LIMIT BRANCH OF SOLUTIONS AS $p \to \infty$ FOR A FAMILY OF SUB-DIFFUSIVE PROBLEMS RELATED TO THE P-LAPLACIAN

FERNANDO CHARRO AND IRENEO PERAL

Abstract. We characterize the limit as $p \to \infty$ of the branches of solutions to

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda\, u^{r(p)} & \text{in } \Omega \subset \mathbb{R}^n \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $\lambda > 0$ and $\lim_{p\to\infty} \dfrac{r(p)}{p-1} = R$, where $R < 1$. We show that the limit set is a curve of positive viscosity solutions of the equation

$$\min\Big\{ |\nabla v(x)| - \Lambda\, v^R(x), -\Delta_\infty v(x) \Big\} = 0 \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0,$$

where $\Delta_\infty u \equiv \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_j \partial x_i} \frac{\partial u}{\partial x_i}$ and $\Lambda > 0$. The key result is a comparison principle for the limit equation from which we deduce uniqueness for the Dirichlet problem and hence the existence of the curve of solutions.

1. Introduction

Our main interest is to study the behavior as $p \to \infty$ of the sequence of positive solutions of the problems

$$(1) \qquad \begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda\, u^{r(p)} & \text{in } \Omega \subset \mathbb{R}^n \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\lambda > 0$ and $\lim_{p\to\infty} \dfrac{r(p)}{p-1} = R$ with $R < 1$ and Ω is a bounded domain. Without loss of generality we shall suppose $r(p) < p-1$ hereafter, this is the sense in which we call problem (1) sub-diffusive. The case $r(p) = p-1$, the eigenvalues case, has been studied in [12].

We observe that for fixed p, the expression

$$(2) \qquad u_{\lambda,p}(x) = \lambda^{\frac{1}{p-1-r(p)}}\, u_{1,p}(x)$$

links the solutions of the problem

$$(3) \qquad \begin{cases} -\operatorname{div}(|\nabla u_{1,p}|^{p-2}\nabla u_{1,p}) = u_{1,p}^{r(p)} & \text{in } \Omega \\ u_{1,p} > 0 & \text{in } \Omega \\ u_{1,p} = 0 & \text{on } \partial\Omega \end{cases}$$

with those of the problem (1). Since $r(p) < p-1$ for p large enough, problem (3) has a unique solution for each fixed λ. The proof is an adaptation of a result by

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Brezis-Oswald in [5], see for instance [1] and the references therein for the details in this framework. Thus, we deduce that the set of positive solutions for fixed p,

$$\Gamma_p = \{(\lambda, u_{\lambda,p}) : \ u_{\lambda,p} \text{ is the nontrivial solution of (1)}\}, \tag{4}$$

is a continuous branch, in fact a smooth curve in $(0,\infty)\times W_0^{1,\infty}(\Omega)$, given by (2).

We are interested in studying both the convergence of those curves of solutions as $p\to\infty$ and the existence of a limit equation satisfied by the elements of the limit set. First of all, we shall state the precise notion of limit set in this setting.

Definition 1. The limit set of the branches Γ_p is defined as

$$\Gamma_\infty = \Big\{(\Lambda, u_\Lambda) : \ \exists \text{ sequence } \big\{(\lambda_p, u_{\lambda_p,p})\big\}_p \text{ s.t. } (\lambda_p, u_{\lambda_p,p}) \in \Gamma_p \text{ and } \lim_{p\to\infty} (\lambda_p)^{1/p} = \Lambda, \ \lim_{p\to\infty} u_{\lambda_p,p} = u_\Lambda \text{ uniformly} \Big\}. \tag{5}$$

Remark 2. It is possible to prove that for fixed $\lambda_p\equiv\mu$ the limit of $(\mu, u_{\mu,p})$ is a couple $(1,v)$ where v is a function independent of μ. The condition $\lim(\lambda_p)^{1/p}=\Lambda$ behaves as a scaling factor for each p that allow a nontrivial limit set to come up. The arguments below work in the same way defining the limit set via the condition

$$\lim_{p\to\infty} (\lambda_p)^{\frac{1}{r(p)}} = \bar\Lambda.$$

Both constants Λ and $\bar\Lambda$ are related by the expression $\bar\Lambda = \Lambda^{1/R}$.

Heuristically, if $(\Lambda, u_\Lambda)\in\Gamma_\infty$, we can take limits as $p\to\infty$ in the expression (2) and conclude that

$$u_\Lambda = \lim_{p\to\infty} u_{\lambda_p,p} = \lim_{p\to\infty} \left(\lambda_p^{\frac{1}{p-1-r(p)}} u_{1,p}(x)\right) = \Lambda^{\frac{1}{1-R}}\, u_1. \tag{6}$$

Thus, it seems that the limit problem should satisfy a scaling property analogous to (2) in problem (1).

In order to make rigorous this formal computation, we show (Section 2 and 3) that there exists a convergent subsequence $\{u_{1,p_i}\}_i$ of solutions to (3) and a limit function $u_1\in\Gamma_\infty$, and then we use such a subsequence to deduce that u_1 is a viscosity solution to the limit problem

$$\begin{cases} \min\big\{|\nabla u_1(x)| - u_1^R(x), -\Delta_\infty u_1(x)\big\} = 0 \quad \text{in } \Omega,\\ u_1 > 0 \text{ in } \Omega\\ u_1 = 0 \text{ on } \partial\Omega, \end{cases} \tag{7}$$

where $\Delta_\infty u \equiv \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j\partial x_i}\frac{\partial u}{\partial x_i}$.

Next, we prove in Section 4 the main result, a comparison principle for the limit problem (7) and, as a consequence, uniqueness of positive solutions to (7). Notice that the comparison and uniqueness results extend to

$$\begin{cases} \min\big\{|\nabla u_\Lambda(x)| - \Lambda\, u_\Lambda^R(x), -\Delta_\infty u_\Lambda(x)\big\} = 0 \quad \text{in } \Omega,\\ u_\Lambda > 0 \text{ in } \Omega\\ u_\Lambda = 0 \text{ on } \partial\Omega, \end{cases} \tag{8}$$

through the re-scaling suggested by (6).

This result could also be read as an extension to the fully nonlinear setting of the uniqueness result by Brezis-Oswald quoted above.

