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Geography of resonances and Arnold diffusion in \textit{a priori} unstable Hamiltonian systems

Amadeu Delshams\textsuperscript{1} and Gemma Huguet\textsuperscript{2}

\textsuperscript{1} Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain
\textsuperscript{2} Centre de Recerca Matemàtica, Apartat 50, 08193 Bellaterra (Barcelona), Spain

E-mail: Amadeu.Delshams@upc.edu and Gemma.Huguet@upc.edu

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Abstract

In this paper we consider the case of a general $C^{r+2}$ perturbation, for $r$ large enough, of an \textit{a priori} unstable Hamiltonian system of $2 + 1/2$ degrees of freedom, and we provide explicit conditions on it, which turn out to be $C^2$ generic and are verifiable in concrete examples, which guarantee the existence of Arnold diffusion.

This is a generalization of the result in Delshams \textit{et al} (2006 \textit{Mem. Am. Math. Soc.}) where the case of a perturbation with a finite number of harmonics in the angular variables was considered.

The method of proof is based on a careful analysis of the geography of resonances created by a generic perturbation and it contains a deep quantitative description of the invariant objects generated by the resonances therein. The scattering map is used as an essential tool to construct transition chains of objects of different topology. The combination of quantitative expressions for both the geography of resonances and the scattering map provides, in a natural way, explicit computable conditions for instability.

Mathematics Subject Classification: 37J40, 37C29, 35B34, 34C29, 37C50
(Some figures in this article are in colour only in the electronic version)

1. Introduction

The goal of this paper is to present a generalization of the geometric mechanism for global instability (popularly known as Arnold diffusion) in \textit{a priori} unstable Hamiltonian systems introduced in \cite{DLS06a}. That paper developed an argument to prove the existence of
global instability in a priori unstable nearly integrable Hamiltonian systems (the unperturbed Hamiltonian presents hyperbolicity, so that it cannot be expressed globally in action-angle variables) and applied it to a model which presented the so-called large gap problem. However, in that case, the perturbation was assumed to be a trigonometric polynomial in the angular variables. In this paper we perform an accurate process of truncation of the Fourier series of the perturbation and we present a deeper study of the geography of resonances. Using this, we are able to extend and simplify some of the results in [DLS06a] and apply them to an a priori unstable Hamiltonian system with a generic perturbation.

The phenomenon of global instability in Hamiltonian systems has attracted the attention of both mathematicians and physicists in the last years due to its remarkable importance for the applications. It deals, essentially, with the question of what is the effect on the dynamics when an autonomous mechanical system is submitted to a small periodic perturbation. More precisely, whether these perturbations accumulate over time giving rise to a long term effect or whether these effects average out.

The instability problem was formulated first by Arnold in 1964. In his celebrated paper [Arn64], Arnold constructed an example for which he proved the existence of trajectories that avoided the obstacles of KAM tori and performed long excursions. The mechanism is based on the existence of transition chains of transition whiskered tori, that is, sequences of tori with transitive dynamics given by a quasi-periodic flow, such that the unstable manifold (whisker) of one of these tori intersects transversally the stable manifold (whisker) of the next one. By an obstruction argument, there is an orbit that follows this transition chain, giving rise to an unstable orbit.

The example proposed in [Arn64] turns out to be rather artificial because the perturbation was chosen in such a way that it preserved exactly the complete foliation of invariant tori existing in the unperturbed system. However, a generic perturbation of size \( \varepsilon \) creates gaps at most of size \( \sqrt{\varepsilon} \) in the foliation of the persisting KAM tori, whereas it moves the whiskers only by an amount \( \varepsilon \). These KAM tori, which are just a continuation of the ones that existed for the integrable system \( (\varepsilon = 0) \) are commonly known as primary tori. The gaps in the foliation of the primary tori are centred around resonances, that is, resonant tori that are destroyed by the perturbation. This is what is known in the literature as the large gap problem (see, for instance, [Moe96] for a discussion about the large gap problem and, indeed, of the problem of diffusion).

In the last ten years there has been a notable progress in the comprehension of the mechanisms that give rise to the phenomenon of instability and a variety of methods has been suggested. As an example of this, we will mention that the large gap problem has been solved simultaneously by a variety of techniques: different geometrical methods [DLS00, DLS06a, DLS06b] (scattering map) and [Tre04, PT07] (separatrix map); topological methods [GL06b, GL06a] and variational methods [CY04, CY09]. For more information regarding the problem of Arnold diffusion in the absence of gaps as well as time estimates, the reader is referred to [DGLS08].

