Game Theory and the Infinity Laplacian

1. Let $p \geq 2$ and consider the **normalized p-Laplacian** operator

$$\Delta_p^N v = |\nabla v|^{2-p} \operatorname{div}\left(|\nabla v|^{p-2} \nabla v\right) = \Delta v + (p-2)\Delta_\infty v,$$

where $\Delta_\infty v = |\nabla v|^{-2} \sum_{i,j} v_{x_i x_j} v_{x_i} v_{x_j}$ is the normalized infinity Laplacian.

(i) Find $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ such that for every $u \in \mathcal{C}^2(\Omega)$ such that

$$-\Delta_p^N u = 0 \qquad \text{in } \Omega \subset \mathbb{R}^n$$

(where Ω is a smooth domain) the following asymptotic mean value formula holds

$$u(x) = \frac{\alpha}{2}\left(\sup_{y \in \overline{B}_\epsilon(x)} u(y) + \inf_{y \in \overline{B}_\epsilon(x)} u(y)\right) + \beta \fint_{B_\epsilon(x)} u(y)\, dy + o(\epsilon^2), \quad \text{as } \epsilon \to 0.$$

You can assume $\nabla u \neq 0$ in Ω for simplicity.

(ii) Describe a game with the following dynamic programming principle

$$\begin{cases} u(x) = \dfrac{\alpha}{2}\left(\sup\limits_{y \in \overline{B}_\epsilon(x)} u(y) + \inf\limits_{y \in \overline{B}_\epsilon(x)} u(y)\right) + \beta \fint_{B_\epsilon(x)} u(y)\, dy & \text{in } \Omega \\ u(x) = F(x) \qquad \text{on } \partial\Omega \end{cases}$$

with $\alpha, \beta \in [0,1]$, $\alpha + \beta = 1$.

2. Two players play **random Tug-of-War** with step ϵ on a domain $\Omega \subset \mathbb{R}^n$ without running payoff but with terminal payoff function $F : \partial\Omega \to \mathbb{R}$. Suppose they follow all the usual rules, except that they use a **biased (unfair) coin** with a probability $p \in (0,1)$ of Player I winning the coin toss at each turn.

(i) Find the Dynamic Programming Principle for this ϵ-game.

(ii) Is there any choice of the probability p in terms of ϵ for which a PDE of the type

$$-\Delta_\infty u + \beta\, |\nabla u| = 0, \qquad \beta \in \mathbb{R}$$

emerges in the limit as $\epsilon \to 0$? You can assume $u \in \mathcal{C}^2$ and $\nabla u \neq 0$ in Ω for simplicity.

3. A function $u \in \mathcal{C}(\Omega)$ is infinity harmonic if an only if

$$-\Delta_\infty u = 0 \quad \text{in } \Omega \subset \mathbb{R}^n \tag{1}$$

in the viscosity sense. It can be checked that if for every $x \in \Omega$ we have

$$u(x) = \frac{1}{2}\left(\max_{y \in B_\epsilon(x)} u(y) + \min_{y \in B_\epsilon(x)} u(y)\right) + o(\epsilon^2) \qquad \text{as } \epsilon \to 0, \tag{2}$$

then u is infinity harmonic (in the viscosity sense).

Prove that the converse does not hold, that is, find an infinity harmonic function $u \in \mathcal{C}(\Omega)$ that violates the asymptotic mean value property (2).

Hint: Notice that u cannot be of class $\mathcal{C}^2(\Omega)$, since a classical solution of (1) must satisfy (2). What explicit examples of "truly viscosity" infinity harmonic functions do you know from the literature?