Moreover, we deduce from the uniqueness result that the set

$$\Gamma_\infty = \{(\Lambda, u_\Lambda) : \ u_\Lambda \text{ is the nontrivial solution of (8)}\},$$

defines a smooth curve of positive solutions of the limit problem given by (6) which verifies (Section 5) the estimate

$$\|u_\Lambda\|_{L^\infty(\Omega)} = \left(\Lambda \cdot \max_{x\in\Omega} \operatorname{dist}(x,\partial\Omega)\right)^{\frac{1}{1-R}}.$$

This is the sense in which we mean the *limit of branches*.

We point out the similarities between the behavior of problem (1) for $p < \infty$ and that of the limit problem (8).

We also provide explicit solutions of problem (7) for a certain class of domains (including the ball and the annulus among others) in Section 6.

Finally, we recall for the reader's convenience the following lemma, stating that weak solutions of our problem are also viscosity solutions. The proof, which we omit here, follows in an analogous way to that of Lemma 1.8 in [12] (see also [2]).

Lemma 3. *If u is a continuous weak solution of* (1)*, then it is a viscosity solution of the same problem, rewritten as*

$$\begin{cases} F_p(\nabla u, D^2 u) = \lambda\, u^{r(p)} & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{9}$$

where

$$F_p(\xi, X) = -\operatorname{trace}\left(\left(Id + (p-2)\frac{\xi\otimes\xi}{|\xi|^2}\right)X\right)\cdot|\xi|^{p-2}.$$

In the sequel we shall always consider the more suitable form of our problem between (1) and (9) without any further reference.

2. A priori bounds.

The following result is an easy consequence of Morrey's estimates.

Lemma 4. *Consider $p > n$ fixed. There exist constants C_1, C_2 independent of p such that the solution $u_{1,p}$ of* (3) *satisfies*

$$\|u_{1,p}\|_{L^\infty(\Omega)} \le C_1, \tag{10}$$

and for every $m > n$,

$$\frac{|u_{1,p}(x) - u_{1,p}(y)|}{|x-y|^{1-\frac{n}{m}}} \le C_2 \qquad \forall x, y \in \Omega \tag{11}$$

if $p > m$.

Proof. 1. Multiplying (1) by $u_{1,p}$ and integrating by parts, we get

$$\int_\Omega |\nabla u_{1,p}|^p\, dx = \int_\Omega |u_{1,p}|^{r+1}\, dx. \tag{12}$$

Morrey's estimates imply that there exists a constant $C > 0$ independent of p such that

$$\|u_{1,p}\|_{L^\infty(\Omega)} \le C\left(\int_\Omega |\nabla u_{1,p}|^p\, dx\right)^{1/p}$$

Combining both expressions above we have

$$\|u_{1,p}\|_{L^\infty(\Omega)} \le C\left(\int_\Omega |u_{1,p}|^{r+1} dx\right)^{1/p} \le C|\Omega|^{\frac{1}{p}}\|u_{1,p}\|_{L^\infty(\Omega)}^{\frac{r+1}{p}},$$

from which we deduce (10) since $\displaystyle\lim_{p\to\infty}\frac{p}{p-r(p)-1} = \frac{1}{1-R}$.

2. Since $p > m$, combining the Hölder inequality and Morrey estimates we have

$$\frac{|u_{1,p}(x) - u_{1,p}(y)|}{|x-y|^{1-\frac{n}{m}}} \leq C \left(\int_\Omega |\nabla u_{1,p}|^m \, dx \right)^{1/m} \leq C \, |\Omega|^{\frac{1}{m} - \frac{1}{p}} \left(\int_\Omega |\nabla u_{1,p}|^p \, dx \right)^{1/p}$$
$$= C \, |\Omega|^{\frac{1}{m} - \frac{1}{p}} \left(\int_\Omega |u_{1,p}|^{r+1} \, dx \right)^{1/p} \leq C \, |\Omega|^{\frac{1}{m}} \, \|u_{1,p}\|_{L^\infty(\Omega)}^{\frac{r+1}{p}},$$

where $C > 0$ is a constant independent of m, p. Then, we obtain (11) from (10). $\square$

We then have the following compactness result.

Proposition 5. *Consider the sequence $\{u_{1,p}\}_p$ of solutions of* (3)*. Then, there exists a subsequence $p_i \to \infty$ and a limit function u_1 such that*

$$\lim_{i\to\infty} u_{1,p_i} = u_1$$

uniformly. Moreover, $u_1(x) > 0$ in Ω.

Proof. The existence of u_1 as uniform limit is a consequence of the previous lemma and the Arzelá-Ascoli compactness criteria.

To prove that $u_1 > 0$, we adapt an idea from [3]. The point is to find a subsolution for problem (1) valid for all p large enough.

Consider the eigenvalue problem for the p-laplacian

$$\begin{cases} -\operatorname{div}(|\nabla \varphi_{1,p}|^{p-2} \nabla \varphi_{1,p}) = \lambda_1(p, \Omega) \, |\varphi_{1,p}|^{p-2} \varphi_{1,p} & \text{in } \Omega \\ \quad \varphi_{1,p} = 0 \quad \text{in } \partial\Omega, \end{cases}$$

where $\lambda_1(p, \Omega)$ and $\varphi_{1,p}$ are the principal eigenvalue and eigenfunction respectively. We take every $\varphi_{1,p}$ normalized in such a way that $\|\varphi_{1,p}\|_\infty = 1$.

From [9] and [12] we know that there exists an *infinity eigenvalue* $\Lambda_\infty(\Omega)$ such that

$$\Lambda_\infty(\Omega) = \lim_{p\to\infty} \big(\lambda_1(p, \Omega)\big)^{1/p} = \Big(\max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega) \Big)^{-1}, \tag{13}$$

and that there exist a sequence of eigenfunctions $\{\varphi_{1,p}\}_p$ with $||\varphi_{1,p}||_\infty = 1$ that converge to an *eigenfunction* $\varphi_{1,\infty}$ solving the limit problem

$$\min \{ |\nabla \varphi_{1,\infty}| - \Lambda_\infty \varphi_{1,\infty}, -\Delta_\infty \varphi_{1,\infty} \} = 0 \quad \text{in } \Omega, \qquad \varphi_{1,\infty} = 0 \text{ on } \partial\Omega.$$

in the viscosity sense. Moreover, it is known that $\varphi_{1,\infty} > 0$.