Of particular interest for this paper are [DLS00, DLS06a, DLS06b]. The strategy in the mentioned papers is based on incorporating in the transition chain new invariant objects, created by the resonances, like secondary tori, which are invariant KAM tori contractible to a periodic orbit, as well as the stable and unstable manifolds of lower dimensional tori. The scattering map, introduced by the same authors (see [DLS08] for a geometric study), is the essential tool for the heteroclinic connections between invariant objects of different topology.

In this paper we extend the geometric mechanism introduced in the mentioned papers to a wider class of model systems for which the perturbation does not need to have a finite number of harmonics in the angular variables. In particular, the Hamiltonian studied in this paper has
Theorem 1.1. Consider the Hamiltonian (1) and assume that hypotheses for Hamiltonian (1), from which we can deduce the following short version:

\[
H_\epsilon(p, q, I, \varphi, t) = \pm \left( \frac{1}{2} p^2 + V(q) \right) + \frac{1}{2} I^2 + \epsilon h(p, q, I, \varphi, t; \epsilon),
\]

where \( p \in (-p_0, p_0) \subset \mathbb{R}, I \in (I_-, I_+) \subset \mathbb{R} \) and \( (q, \varphi, t) \in \mathbb{T}^3 \).

The main result of this paper is theorem 2.1, stated in section 2.2 with the concrete hypotheses for Hamiltonian (1), from which we can deduce the following short version:

**Theorem 1.1.** Consider the Hamiltonian (1) and assume that \( V \) and \( h \) are \( C^{r+2} \) functions which are \( C^2 \) generic, with \( r > r_0 \), large enough. Then there is \( \epsilon^* > 0 \) such that for \( 0 < |\epsilon| < \epsilon^* \) and for any interval \([I_-, I_+]\) \( (I_-, I_+) \), there exists a trajectory \( \tilde{x}(t) \) of the system with Hamiltonian (1) such that for some \( T > 0 \)

\[
I(\tilde{x}(0)) \leq I_0^*; \quad I(\tilde{x}(T)) \geq I_1^*.
\]

**Remark 1.2.** A value of \( r_0 \) which follows from our argument is \( r_0 = 242 \) (see remark 2.3).

Our strategy for the proof follows the geometric mechanism proposed in [DLS06a]. Indeed, in order to organize the different invariant objects that we will use to construct a transition chain, we will first identify the normally hyperbolic invariant manifold (NHIM) present in the system. This NHIM will have associated stable and unstable invariant manifolds that, generically, intersect transversally. Therefore, we can associate with this object two types of dynamics: the inner and the outer. The outer dynamics takes into account the asymptotic motions to the NHIM and is described by the scattering map. The inner dynamics is the one restricted to the NHIM and contains Cantor families of primary and secondary KAM tori. Since generically these families of KAM tori, invariant for the inner dynamics, are not invariant for the outer dynamics, the combination of both dynamics will allow us to construct a transition chain.

The results in [DLS06a] can be applied straightforwardly for the persistence of the NHIM and the transversality of its associated stable and unstable manifolds. The arguments presented in this paper focus on the inner dynamics and the study of the invariant objects present in the NHIM.

For Hamiltonian (1), resonances correspond to the places where the frequency \( I = -l/k \) for \((k, l) \in \mathbb{Z}^2\) is rational and the associated Fourier coefficient \( h_{k,l} \) of the perturbation \( h \) is non-zero. On these resonances, the foliation of KAM tori in the NHIM is destroyed and gaps between the Cantor family of invariant tori in the NHIM of size \( O(\epsilon^{1/2}|h_{k_0,l_0}|^{1/2}) \) are created, for \((k_0, l_0)\) such that \( l/k = l_0/k_0 \) and \( \gcd(k_0, l_0) = 1 \) (see equation (88)). For a perturbation \( h \) which is a \( C^{r+2} \) function and \( C^2 \) generic, when we restrict it to the NHIM and we write it in adequate coordinates we are left with a \( C^r \) perturbation (see the subsection ‘restriction to NHIM’ in section 2.3.3), so that \(|h_{k,l}| \sim |(k,l)|^{-r} \), and therefore the above gaps are of size \( O(\epsilon^{1/2}|(k_0,l_0)|^{-r/2}) \). Moreover, other invariant objects, like secondary tori and lower dimensional tori, are created inside the gap. They correspond to invariant objects of different topology that were not present in the unperturbed system but are generated by the resonances.