Remark: A function $u \in \mathcal{C}(\Omega)$ is infinity harmonic in the viscosity sense if and only if it satisfies the asymptotic mean value property (2) *in the viscosity sense.*

4. Let $1 < q < \infty$ and $x \in \mathbb{R}^n$. Then, we define its ℓ^q-norm as $|x|_q = \left(\sum_{i=1}^n |x_i|^q\right)^{1/q}$, with $q = 2$ the usual Euclidean case. Two players play **random Tug-of-War** on a domain $\Omega \subset \mathbb{R}^n$ without running payoff but with terminal payoff function $F : \partial\Omega \to \mathbb{R}$. Suppose they follow all the usual rules, except that instead of choosing the next game position from an Euclidean ball of radius ϵ, they pick the new game position from an ℓ^q**-ball of radius** ϵ, namely,

$$x_{k+1} \in B_\epsilon^{\ell^q}(x_k) = \left\{x \in \mathbb{R}^n : \ |x - x_k|_q \le \epsilon\right\}, \qquad 1 < q < \infty.$$

(i) Find the Dynamic Programming Principle that the ϵ-value u_ϵ satisfies.

(ii) For $u_\epsilon \in \mathcal{C}^2$ and x_0 such that $\nabla u_\epsilon(x_0) \neq 0$. Show that the points $x^\epsilon_{\max}, x^\epsilon_{\min} \in \overline{B_\epsilon^{\ell^q}}$ such that

$$u_\epsilon(x^\epsilon_{\max}) = \max_{y \in \overline{B_\epsilon^{\ell^q}}} u_\epsilon(y), \qquad u_\epsilon(x^\epsilon_{\min}) = \min_{y \in \overline{B_\epsilon^{\ell^q}}} u_\epsilon(y)$$

are given by

$$x^\epsilon_{\max} = x_0 + \epsilon\left[J\big(\nabla u_\epsilon(x_0)\big) + o(1)\right], \qquad x^\epsilon_{\min} = x_0 - \epsilon\left[J\big(\nabla u_\epsilon(x_0)\big) + o(1)\right],$$

as $\epsilon \to 0$, where

$$J(y) = |y|_{q^*}^{-q^*/q}\big(|y_1|^{q^*-2}y_1, \ldots, |y_n|^{q^*-2}y_n\big), \qquad \text{for } y \neq 0.$$

Hint: Note that by Hölder's inequality, for every $1 < q < \infty$,

$$\langle x, y\rangle = \sum_{i=1}^n x_i y_i \le \left(\sum_{i=1}^n |x_i|^q\right)^{1/q} \cdot \left(\sum_{i=1}^n |y_i|^{q^*}\right)^{1/q^*} \qquad \text{for } \frac{1}{q} + \frac{1}{q^*} = 1$$

On the other hand, $\hat{x} = J(y)$ verifies $|\hat{x}|_q = 1$ and $\langle \hat{x}, y\rangle = |y|_{q^*}$.

(iii) Let u be the uniform limit of the u_ϵ as $\epsilon \to 0$. Show that at points of twice differentiability of u where $\nabla u \neq 0$, we have

$$-\left\langle D^2 u\, J(\nabla u), J(\nabla u)\right\rangle = 0. \tag{3}$$

Remark: Equation (3) is the analogous in the case of a general ℓ^q-norm of the equation $-\Delta_\infty u = 0$ in the Euclidean case, and has applications to **absolutely minimizing Lipschitz extensions with general norms**.

Optimization Equations, Transport, and Monge-Ampère

5. Find the optimal transport map for the quadratic cost $c(x, y) = |x - y|^2$ between $\mu = f(x)\,dx$ and $\nu = g(y)\,dy$ in two dimensions, where $f(x) = \frac{1}{\pi}\mathbb{1}_{B_{(0,1)}(x)}$ and $g(y) = \frac{1}{8\pi}(4 - |y|^2)$.

Hint: Since both densities are radially symmetric the transport map has to be radial and the problem is reduced to an optimal transport problem in one dimension.

6. Let $R : \mathbb{R}^n \to \mathbb{R}^n$ be given by $R(x) = -x$. Characterize the probability measures μ such that R is an optimal transport map between μ and $\nu = R_{\#}\mu = \mu \circ R^{-1}$ for the quadratic cost.

Hint: By cyclical monotonicity of optimal maps, in the quadratic case you know that $\langle R(x) - R(y), x - y\rangle \geq 0$ for all x, y in the support of μ (in general you have $\sum_i c(x_i, R(x_i)) \leq \sum_i c(x_i, R(x_{i+1}))$, here you take two points in the sum and c as the quadratic cost).

7. Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{M}$ be the space of $n \times n$ real matrices. For $F : \Omega \times \mathcal{M} \to \mathbb{R}$ of class $\mathcal{C}^1$, the equation

$$F\big(x, D^2 u(x)\big) = 0 \qquad \text{in } \Omega, \tag{4}$$

is uniformly elliptic with ellipticity constants $0 < \theta \leq \Theta$ if and only if

$$\theta|\xi|^2 \leq F_{ij}(x, M)\,\xi_i\xi_j \leq \Theta|\xi|^2 \qquad \forall x \in \Omega,\ \forall M \in \mathcal{M} \text{ symmetric and } \forall \xi \in \mathbb{R}^n, \tag{5}$$

where $F_{ij}(x, A) = \partial_{a_{ij}} F(x, A)$. In particular, consider the following Monge-Ampère equation

$$\det\big(D^2 u(x)\big) = f(x), \qquad \text{in } \Omega. \tag{6}$$

(i) For $F(x, M) = \det(M) - f(x)$, show that

$$\big(F_{ij}(x, M)\big)_{1\leq i,j\leq n} = \operatorname{cof}(M)$$

where $\operatorname{cof}(M)$ is the matrix of cofactors of M. Then find a symmetric matrix $M \in \mathcal{M}$ that violates (5).

Hint: When computing F_{ij} you can use Laplace's formula and write the determinant as an expansion across row i or column j, i.e.

$$\det(M) = \sum_{j=1}^{n} m_{ij}\,(\operatorname{cof}(M))_{ij} = \sum_{i=1}^{n} m_{ij}\,(\operatorname{cof}(M))_{ij},$$

where $M = (m_{ij})_{1\leq i,j\leq n}$ and $\operatorname{cof}(M)$ is the matrix of cofactors of M.

(ii) However, equation (6) is uniformly elliptic if D^2u is in an appropriately restricted space of matrices. Show that if we only consider solutions of (6) that satisfy

$$0 < D^2u(x) \leq C\,\mathrm{Id} \quad \text{and} \quad 0 < \mu \leq f(x) \quad \forall x \in \Omega, \tag{7}$$

for some positive constants C, μ, then equation (6) is uniformly elliptic in the class of solutions that satisfy (7).

Hint: Keep in mind that the determinant of a matrix is the product of its eigenvalues. Also, the following formula for the inverse of a nonsingular matrix M can be useful:

$$M^{-1} = \frac{1}{\det(M)}\big(\mathrm{cof}(M)\big)^t,$$

where t denotes the transpose of a matrix.

8. Use optimal transport to prove the following form of the **isoperimetric inequality** with sharp constant: For $M \subset \mathbb{R}^n$

$$\mathrm{Vol}(M) = \mathrm{Vol}(B_1) \quad \Rightarrow \quad \mathcal{H}^{n-1}(\partial M) \geq \mathcal{H}^{n-1}(\partial B_1)$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

Complete the following sketch of the proof.

Proof.

(i) Take $f = \mathbb{1}_M$ and $g = \mathbb{1}_{B_1}$. Brenier's theorem then gives a volume-preserving map $G = \nabla u$ between M and B_1 such that

$$\int_M \phi(\nabla u(x))f(x)dx = \int_{B_1} \phi(y)g(y)dy.$$

In particular,

$$1 = \det{}^{1/n}(D^2u(x)) \qquad \text{a.e. } x \in M.$$

(ii) The arithmetic-geometric mean inequality yields

$$\mathrm{Vol}(M) \leq \frac{1}{n}\int_M \Delta u\, dx.$$

(iii) Integrating by parts (divergence theorem) we have

$$\mathrm{Vol}(B_1) = \mathrm{Vol}(M) \leq \frac{1}{n}\int_{\partial M} 1\, d\mathcal{H}^{n-1} = \frac{1}{n}\mathcal{H}^{n-1}(\partial M).$$

(iv) In the special case $M = B_1$, Brenier's map coincides with the identity map so equalities hold throughout the previous argument, yielding the optimal constant. □