Now, fix $\varepsilon > 0$ and define $\phi_{1,p} = t \, \varphi_{1,p}$, with $t = \min\big\{1, \Lambda_\infty^{\frac{-1}{1-R}} - \varepsilon\big\}$. Then we have $\|\phi_{1,p}\|_\infty = t \, \|\varphi_{1,p}\|_\infty \leq 1$ and, from (13),

$$t \leq \Lambda_\infty^{\frac{-1}{1-R}} - \varepsilon < \lambda_1(p, \Omega)^{\frac{-1}{p-1-r(p)}} \qquad \text{for } p \text{ large enough.}$$

Thus, for that range of p,

$$-\Delta_p \phi_{1,p} = \lambda_1(p, \Omega) \phi_{1,p}^{p-1} \leq \lambda_1(p, \Omega) \, t^{p-1-r(p)} \phi_{1,p}^{r(p)} < \phi_{1,p}^{r(p)},$$

and by [4] we conclude that $\phi_{1,p} \leq u_{1,p}$. Passing to the limit (up to a subsequence)

$$u_1(x) \geq t \varphi_{1,\infty} > 0. \qquad \square$$

Remark 6. The limit function u_1 is Lipschitz continuous since, by letting $p \to \infty$ and then $m \to \infty$ in (11), we obtain

$$\frac{|u_1(x) - u_1(y)|}{|x-y|} \leq C_2.$$

Remark 7. We will prove below that the limit function u_1 is the unique solution of problem (15). As a consequence, not only a subsequence but the whole sequence $u_{1,p}$ converges to u_1. Moreover, for every sequence $\{\lambda_p\}_p$ such that $\lim_{p\to\infty}(\lambda_p)^{1/p} = \Lambda$, we deduce from (2) that the solution $u_{\lambda_p,p}$ of (1) with $\lambda = \lambda_p$ converges to a function u_Λ uniformly in Ω, and that

$$u_\Lambda(x) = \Lambda^{\frac{1}{1-R}} u_1(x). \tag{14}$$

3. The limit problem.

In the present section, we characterize limits of solutions of (3) (elements of Γ_∞) as solutions of a PDE. See [9] and [12] for related results in the eigenvalue case.

Proposition 8. *The limit function u_1 in Proposition* (5) *is a viscosity solution of the problem*

$$\begin{cases} \min\left\{|\nabla v(x)| - v^R(x), -\Delta_\infty v(x)\right\} = 0 \quad \text{in } \Omega, \\ v > 0 \ \text{in } \Omega \\ v = 0 \ \text{on } \partial\Omega. \end{cases} \tag{15}$$

Proof. If the solutions of our problem (1) were of class $\mathcal{C}^2$ the p-laplacian can be expanded to

$$\begin{aligned} \Delta_p u &= \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}\Delta u + (p-2)\,|\nabla u|^{p-4}\langle D^2u\nabla u, \nabla u\rangle \\ &= (p-2)|\nabla u|^{p-4}\left\{\frac{1}{p-2}|\nabla u|^2\Delta u + \langle D^2u\nabla u, \nabla u\rangle\right\}. \end{aligned}$$

But the solutions of our problem are only $\mathcal{C}^{1,\gamma}$, and we have to reinterpret the previous calculation in the viscosity sense.

Consider a point $x_0 \in \Omega$ and a function $\phi \in \mathcal{C}^2(\Omega)$ such that $u_1 - \phi$ has a strict local minimum at x_0. As u_1 is the uniform limit of u_{1,p_i}, there exists a sequence of points $x_i \to x_0$ such that $(u_{1,p_i} - \phi)(x_i)$ is a local minimum for each i. Then, as u_{1,p_i} is a viscosity solution and so a supersolution, we get

$$\begin{aligned} -(p_i-2)|\nabla\phi(x_i)|^{p_i-4}&\left\{\frac{|\nabla\phi(x_i)|^2}{p_i-2}\Delta\phi(x_i) + \langle D^2\phi\,(x_i)\nabla\phi(x_i), \nabla\phi(x_i)\rangle\right\} \\ &= -\Delta_{p_i}\phi(x_i) \geq u_{1,p_i}^{r(p_i)}(x_i). \end{aligned}$$

Rearranging terms, we obtain

$$\begin{aligned} -(p_i-2)\left[\frac{|\nabla\phi(x_i)|}{(u_{1,p_i}(x_i))^{\frac{r(p_i)}{p_i-4}}}\right]^{p_i-4} &\left\{\frac{|\nabla\phi(x_i)|^2}{p_i-2}\ \Delta\phi(x_i)\right. \\ &\left. + \langle D^2\ \phi(x_i)\nabla\phi(x_i), \nabla\phi(x_i)\rangle\right\} \geq 1. \end{aligned}$$

Notice that $u_1(x_0) > 0$ by Proposition 5. Then, if we suppose that

$$|\nabla\phi(x_0)| < u_1^R(x_0)$$

we obtain a contradiction letting $i \to \infty$ in the previous inequality. Thus, it is

$$|\nabla\phi(x_0)| - u_1^R(x_0) \geq 0. \tag{16}$$

We also have that

$$-\Delta_\infty\phi(x_0) = -\langle D^2\phi(x_0)\nabla\phi(x_0), \nabla\phi(x_0)\rangle \geq 0, \tag{17}$$

because we would get a contradiction otherwise.

Therefore, we can put together (16) and (17) writing

$$\min\left\{|\nabla\phi(x_0)| - u_1^R(x_0), -\Delta_\infty\phi(x_0)\right\} \geq 0.$$

Hence, we conclude that u_1 is a viscosity supersolution of equation (15).

It remains to be shown that u_1 is a viscosity subsolution of the limit equation (15), i.e. we have to show that, for each $x_0 \in \Omega$ and $\phi \in \mathcal{C}^2(\Omega)$ such that $u_1 - \phi$ attains a strict local maximum at x_0 (note that x_0 and ϕ are not the same than before) we have

$$\min\left\{|\nabla\phi(x_0)| - u_1^R(x_0), -\Delta_\infty\phi(x_0)\right\} \leq 0.$$

We can suppose that

$$|\nabla\phi(x_0)| > u_1^R(x_0),$$

because otherwise, we are done. As we did before, the uniform convergence of u_{1,p_i} to u_1 provides us with a sequence of points $x_i \to x_0$ which are local maxima of $u_{1,p_i} - \phi$. Recalling the definition of viscosity subsolution we have

$$-(p_i-2)\left[\frac{|\nabla\phi(x_i)|}{(u_{1,p_i}(x_i))^{\frac{r(p_i)}{p_i-4}}}\right]^{p_i-4}\left\{\frac{|\nabla\phi(x_i)|^2}{p_i-2}\,\Delta\phi(x_i)\right.$$
$$\left.+\langle D^2\,\phi(x_i)\nabla\phi(x_i),\nabla\phi(x_i)\rangle\right\} \leq 1,$$

for each fixed p. Letting $i \to \infty$ we obtain $-\Delta_\infty\phi(x_0) \leq 0$ because in other case we get a contradiction. □

In the following Lemma we show that u_Λ given by relation (14) is a viscosity solution of a reescaled version of (15).