In order to study their existence and give an approximate expression for them we will combine \( m \) steps of averaging plus a KAM theorem. Note that in our case, since the perturbation is generic, we will have an infinite number of resonances. Our approach for this study will be to consider an adequate truncation up to some order \( M \), depending on \( \epsilon \), of the Fourier series of the perturbation \( h \) in such a way that we deal only with a finite number of harmonics \(|(k, l)| \leq M \) and therefore of resonances.

Another remarkable difference with respect to the results obtained in [DLS06a] is that in that case the size of the gaps created in the foliation of invariant tori was uniform, whereas in our case, since the size is \( O(\epsilon^{1/2}|(k_0,l_0)|^{-r/2}) \), we have a heterogeneous sea of gaps of
different sizes. Among them, we will distinguish between small gaps and big gaps, which are strongly related to the mentioned large gap problem. Indeed, big gaps are those of size bigger than or equal to $\varepsilon$ and therefore they are generated by resonances $-l_0/k_0$ of order one, such that $|(k_0, l_0)| < \varepsilon^{-1/r}$ or, equivalently, $|(k_0, l_0)|^{-r/2} \geq \varepsilon^{1/2}$ (see section 3.3.3 for precise results).

From a more technical point of view (see section 3.2 for details), we would like to remark that the main difficulties arise from the fact that in order to perform a resonant averaging procedure, we need to isolate resonances corresponding to $|(k, l)| \leq M$, for $M$ depending on $\varepsilon$. Consequently, the width $L$ of the resonant domain cannot be chosen independently of $\varepsilon$, as was the case in [DLS06a]. Moreover, along the averaging procedure we need to keep track of the $C^r$ norms of the averaged terms and the remainders, and these blow up as a negative power of $L$. Hence, we will see that a good choice for $L$ around a resonance $I = -l/k$ will be $L = L_0 \sim \varepsilon^{1/n}/|k|$ (see hypotheses of theorem 3.11), where $n$ is the required regularity to apply KAM theorem after the averaging procedure. Note that $L$ is not uniform along the resonances but depends on the value $|k|$ of the resonance.

Finally, after $m$ steps of averaging, we will show that the remainder tail, that is, the Fourier coefficients $h_{k,l}$ such that $|(k,l)| > M$ can be neglected. This will be ensured by a fast enough decreasing rate of the coefficients and therefore a large enough regularity $r$ of the perturbation. Thus, the required regularity $r$ will be determined according to the number $m$ of steps of averaging performed, as well as the needed regularity $n$ to apply KAM theorem after the averaging procedure.

We are using a version of the KAM theorem that requires to have the Hamiltonian system written in action angle variables. Since near the resonances we approximate the system by the one which is close to a pendulum, the action variable becomes singular on the separatrix. This fact, together with the requirement to have the invariant objects close enough (at a distance smaller than $\varepsilon$), implies that the perturbation of the averaged Hamiltonian has to be extremely small in the resonant regions. The immediate consequence of this fact is that, in the case we are studying, one has to perform at least $m = 10$ steps of averaging (see theorem 3.28). The needed regularity $n$ to apply KAM theorem after $m$ averaging steps is $n = 2m + 6$ (see proposition 3.24). Since the regularity $r$ required to ensure that the remainder tail is smaller than the averaging remainder turns out to be $r > (n - 2)m + 2$, see remark 3.20, one has to impose $r > r_0 = 242$.

We do not claim that this is an optimal result. Actually, another version of the KAM theorem that allowed us to avoid the change into action-angle variables like [LGJV05, FLS09] could improve the results in terms of the needed regularity (see also [HLS09] for a numerical implementation). However, it is worth mentioning that we managed to decrease the required steps of averaging in the resonant domains with respect to the results in [DLS06a]. Since in the resonances the behaviour of KAM tori is different depending on how close they are to the separatrix (tori are flatter as they are further from the separatrix), we consider different regions where we perform different scalings. This strategy, which was already introduced in [DLS06a], has been improved in this paper introducing a new sequence of domains in theorem 3.30. When applied to the case with a finite number of resonances as in [DLS06a], $m = 9$ steps of averaging and $r \geq 26$ are enough (see remark 3.32). This clearly improves the needed regularity $r$ which was $r \geq 60$ in [DLS06a] because $m$ was chosen as $m = 26$.

