\big) = 0 \quad \text{in } \Omega, \tag{4.17}$$

where $f(\cdot)$ satisfies hypothesis (2.2) with $m = 1$. Assume that u, v are strictly positive in Ω and $u \le v$ on $\partial\Omega$. Then, $u \le v$ in $\overline{\Omega}$.

Let us point out that we are not under the scope of [16] since we are not assuming the properness of F.

The proof is an adaptation of the one to Theorem 2.1. In this context, Lemma 2.3 reads as follows.

Lemma 4.14. *Let $w > 0$ be a supersolution (subsolution) of problem (4.17) in Ω and q as in (2.2). Then,*

$$\tilde{w}(x) = \frac{1}{1-q}\cdot w^{1-q}(x)$$

is a viscosity supersolution (subsolution) of

$$F\left(\frac{f\left(\left[(1-q)\,\tilde{w}(x)\right]^{\frac{1}{1-q}}\right)}{\left[(1-q)\,\tilde{w}(x)\right]^{\frac{q}{1-q}}}, \nabla\tilde{w}, D^2\tilde{w} + \frac{q}{1-q}\frac{\nabla\tilde{w}\otimes\nabla\tilde{w}}{\tilde{w}}\right) = 0, \tag{4.18}$$

in every Ω^* *such that* $\overline{\Omega^*}\subset\Omega$.

The rest of the proof of Theorem 3.2 applies almost unchanged.

Appendix A. Hopf's Lemma for uniformly elliptic equations

We recall the Hopf boundary lemma, used in the proof of the analogous to Lemma 3.5. For further refinements, see [24] (see also [3] and [31]).

Proposition A.1 (Hopf's Lemma)**.** *Let* Ω *be a bounded domain and* u *a viscosity solution of*

$$F(\nabla u, D^2 u) \le 0 \quad \text{in } \Omega,$$

where F *satisfies* $(F1')$, $(F2)$, *and* $(F3)$. *In addition, let* $x_0\in\partial\Omega$ *satisfy*

i) $u(x_0) > u(x)$ *for all* $x\in\Omega$.

ii) $\partial\Omega$ *satisfies an interior sphere condition at* x_0.

Then, for every nontangential direction ξ *pointing into* Ω,

$$\lim_{t\to 0^+}\frac{u(x_0+t\xi)-u(x_0)}{t} < 0.$$

Proof. Our proof follows [19, Section 3.2]. Since Ω satisfies an interior sphere condition at x_0, there exists a ball $B = B_R(y)\subset\Omega$ with $x_0\in\partial B$. For $0<\rho<R$, we consider $A = \{x\in\Omega:\ \rho<|x-y|<R\}$ and define

$$v(x) = e^{\frac{-\alpha|x-y|^2}{2}} - e^{\frac{-\alpha R^2}{2}},$$

and

$$w(x) = u(x) - u(x_0) + \epsilon\, v(x),$$

for $x\in A$, where $\alpha,\epsilon>0$ are constants yet to be determined. Then,
1. $\mathcal{P}^-_{\theta,\Theta}(D^2 w) - \gamma\,|\nabla w| \le 0$ in A for α large enough. Let $\phi\in\mathcal{C}^2$ and $\hat{x}\in A$ such that $w-\phi$ has a local maximum at $\hat{x}$. It is easy to see that $u-\Phi$ has a local maximum at $\hat{x}$, with $\Phi(x) = \phi(x) - \epsilon\, v(x)$. Since $v\in\mathcal{C}^2$, so it is Φ, and the definition of u and the structure condition $(F3)$ imply

$$\begin{aligned} 0 &\ge F(\nabla\Phi(\hat{x}), D^2\Phi(\hat{x})) = F\big(\nabla\phi(\hat{x}) - \epsilon\,\nabla v(\hat{x}), D^2\phi(\hat{x}) - \epsilon\, D^2 v(\hat{x})\big)\\ &\ge F(\nabla\phi(\hat{x}), D^2\phi(\hat{x})) + \mathcal{P}^-_{\theta,\Theta}(-\epsilon\, D^2 v(\hat{x})) - \gamma\,\epsilon\,|\nabla v(\hat{x})|\\ &\ge F(0,0) + \mathcal{P}^-_{\theta,\Theta}(D^2\phi(\hat{x})) - \gamma\,|\nabla\phi(\hat{x})| - \epsilon\,\mathcal{P}^+_{\theta,\Theta}(D^2 v(\hat{x})) - \gamma\,\epsilon\,|\nabla v(\hat{x})|. \end{aligned}$$

By direct computation,

$$\begin{aligned} \mathcal{P}^+_{\theta,\Theta}(\,D^2 v(\hat{x})) &= e^{\frac{-\alpha|x-y|^2}{2}}\,\mathcal{P}^+_{\theta,\Theta}\big(\alpha^2(x-y)\otimes(x-y) - \alpha\, I\big)\\ &\le e^{\frac{-\alpha|x-y|^2}{2}}\big(-\alpha^2\rho^2\theta + \alpha\, n\,\Theta\big),\\ |\nabla v(\hat{x})| &\le \alpha\, R\, e^{\frac{-\alpha|x-y|^2}{2}}. \end{aligned}$$

Combining the expressions above,

$$\mathcal{P}^-_{\theta,\Theta}(D^2\phi(\hat{x})) - \gamma\,|\nabla\phi(\hat{x})| \le \epsilon\, e^{\frac{-\alpha|x-y|^2}{2}}\big(-\alpha^2\rho^2\theta + \alpha\,(n\,\Theta - \gamma\, R)\big) \le 0,$$

for α large enough.

2. $w \le 0$ on ∂A for $\epsilon > 0$ small enough. Since $u - u(x_0) < 0$ on $\partial B_\rho(y)$, we can choose $\epsilon > 0$ small enough such that

$$\big(u - u(x_0) + \epsilon v\big) \le 0 \tag{A.1}$$

on $\partial B_\rho(y)$. Moreover, since $v = 0$ on $\partial B_R(y)$, (A.1) also holds in the outer boundary. Hence, the ABP estimate, (see for instance [13, Proposition 2.12]) implies $w \le 0$ in the whole A. Hence, for every nontangential direction ξ pointing into Ω, one has

$$\lim_{t\to 0^+} \frac{u(x_0 + t\xi) - u(x_0)}{t} \le -\epsilon \frac{\partial v}{\partial \xi}(x_0) = \epsilon\, \alpha\, e^{\frac{-\alpha R^2}{2}} \langle (x_0 - y), \xi\rangle < 0. \qquad \square$$

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