Lemma 9. *The solutions of the problem*

$$\text{(18)} \qquad \begin{cases} \min\left\{|\nabla u_\Lambda(x)| - \Lambda\, u_\Lambda^R(x), -\Delta_\infty u_\Lambda(x)\right\} = 0 & \text{in } \Omega \\ u_\Lambda = 0 \quad \text{in } \partial\Omega, \end{cases}$$

and those of the problem

$$\text{(19)} \qquad \begin{cases} \min\left\{|\nabla u_1(x)| - u_1^R(x), -\Delta_\infty u_1(x)\right\} = 0 & \text{in } \Omega \\ u_1 = 0 \quad \text{in } \partial\Omega, \end{cases}$$

are related by the expression $u_\Lambda(x) = \Lambda^{\frac{1}{1-R}}\, u_1(x)$.

We omit the proof since it is standard.

4. Uniqueness of solutions for the limit problem.

The main result in the present section is the following Comparison Principle for equation (15), from which we deduce uniqueness of positive solutions of the limit problem.

Notice that then the whole sequence $u_{1,p}$ converges uniformly to u_1 as $p \to \infty$. Hence, we deduce from (2) that the whole sequence $u_{\lambda_p,p}$ converges uniformly to some u_Λ provided $\lim\limits_{p\to\infty}(\lambda_p)^{1/p} = \Lambda$ (see Remark 7). Lemma 9 then implies that u_Λ is the unique positive viscosity solution of (18).

Theorem 10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and consider a subsolution u and a supersolution v of*

$$\text{(20)} \qquad \min\left\{|\nabla w(x)| - w^R(x), -\Delta_\infty w(x)\right\} = 0 \quad \text{in } \Omega.$$

Suppose that both, u and v are strictly positive in Ω, continuous up to the boundary and satisfy $u \leq v$ on $\partial\Omega$. Then, $u \leq v$ in $\overline{\Omega}$.

Remark 11. Every nontrivial solution u of (19) is $\infty-$superharmonic. The Harnack inequality for $\infty-$superharmonic functions (see [14] and [15]) implies $u > 0$ inside Ω.

Once we have proved the above Comparison Principle, we deduce from Remark 11 that there exists a unique solution to problem (19). Hence, from Lemma 9 the set Γ_∞ is a smooth curve of solutions parametrized by Λ. Notice that we can suppose $R \neq 0$, since the case $R = 0$ has been already studied in [10] and [11].

Our aim is to study the uniqueness of solutions for the problem (19) following the techniques in [7]. Equation in (19) can be written as $F(u, \nabla u, D^2 u) = 0$, where F is given by

$$\begin{array}{rcl} F : \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n & \longrightarrow & \mathbb{R} \\ (r, p, X) & \longrightarrow & \min\left\{|p| - r^R, -\langle Xp, p\rangle\right\} \end{array}$$

which, in the notation of [7], is *degenerate elliptic* in the sense that $F(r, p, X) \leq F(r, p, Y)$ whenever $Y \leq X$, but not *proper*, namely, it is non-increasing in r.

Thus, equation (19) is not in the framework of [7]. Nevertheless, it is possible to transform our equation into a valid one by mean of a change of variables. Following [12], we prove the next lemma.

Lemma 12. *Let u be a strictly positive supersolution (subsolution) of problem* (19) *in Ω. Then, $v(x) = (1-R)^{-1} u^{1-R}(x)$ is a viscosity supersolution (subsolution) of*

$$\min\left\{|\nabla v(x)| - 1, -\Delta_\infty v(x) - \frac{R}{1-R}\left(\frac{|\nabla v(x)|^4}{v(x)}\right)\right\} = 0, \tag{21}$$

in every Ω^* such that $\overline{\Omega^*} \subset \Omega$.

Proof. Let $\phi \in \mathcal{C}^2(\Omega)$ be a function touching v from below at $x_0 \in \Omega$. We define $\Phi(x) = \Big((1-R)\,\phi(x)\Big)^{\frac{1}{1-R}}$ which touches u from below at x_0. Notice that $\Phi(x)$ is $\mathcal{C}^2$ in a neighborhood of x_0 since $u > 0$ in Ω implies $\phi(x) > 0$ near x_0. We can compute the derivatives of $\Phi(x)$ in terms of those of $\phi(x)$

$$\begin{aligned} \nabla\Phi(x_0) &= \Big((1-R)\,\phi(x_0)\Big)^{\frac{R}{1-R}} \nabla\phi(x_0), \\ D^2\Phi(x_0) &= \Big((1-R)\,\phi(x_0)\Big)^{\frac{R}{1-R}} D^2\phi(x_0) \\ &\qquad + R\Big((1-R)\,\phi(x_0)\Big)^{\frac{2R-1}{1-R}} \nabla\phi(x_0) \otimes \nabla\phi(x_0), \end{aligned}$$

As u is a viscosity solution of (19), we have that

$$\begin{aligned} 0 &\leq \min\left\{|\nabla\Phi(x_0)| - \Phi^R(x_0), -\langle D^2\Phi(x_0)\nabla\Phi(x_0), \nabla\Phi(x_0)\rangle\right\} \\ &\leq \min\Bigg\{\Big((1-R)\,\phi(x_0)\Big)^{\frac{R}{1-R}}\Big(|\nabla\phi(x_0)| - 1\Big), \\ &\qquad -\Big((1-R)\,\phi(x_0)\Big)^{\frac{3R}{1-R}}\left(\Delta_\infty\phi(x_0) + \frac{R}{1-R}\frac{|\nabla\phi(x_0)|^4}{\phi(x_0)}\right)\Bigg\}. \end{aligned}$$