Sections 3.3.3, 3.3.4 and 3.3.5 contain a quantitative description of the geography of resonances and a detailed study of the invariant objects generated by the resonances. The effect of the resonances in a system constitutes a fundamental problem not only for diffusion but also for many other physical applications and it has been an important object of study in the physical literature, see for instance [Chi79, Ten82]. The study performed in this paper
contributes to a better understanding of the different types of resonances and the geometric objects that one can find therein under an arbitrarily small value of the perturbation parameter $\varepsilon$. Therefore, this study can be very helpful in many physical problems, although in concrete problems, the size of $\varepsilon$ is not necessarily very small, and other diffusion mechanisms can take place, like Chirikov’s overlapping of resonances [Chi79].

Moreover, we think that this study can be extended to a class of models that presents multiple resonances, see [DLS07].

We would like to emphasize that in our case, and this is different from the results in [DLS06a], only the resonances of order 1, that is the ones that appear at the first step of averaging, create big gaps; whereas in [DLS06a], both resonances of orders 1 and 2 could generate big gaps. This is because we are dealing with a perturbation that generically will have all the harmonics different from zero. This means that the effect of the resonances associated with the biggest Fourier coefficients (low frequencies) will be detected at the first step of averaging. Since the size of the gap depends on both the order of the resonance and the size of the Fourier coefficient associated with that resonance, the ones that appear at the second step of averaging already correspond to small Fourier coefficients and the size of their gap will be smaller than $\varepsilon$. The immediate consequence of this fact is that in the forthcoming theorem 2.1 we can give all conditions explicitly in terms of the original perturbation $h$.

The paper is organized in the following way. In section 2 we state theorem 2.1, which establishes the existence of diffusing orbits for the model considered under precise conditions. Since the required hypotheses are checked to be $C^2$ generic, theorem 1.1 follows straightforwardly. The proof of theorem 2.1 is given in section 2, except for two technical results, theorem 3.1 and proposition 4.1, which are postponed to the following sections.

Thus, in section 3 we prove theorem 3.1, which provides a quantitative existence of invariant objects for the inner dynamics in the NHIM following the steps indicated in section 2. In section 4, we use the scattering map to prove proposition 4.1 about the existence of heteroclinic connections between the invariant objects obtained in the previous section.

We would like to remark that, in contrast to [DLS06a], and thanks to the new results about the scattering map obtained in [DLS08], we use the Hamiltonian function generating the deformation of the scattering map instead of the scattering map itself, in order to compute the images of the leaves of a certain foliation under the scattering map.

Finally, in section 5 we have included for illustration a concrete example, for which we sketch how the hypotheses of theorem 2.1 can be checked. We plan to come back to this example in a future paper for a more detailed description of the mechanism. In the appendix, we have brought some technical results used in the paper.

2. Statement of results

Before stating the main result in this paper we need to introduce some notation.

2.1. Notation and preliminaries

Let $r$ be a positive integer and $D \subset \mathbb{R}^n$ a compact set with non-empty interior $\hat{D}$. We will denote the set of $C^r$ functions from $\hat{D}$ to $\mathbb{R}^m$ and continuous on $D$ by $C^r(D, \mathbb{R}^m)$. When $m = 1$, we simply write $C^r(D)$ instead of $C^r(D, \mathbb{R})$. Given $f \in C^r(D, \mathbb{R}^m)$, we consider the standard $C^r$ norm,

$$|f|_{C^r(D)} = \sum_{i=1}^{m} \sum_{\ell=0}^{r} \sum_{|\alpha| = \ell} \sup_{x \in D} \frac{|D^\alpha f_i(x)|}{\alpha!},$$
where \( f_i \) denotes the \( i \)-th component of the function \( f \), for \( i = 1, \ldots, m \). We omit the domain in the notation when it does not lead to confusion.