Notice that, $\phi(x_0) = v(x_0) > 0$. We deduce that

$$\Big((1-R)\,\phi(x_0)\Big)^{\frac{R}{1-R}}\Big(|\nabla\phi(x_0)| - 1\Big) \geq 0, \quad \text{and}$$

$$-\Big((1-R)\,\phi(x_0)\Big)^{\frac{3R}{1-R}}\left(\Delta_\infty\phi(x_0) + \frac{R}{1-R}\frac{|\nabla\phi(x_0)|^4}{\phi(x_0)}\right) \geq 0.$$

It follows that

$$\min\left\{|\nabla\phi(x_0)| - 1, -\Delta_\infty\phi(x_0) - \frac{R}{1-R}\frac{|\nabla\phi(x_0)|^4}{\phi(x_0)}\right\} \geq 0.$$

We have proved that v is a viscosity supersolution of (21). The subsolution case is analogous. $\square$

Equation (21) is given by the function

$$\begin{array}{rcl} \tilde{F} : \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{S}^n & \longrightarrow & \mathbb{R} \\ (r, p, X) & \longrightarrow & \min\left\{ |p| - 1, -\langle Xp, p\rangle - \dfrac{R}{1-R}\, \dfrac{|p|^4}{r} \right\}, \end{array} \tag{22}$$

which is both degenerate elliptic and proper. Notice that the equation is singular in $r = 0$ but, as we are dealing with positive solutions, there are no sign changes in the interior of Ω, and we can proceed.

As it is pointed out in [7, Section 5.C], it is possible to establish comparison when either u or v are strict. This is not our case, but we are going to show that it is possible to construct strict supersolutions of (21) starting from any positive supersolution. In [11] and [12] we find related constructions in the eigenvalue case.

Lemma 13. *Let $v(x)$ be a strictly positive viscosity supersolution of (21) in Ω^* such that $\overline{\Omega^*} \subset \Omega$. Then,*

$$\tilde{v}(x) = a \cdot \big(v(x) + \kappa\big), \tag{23}$$

is a strict supersolution of the same equation for any fixed $a > 1$ and $\kappa > 0$.

Proof. Let $\phi \in \mathcal{C}^2$ be a function touching $\tilde{v}(x)$ from below in some $x_0 \in \Omega^*$. We define

$$\Phi(x) = \frac{1}{a}\, \phi(x) - \kappa,$$

which clearly touches $v(x)$ from below in x_0. We can compute the derivatives of $\Phi(x)$ in terms of those of $\phi(x)$, this is

$$\nabla\Phi(x_0) = a^{-1}\,\nabla\phi(x_0) \quad \text{and} \quad D^2\Phi(x_0) = a^{-1}\, D^2\phi(x_0). \tag{24}$$

Since $v(x)$ is a viscosity supersolution of (21) in Ω^*, we deduce

$$|\nabla\Phi(x_0)| - 1 \geq 0, \tag{25}$$

and

$$-\langle D^2\Phi(x_0)\nabla\Phi(x_0), \nabla\Phi(x_0)\rangle - \frac{R}{1-R}\, \frac{|\nabla\Phi(x_0)|^4}{v(x_0)} \geq 0. \tag{26}$$

From (24) and (25), we have

$$|\nabla\phi(x_0)| - 1 = a\,|\nabla\Phi(x_0)| - 1 \geq a - 1, \tag{27}$$

and from (24) and (26) we get

$$\begin{aligned} -\langle D^2\phi(x_0)\nabla\phi(x_0),\nabla\phi(x_0)\rangle - \frac{R}{1-R}\, \frac{|\nabla\phi(x_0)|^4}{\tilde{v}(x_0)} &\geq \\ \geq a^3 \frac{R}{1-R}\, |\nabla\Phi(x_0)|^4 \left(\frac{1}{v(x_0)} - \frac{1}{v(x_0)+\kappa} \right) \end{aligned}$$

All the terms involved in the last expression are positive. From (25) and $a > 1$ we get

$$\begin{aligned} -\langle D^2\phi(x_0)\nabla\phi(x_0), \nabla\phi(x_0)\rangle - \frac{R}{1-R}\, \frac{|\nabla\phi(x_0)|^4}{\tilde{v}(x_0)} &\geq \\ \geq a^3 \frac{R}{1-R} \left(\frac{\kappa}{v(x_0)\,(v(x_0)+\kappa)} \right) \geq \frac{R}{1-R} \left(\frac{\kappa}{\|v\|_\infty\,(\|v\|_\infty + \kappa)} \right). \end{aligned} \tag{28}$$

Finally, we can put together (27) and (28) in the following way

$$\begin{aligned} \min\ &\left\{ |\nabla\phi(x_0)| - 1, -\langle D^2\phi(x_0)\nabla\phi(x_0), \nabla\phi(x_0)\rangle - \frac{R}{1-R}\, \frac{|\nabla\phi(x_0)|^4}{\tilde{v}(x_0)} \right\} \\ &\geq \min\left\{ a - 1, \frac{R}{1-R} \left(\frac{\kappa}{\|v\|_\infty\,(\|v\|_\infty + \kappa)} \right) \right\} = \mu(a, \kappa, R, \|v\|_\infty) > 0. \quad \square \end{aligned}$$

Remark 14. For any $a(\varepsilon) > 1$ such that $a(\varepsilon) \to 1$, and $\kappa(\varepsilon)$ such that $\kappa(\varepsilon) \to 0$ (for example $a(\varepsilon) = 1 + \varepsilon$ and $\kappa(\varepsilon) = \varepsilon$), expression (23) defines a family of approximations of the identity as $\varepsilon \to 0$ since

$$\|\tilde{v} - v\|_{L^\infty(\Omega^*)} \leq (a(\varepsilon) - 1)\, \|v\|_{L^\infty(\Omega^*)} + a(\varepsilon)\kappa(\varepsilon).$$

We shall use the comparison principle for semicontinuous functions, due to M.G. Crandall and H. Ishii, which can be find in [6] and in the Appendix of [7]. We include here a simplified version

Lemma 15. *Let $\tau > 0$. Let $u, -v$ be real-valued, upper-semicontinuous functions in Ω, and $(x_\tau, y_\tau) \in \Omega \times \Omega$ be a local maximum point of the function $u(x) - v(y) - \frac{\tau}{2}|x - y|^2$. Then, there exist symmetric matrices X_τ and Y_τ such that*

$$\big(\tau(x_\tau - y_\tau), X_\tau\big) \in \overline{J}^{2+} u(x_\tau) \qquad \text{and} \qquad \big(\tau(x_\tau - y_\tau), Y_\tau\big) \in \overline{J}^{2-} v(y_\tau),$$

where $\overline{J}^{2+} u(x_\tau)$ and $\overline{J}^{2-} v(y_\tau)$ are the closures of the upper and lower semijets of u and v, respectively (see [7]). We also have

$$-3\tau \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_\tau & 0 \\ 0 & -Y_\tau \end{pmatrix} \leq 3\tau \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{29}$$

We shall also need the following lemma, whose proof can be find in [7, Proposition 3.7].