We use the standard multi-index notation: if \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) one sets
\[
|\alpha| = \alpha_1 + \cdots + \alpha_n,
\]
\[
\alpha! = \alpha_1! \cdots \alpha_n!,
\]
\[
D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.
\]

In the case that the function \( f \) depends only on a few of the variables, we will denote it in the same way, that is \( |f|_C^r = |f|_C^r(D) \), and consider it as a function of more variables defined in the appropriate domain.

Note that we denote
\[
|f|_C^0 = \sup_{x \in D} |f(x)|,
\]
which is the standard supremum norm, so the \( |\cdot|_C^r(D) \) norm can be expressed, equivalently, as
\[
|f|_C^r(D) := \sum_{i=1}^m \sum_{\ell=0}^r \sum_{|\alpha|=\ell} \frac{|D^\alpha f|_C^0(D)}{\alpha!}.
\]

The space of \( C^r(D) \) functions endowed with the \( C^r \) norm is a Banach algebra (see [Con90]), that is it is a Banach space with the property that given any two functions \( f, g \) in \( C^r(D) \), they satisfy
\[
|fg|_C^r \leq |f|_C^r |g|_C^r.
\]

Since we will also deal with \( C^r \) functions defined on a compact domain \( D = I \times T^n \), where \( I \subset \mathbb{R}^n \) is a compact set with non-empty interior, we can also consider the following seminorm, that takes into account the different regularities and the estimates for the derivatives in each type of variable:
\[
|f|_{\ell_1, \ell_2} := \sum_{m_1=0}^{\ell_1} \sum_{m_2=0}^{\ell_2} \sum_{|\alpha|_1 = m_1, |\alpha|_2 = m_2} \frac{1}{\alpha_1! \alpha_2!} \sup_{(I,\varphi) \in D} \left| \frac{\partial^{m_1+m_2} f(I, \varphi)}{\partial I^{\alpha_1} \partial \varphi^{\alpha_2}} \right|,
\]
for \( 0 \leq \ell_1 + \ell_2 \leq r \).

We will use the following notation, which is rather usual. Given \( \alpha = \alpha(\varepsilon) \) and \( \beta = \beta(\varepsilon) \), we will write \( \alpha \leq \beta \) and also \( \alpha = O(\beta) \) if there exist \( \varepsilon_0 \) and a constant \( C \) independent of \( \varepsilon \), such that \( |\alpha(\varepsilon)| \leq C |\beta(\varepsilon)| \), for \( |\varepsilon| \leq \varepsilon_0 \). When we have \( \alpha \leq \beta \) and \( \beta \leq \alpha \) we will write \( \alpha \sim \beta \). However, in some informal discussions we will abuse notation and we will say that \( \alpha \) is of order \( \varepsilon^p \) if and only if \( \alpha \sim \varepsilon^p \).

We will say that a function \( f = O_{C^r(D)}(\beta) \) when
\[
|f|_{C^r(D)} \leq \beta.
\]

2.2. Set up and main result

We consider a \( 2\pi \)-periodic in time perturbation of a pendulum and a rotor described by the non-autonomous Hamiltonian (1),
\[
H_\varepsilon(p, q, I, \varphi, t) = H_0(p, q, I) + \varepsilon h(p, q, I, \varphi, t; \varepsilon) = P_\pm(p, q) + \frac{1}{2} I^2 + \varepsilon h(p, q, I, \varphi, t; \varepsilon),
\]
(4)
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where

\[ P_{\pm}(p, q) = \pm \left( \frac{1}{2} p^2 + V(q) \right) \]

and \( V(q) \) is a 2\( \pi \)-periodic function. We will refer to \( P_{\pm}(p, q) \) as the pencil.

The term \( \frac{1}{2} I^2 \) describes a rotor and the final term \( \varepsilon h \) is the perturbation term and depends periodically on time, so that \( h \) can be expressed via its Fourier series in the variables \( (\varphi, t) \):

\[ h(p, q, I, \varphi, t; \varepsilon) = \sum_{(k, l) \in \mathbb{Z}^2} h_{k,l}(p, q, I; \varepsilon) e^{i(k\varphi + lt)}. \]

It will be convenient to consider the autonomous system by introducing the extra variables \( (A, s) \):

\[
\tilde{H}_\varepsilon(p, q, I, \varphi, A, s) = A + H_0(p, q, I) + \varepsilon h(p, q, I, \varphi, s; \varepsilon) = A + P_{\pm}(p, q) + \frac{1}{2} I^2 + \varepsilon h(p, q, I, \varphi, s; \varepsilon)
\]

where the pairs \( (p, q) \in \mathbb{R} \times \mathbb{T}, (I, \varphi) \in \mathbb{R} \times \mathbb{T} \) and \( (A, s) \in \mathbb{R} \times \mathbb{T} \) are symplectically conjugate.