Lemma 16. *Suppose that u and $-v$ are real-valued, upper-semicontinuous functions in $\overline{\Omega}$. For every $\tau > 0$, we denote*

$$M_\tau = \sup_{\overline{\Omega}\times\overline{\Omega}} \left(u(x) - v(y) - \frac{\tau}{2}\,|x - y|^2 \right).$$

If $(x_\tau, y_\tau) \in \overline{\Omega} \times \overline{\Omega}$ is such that $M_\tau = u(x_\tau) - v(y_\tau) - \frac{\tau}{2}\,|x_\tau - y_\tau|^2$, then
1. $\lim_{\tau \to \infty} \tau |x_\tau - y_\tau|^2 = 0$.
2. $\lim_{\tau \to \infty} M_\tau = u(\hat{x}) - v(\hat{x}) = \sup_{x \in \overline{\Omega}} (u(x) - v(x))$ *whenever $\hat{x}$ is a limit point of x_τ.*

Now, we are ready to start with the proof of the Comparison Principle.

Proof of Theorem 10. As $u - v \in \mathcal{C}(\overline{\Omega})$ and $\overline{\Omega}$ is compact, $u - v$ attains a maximum at $\overline{\Omega}$. In order to arrive at a contradiction, we suppose that $\max_{\overline{\Omega}}(u - v) > 0$. Consider

$$\tilde{u}(x) = \frac{u^{1-R}(x)}{1 - R} \qquad \text{and} \qquad \tilde{v}(x) = \frac{v^{1-R}(x)}{1 - R}.$$

We have proved in Lemma 12 that $\tilde{u}$ and $\tilde{v}$ are respectively a subsolution and a supersolution of (21) in every open Ω^* such that $\overline{\Omega^*} \subset \Omega$. Moreover, Lemma 13 implies that

$$\tilde{v}_\varepsilon(x) = (1 + \varepsilon) \cdot \big(\tilde{v}(x) + \varepsilon\big),$$

is a strict supersolution of (21) in Ω^*. Notice that, since $(u - v)|_{\partial\Omega} \leq 0$

$$\tilde{u} - \tilde{v}_\varepsilon = \tilde{u} - (1 + \varepsilon)\, \tilde{v} - (1 + \varepsilon)\, \varepsilon < 0 \qquad \text{on } \partial\Omega.$$

Moreover, Remark 14 implies $\max_{\overline{\Omega}}(\tilde{u} - \tilde{v}_\varepsilon) > 0$ for ε small enough. Thus, we can suppose that Ω^* contains all the maximum points of $\tilde{u} - \tilde{v}_\varepsilon$ for ε fixed.

Now, for each $\tau > 0$, let (x_τ, y_τ) be a maximum point of $\tilde{u}(x) - \tilde{v}_\varepsilon(y) - \frac{\tau}{2}|x - y|^2$ in $\overline{\Omega} \times \overline{\Omega}$. By the compactness of $\overline{\Omega}$, we can suppose that $x_\tau \to \hat{x}$ as $\tau \to \infty$ for some $\hat{x} \in \overline{\Omega}$ (notice that also $y_\tau \to \hat{x}$). Lemma 16 implies that $\hat{x}$ is a maximum point of $\tilde{u} - \tilde{v}_\varepsilon$ and, consequently, it is an interior point of Ω^*. We also have

$$\lim_{\tau \to \infty} \left(\tilde{u}(x_\tau) - \tilde{v}_\varepsilon(y_\tau) - \frac{\tau}{2}|x_\tau - y_\tau|^2 \right) = \tilde{u}(\hat{x}) - \tilde{v}_\varepsilon(\hat{x}) > 0.$$

Thus, for τ large enough, we have that both x_τ and y_τ are interior points of Ω^* and

$$\tilde{u}(x_\tau) - \tilde{v}_\varepsilon(y_\tau) - \frac{\tau}{2}|x_\tau - y_\tau|^2 > 0. \tag{30}$$

Applying lemma 15, there exist two symmetric matrices X_τ, Y_τ such that

$$\big(\tau(x_\tau - y_\tau), X_\tau\big) \in \overline{J}^{2+}\tilde{u}(x_\tau), \quad \text{and} \quad \big(\tau(x_\tau - y_\tau), Y_\tau\big) \in \overline{J}^{2-}\tilde{v}_\varepsilon(y_\tau),$$

and

$$\langle X_\tau \xi, \xi\rangle - \langle Y_\tau \eta, \eta\rangle \le 3\tau\,|\xi - \eta|^2 \quad \forall \xi, \eta \in \mathbb{R}^n. \tag{31}$$

Thus, we have

$$\min\left\{\tau\,|x_\tau - y_\tau| - 1, -\tau^2\langle X_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle - \frac{R}{1-R}\,\frac{\tau^4|x_\tau - y_\tau|^4}{\tilde{u}(x_\tau)}\right\} \le 0.$$

and

$$\min\left\{\tau\,|x_\tau - y_\tau| - 1, -\tau^2\langle Y_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle - \frac{R}{1-R}\,\frac{\tau^4|x_\tau - y_\tau|^4}{\tilde{v}_\varepsilon(y_\tau)}\right\}$$
$$\ge \min\left\{\varepsilon, \frac{R}{1-R}\left(\frac{\varepsilon}{\|v\|_\infty\,(\|v\|_\infty + \varepsilon)}\right)\right\} = \mu(\varepsilon, R, \|v\|_\infty) > 0.$$

Subtracting the first equation from the second one, we have

$$0 < \mu(\varepsilon, R, \|v\|_\infty) \le \min\Big\{\tau\,|x_\tau - y_\tau| - 1,$$
$$-\tau^2\langle Y_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle - \frac{R}{1-R}\,\frac{\tau^4|x_\tau - y_\tau|^4}{\tilde{v}_\varepsilon(y_\tau)}\Big\} \tag{32}$$
$$-\min\Big\{\tau\,|x_\tau - y_\tau| - 1,$$
$$-\tau^2\langle X_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle - \frac{R}{1-R}\,\frac{\tau^4|x_\tau - y_\tau|^4}{\tilde{u}(x_\tau)}\Big\}. \tag{33}$$

Now, there are four cases to be considered depending on the values attained in the minima (32) and (33). The key point in the foregoing is that we have $\tilde{v}_\varepsilon(y_\tau) \le \tilde{u}(x_\tau)$ and $X_\tau \le Y_\tau$ from (30) and (31) respectively.