The extra variable \( A \) does not play any dynamical role and it simply makes the system autonomous. So, we are only interested in studying the dynamics of variables \( (p, q, I, \varphi, s) \) given by the following system of equations:

\[
\dot{p} = \mp V'(p) - \frac{\varepsilon}{\partial q} h(p, q, I, \varphi, s; \varepsilon),
\]

\[
\dot{q} = \pm p + \varepsilon \frac{\partial h}{\partial p}(p, q, I, \varphi, s; \varepsilon),
\]

\[
\dot{I} = -\varepsilon \frac{\partial h}{\partial \varphi}(p, q, I, \varphi, s; \varepsilon),
\]

\[
\dot{\varphi} = I + \varepsilon \frac{\partial h}{\partial I}(p, q, I, \varphi, s; \varepsilon),
\]

\[
\dot{s} = 1.
\]

The domain of definition we consider is a compact set of type

\[ \mathcal{D} := \mathcal{S} \times [I_-, I_+] \times \mathbb{T}_2 \times [-\varepsilon_0, \varepsilon_0], \]

where \( \mathcal{S} \subset \mathbb{R} \times \mathbb{T} \) is a neighbourhood of the separatrix \( P_{\pm}^{-1}(0) \) of the pencil.

Then, the main theorem of this paper is

**Theorem 2.1.** Consider a Hamiltonian of the form (1) where \( V \) and \( h \) are \( C^{r+2} \) in \( \mathcal{D} \), with \( r > r_0 \), sufficiently large. Assume also that

**H1** The potential \( V : \mathbb{T} \to \mathbb{R} \) has a unique global maximum, say at \( q = 0 \), which is non-degenerate (i.e. \( V''(0) < 0 \)). We denote by \( (p_0(t), q_0(t)) \) an orbit of the pencil \( P_{\pm}(p, q) \) in (1), homoclinic to \( (0, 0) \).

**H2** Consider the Poincaré function, also called Melnikov potential, associated with \( h \) (and with the homoclinic orbit \( (p_0, q_0) \)):

\[
\mathcal{L}(I, \varphi, s) = -\int_{-\infty}^{\infty} (h(p_0(\sigma), q_0(\sigma), I, \varphi + I\sigma, s + \sigma; 0) - h(0, 0, I, \varphi + I\sigma, s + \sigma; 0)) d\sigma
\]
Given real numbers $I_-, I_*$, assume that, for any value of $I \in (I_-, I_*)$, there exists an open set $\mathcal{F}_1 \subset \mathbb{T}^2$, with the property that when $(I, \varphi, s) \in H$, where

$$ H = \bigcup_{l \in (I_-, I_*)} \{ I \} \times \mathcal{F}_1 \subset (I_-, I_*) \times \mathbb{T}^2, $$

(10)

the map

$$ \tau \in \mathbb{R} \mapsto \mathcal{L}(I, \varphi - I \tau, s - \tau) $$

has a non-degenerate critical point $\tau$ which is locally given by the implicit function theorem in the form $\tau = \tau^*(I, \varphi, s)$, with $\tau^*$ a smooth function.

Introduce the reduced Poincaré function $\mathcal{L}^*$ defined by

$$ \mathcal{L}^*(I, \varphi) := \mathcal{L}(I, \varphi - I \tau^*(I, \varphi, 0), -\tau^*(I, \varphi, 0)), $$

(11)

on

$$ H^* = \{(I, \tilde{\varphi}) : \tilde{\varphi} = \varphi - Is, (I, \varphi, s) \in H\} = \bigcup_{l \in (I_-, I_*)} \{ I \} \times \mathcal{F}_1^*, $$

(12)

and assume that there exists an open set $\mathcal{F}_1^{**} \subset \mathcal{F}_1^*$ (respectively $\mathcal{F}_1^{**} \subset \mathcal{F}_1^*$), such that for $\tilde{\varphi} = \varphi - Is \in \mathcal{F}_1^{**}$, the function

$$ \tilde{\varphi} \mapsto \frac{\partial \mathcal{L}^*}{\partial \tilde{\varphi}}(I, \tilde{\varphi}) $$

(13)

is non-constant and positive (respectively negative). We denote

$$ H^*_+ = \bigcup_{l \in (I_-, I_*)} \{ I \} \times \mathcal{F}_1^{*+} $$

(respectively, $H^*_+ = \bigcup_{l \in (I_-, I_*)} \{ I \} \times \mathcal{F}_1^{*-}$).