(1) Both minima are equal to $\tau\,|x_\tau - y_\tau| - 1$ and the difference is 0.

(2) The minimum at (33) is $-\tau^2\langle X_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle - \frac{R}{1-R}\,\frac{\tau^4|x_\tau - y_\tau|^4}{\tilde{u}(x_\tau)}$ and the one in (32) is $\tau\,|x_\tau - y_\tau| - 1$. Actually, it is impossible for this alternative to hold unless both quantities are equal because

$$\begin{aligned}\tau\,|x_\tau - y_\tau| - 1 &= \min(32) \le \\ &\le -\tau^2\langle Y_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle - \frac{R}{1-R}\,\frac{\tau^4|x_\tau - y_\tau|^4}{\tilde{v}_\varepsilon(y_\tau)} \\ &\le -\tau^2\langle X_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle - \frac{R}{1-R}\,\frac{\tau^4|x_\tau - y_\tau|^4}{\tilde{u}(x_\tau)} \\ &= \min(33) \le \tau\,|x_\tau - y_\tau| - 1.\end{aligned}$$

(3) The minimum in (32) is $-\tau^2\langle Y_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle - \frac{R}{1-R}\,\frac{\tau^4|x_\tau - y_\tau|^4}{\tilde{v}_\varepsilon(y_\tau)}$ and the one in (33) is $\tau\,|x_\tau - y_\tau| - 1$. Using the fact that

$$-\tau^2\langle Y_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle - \frac{R}{1-R}\,\frac{\tau^4|x_\tau - y_\tau|^4}{\tilde{v}_\varepsilon(y_\tau)} \le \tau\,|x_\tau - y_\tau| - 1,$$

we find out that $\min(32) - \min(33) \le 0$.

(4) The minimum in (32) is $-\tau^2\langle Y_\tau(x_\tau-y_\tau),(x_\tau-y_\tau)\rangle-\frac{R}{1-R}\frac{\tau^4|x_\tau-y_\tau|^4}{\tilde{v}_\varepsilon(y_\tau)}$ and the one in (33) is $-\tau^2\langle X_\tau(x_\tau-y_\tau),(x_\tau-y_\tau)\rangle-\frac{R}{1-R}\frac{\tau^4|x_\tau-y_\tau|^4}{\tilde{u}(x_\tau)}$. Therefore

$$\begin{aligned}\min(32)-\min(33) &= -\tau^2\big\langle (Y_\tau-X_\tau)(x_\tau-y_\tau),(x_\tau-y_\tau)\big\rangle\\ &\quad -\frac{R}{1-R}\tau^4|x_\tau-y_\tau|^4\left(\frac{1}{\tilde{v}_\varepsilon(y_\tau)}-\frac{1}{\tilde{u}(x_\tau)}\right)\le 0.\end{aligned}$$

Thus, all the alternatives lead to

$$0<\mu(\varepsilon,R,\|v\|_\infty)\le \min(32)-\min(33)\le 0$$

which is a contradiction. □

5. Estimates for $\|u_\Lambda\|_{L^\infty}$.

Our goal in this section is to prove estimates for the limit function u_Λ. To this aim, we consider the family of problems

$$\min\big\{|\nabla u(x)|-\Lambda_\infty u^R(x),-\Delta_\infty u(x)\big\}=0\quad\text{in }\Omega,\quad u|_{\partial\Omega}=0 \tag{34}$$

for $R\in[0,1]$, where

$$\Lambda_\infty(\Omega)=\big(\operatorname{maxdist}_{x\in\Omega}(x,\partial\Omega)\big)^{-1} \tag{35}$$

is the principal ∞-eigenvalue (see [12]). It is known (consult [10] and [11, Lemma 6.10]) that the unique solution to problem (34) when $R=0$ is

$$\delta(x)=\frac{\operatorname{dist}(x,\partial\Omega)}{\operatorname{maxdist}_{x\in\Omega}(x,\partial\Omega)}.$$

For $R=1$, we consider the maximal $\infty-$eigenfunction v with $\|v\|_{L^\infty}=1$. Then, we have the following result.

Proposition 17. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain and $R<1$. Consider $\Lambda>0$ and u_Λ the nontrivial solution of*

$$\min\big\{|\nabla u_\Lambda(x)|-\Lambda u_\Lambda^R(x),-\Delta_\infty u_\Lambda(x)\big\}=0\quad\text{in }\Omega,\quad u_\Lambda|_{\partial\Omega}=0.$$

Then we have

$$\Big(\Lambda\cdot\max_{y\in\Omega}\operatorname{dist}(y,\partial\Omega)\Big)^{\frac{1}{1-R}}v(x)\le u_\Lambda(x)\le\Big(\Lambda\cdot\max_{y\in\Omega}\operatorname{dist}(y,\partial\Omega)\Big)^{\frac{1}{1-R}}\delta(x), \tag{36}$$

for every $x\in\Omega$, where $v(x)$ and $\delta(x)$ are defined above. Indeed,

$$\|u_\Lambda\|_{L^\infty(\Omega)}=\Big(\Lambda\cdot\max_{x\in\Omega}\operatorname{dist}(x,\partial\Omega)\Big)^{\frac{1}{1-R}}. \tag{37}$$

Lemma 18. *For every $R\in(0,1)$, the functions $v(x)$ and $\delta(x)$ are respectively a sub- and a supersolution of problem* (34).

Once we have proved Lemma 18, Proposition 17 follows easily from the Comparison Principle. Indeed, by comparison, for every fixed $R\in(0,1)$ we have

$$v(x)\le u_{\Lambda_\infty}(x)\le\delta(x)\qquad\forall x\in\Omega,$$

where u_{Λ_∞} is the solution of (34). Formula (14) implies

$$u_\Lambda(x)=\big(\Lambda\cdot\Lambda_\infty^{-1}\big)^{\frac{1}{1-R}}\ u_{\Lambda_\infty}(x), \tag{38}$$

and then we can combine both expressions above and (35) to get (36), from which we deduce (37) since $\|v\|_{L^\infty(\Omega)}=\|\delta\|_{L^\infty(\Omega)}=1$.

Proof of Lemma 18. 1. Consider a point $x_0 \in \Omega$ and a function $\phi \in \mathcal{C}^2$ such that $(v - \phi)$ has a maximum in x_0. As v is an $\infty-$eigenfunction, it satisfies

$$\min\{|\nabla\phi(x_0)| - \Lambda_\infty v(x_0), -\Delta_\infty\phi(x_0)\} \leq 0 \quad \text{in } \Omega.$$

We can consider $-\Delta_\infty\phi(x_0) > 0$ and $|\nabla\phi(x_0)| - \Lambda_\infty v(x_0) \leq 0$ since we are done otherwise. Clearly

$$|\nabla\phi(x_0)| - \Lambda_\infty v(x_0)^R \leq \Lambda_\infty\big(v(x_0) - v(x_0)^R\big) \leq 0,$$

since $\|v\|_\infty = 1$, and then

$$\min\big\{|\nabla\phi(x_0)| - \Lambda_\infty v(x_0)^R, -\Delta_\infty\phi(x_0)\big\} \leq 0 \quad \text{in } \Omega.$$

2. Now, consider a point $x_0 \in \Omega$ and a function $\phi \in \mathcal{C}^2$ such that $(\delta - \phi)$ has a minimum in x_0. As δ is a solution of problem (34) when $R = 0$, in particular it satisfies

$$\min\{|\nabla\phi(x_0)| - \Lambda_\infty, -\Delta_\infty\phi(x_0)\} \geq 0 \quad \text{in } \Omega.$$

Then $-\Delta_\infty\phi(x_0) \geq 0$ and $|\nabla\phi(x_0)| \geq \Lambda_\infty$ and we have

$$|\nabla\phi(x_0)| - \Lambda_\infty\delta(x_0)^R \geq \Lambda_\infty\big(1 - \delta(x_0)^R\big) \geq 0$$

since $\|\delta\|_\infty = 1$. Thus, we conclude

$$\min\big\{|\nabla\phi(x_0)| - \Lambda_\infty\delta(x_0)^R, -\Delta_\infty\phi(x_0)\big\} \geq 0 \quad \text{in } \Omega. \qquad \square$$

6. Explicit solutions of the limit problem.

Now, we are going to compute the explicit solution of the limit problem for a class of domains including the ball and the annulus among others. Following [11], we define the ridge set of Ω as

$$\begin{aligned}\mathcal{R} &= \{x \in \Omega : \text{dist}(x, \partial\Omega) \text{ is not differentiable at } x\}\\ &= \{x \in \Omega :\ \exists x_1, x_2 \in \partial\Omega,\ x_1 \neq x_2,\ \text{s.t. } |x - x_1| = |x - x_2| = \text{dist}(x, \partial\Omega)\}\end{aligned}$$

and its subset $\mathcal{M} = \{x \in \Omega :\ \text{dist}(x, \partial\Omega) = \text{maxdist}_{x\in\Omega}(x, \partial\Omega)\}$, the set of maximal distance. We have

Proposition 19. *Given $R \in (0,1)$ and $\Lambda > 0$, if $\mathcal{M} \equiv \mathcal{R}$, then*

$$u_\Lambda(x) = \Lambda^{\frac{1}{1-R}} \cdot \big(\max_{x\in\Omega} \text{dist}(x, \partial\Omega)\big)^{\frac{R}{1-R}} \cdot \text{dist}(x, \partial\Omega), \tag{39}$$

is the unique positive solution of

$$\min\big\{|\nabla u_\Lambda(x)| - \Lambda\, u_\Lambda^R(x), -\Delta_\infty u_\Lambda(x)\big\} = 0 \quad \text{in } \Omega, \quad u_\Lambda|_{\partial\Omega} = 0. \tag{40}$$

Proof. From Proposition 17, we have that

$$\Big(\Lambda \cdot \max_{y\in\Omega} \text{dist}(y, \partial\Omega)\Big)^{\frac{1}{1-R}} v(x) \leq u_\Lambda(x) \leq \Big(\Lambda \cdot \max_{y\in\Omega} \text{dist}(y, \partial\Omega)\Big)^{\frac{1}{1-R}} \delta(x), \quad \forall x \in \Omega,$$

where $v(x)$ and $\delta(x)$ are the same as in the previous section. Notice that we have this inequality without any hypothesis on Ω.

It is a well-known result (see [11] and [13] for example) that if $\mathcal{M} = \mathcal{R}$, then $v(x) = \delta(x)$ is the maximal solution of the $\infty-$eigenvalue problem with $\|v\|_{L^\infty} = 1$, from which we deduce (39). $\square$

Remark 20. In fact, (39) defines a branch of solutions of (40) even if $R > 1$ whenever $\mathcal{M} \equiv \mathcal{R}$. Indeed, as a consequence of [11, Lemma 6.10], we have

$$\begin{aligned} 0 &= \min\Big\{|\nabla \mathrm{dist}(x,\partial\Omega)| - 1, -\Delta_\infty \mathrm{dist}(x,\partial\Omega)\Big\} \\ &= \min\Big\{\Lambda^{\frac{1}{1-R}} \cdot \max_{x\in\Omega} \mathrm{dist}(x,\partial\Omega)^{\frac{R}{1-R}} \cdot \big(|\nabla \mathrm{dist}(x,\partial\Omega)| - 1\big), \\ &\qquad \Lambda^{\frac{3}{1-R}} \cdot \max_{x\in\Omega} \mathrm{dist}(x,\partial\Omega)^{\frac{3R}{1-R}} \cdot \big(-\Delta_\infty \mathrm{dist}(x,\partial\Omega)\big)\Big\} \\ &= \min\Big\{|\nabla u_\Lambda(x)| - \Lambda\, u_\Lambda^R(x), -\Delta_\infty u_\Lambda(x)\Big\} \qquad \forall x \in \mathcal{M}, \end{aligned}$$

while u is $\infty-$harmonic and satisfies $|\nabla u| - \Lambda\, u^R > 0$ outside $\mathcal{R}$ (the proof is analogous to that of the "$\Lambda-$lemma" in [13]).

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Departamento de Matematicas, U. Autonoma de Madrid, 28049 Madrid, Spain.
E-mail address: `fernando.charro@uam.es, ireneo.peral@uam.es`