Fix $1/(r/6 - 1) < \nu < 1/16$, for any $0 < \varepsilon < 1$ and for any $(k_0, l_0) \in \mathbb{Z}^2$ with $\gcd(k_0, l_0) = 1$ and $|(k_0, l_0)| < M_{BG}$, where $|(k_0, l_0)| = \max(|k_0|, |l_0|)$ and $M_{BG} = \varepsilon^{-(1+\nu)/r}$, introduce the $2\pi$-periodic function

$$ U^{k_0, l_0}(\theta) = \sum_{t \in \mathbb{Z} - \{0\}, \ |t||k_0, l_0| < M} h_{k_0, l_0}(0, 0, -l_0/k_0, 0)e^{i\theta t}, $$

where $\theta = k_0 \varphi + l_0 s$ and $M = \varepsilon^{-(1/(2\nu)+\delta)}$, for $0 < \delta < 1/10$, for which we assume

$\mathbf{H3}$ The function $U^{k_0, l_0}$ has a non-degenerate global maximum.

$\mathbf{H3}'$ For $|(k_0, l_0)| < C_1 \varepsilon^{-1/r}$, where $C_1$ is given explicitly in (153), we assume that the $2\pi k_0$-periodic function $f$ given by

$$ f(\theta) = \frac{k_0 U^{k_0, l_0}(\theta) \frac{2\pi}{k_0} \left( \frac{k_0}{k_0}, \frac{\theta}{k_0} \right) + 2U^{k_0, l_0}(\theta) \frac{2\pi}{k_0} \left( \frac{k_0}{k_0}, \frac{\theta}{k_0} \right)} {2 \frac{2\pi}{k_0} \left( \frac{k_0}{k_0}, \frac{\theta}{k_0} \right)} $$

(15)

is non-constant for $\theta/k_0 \in \mathcal{F}_1^{**}/k_0$.

$\mathbf{H3}''$ For $C_1 \varepsilon^{-1/r} \leq |(k_0, l_0)| \leq C_2 \varepsilon^{-1/r}$, where $C_1$ and $C_2$ are given explicitly in (153) and (155) respectively, we assume the non-degeneracy condition stated explicitly in equation (156).

Then, there exists $\varepsilon^* > 0$ such that for $0 < |\varepsilon| < \varepsilon^*$ and for any interval $[I^*, I^*] \subset (I_-, I_*)$, there exists a trajectory $\tilde{x}(t)$ of the system (1) such that for some $T > 0$

$$ I(\tilde{x}(0)) \leq I^*; \quad I(\tilde{x}(T)) \geq I^*. $$

(respectively,

$$ I(\tilde{x}(0)) \geq I^*; \quad I(\tilde{x}(T)) \leq I^*. $$
Remark 2.2. As a matter of fact, the only restriction needed for diffusion in $H^2$ is that the function $\theta \mapsto L^*(I, \theta)$ has to be non-constant. Typically, when it is non-constant there are some domains $H^+$ and $H^-$ where its derivative is positive and negative, respectively. These are the sets chosen to have an increasing and decreasing diffusion in $I$, respectively.

Remark 2.3. $r_0$ depends on the number $m$ of some averaging steps performed in the proof: $r_0 = 2(m + 1)^2$ and $m \geq 10$ (see hypotheses of theorem 3.1 in section 3). If we take just $m = 10$ then $r_0 = 242$ is enough.

Remark 2.4. The truncation order $M$ in hypotheses H3 depends on the regularity $n$ required for the application of the KAM theorem along the proof: $M = \varepsilon^{-1/(n+4)}$, for $n = 2m + 6$ and $0 < \delta < 1/m$, where $m$ is the number of averaging steps performed in the proof and is such that $m \geq 10$ (see hypotheses of theorems 3.11 and 3.1 and remark 3.20). Hence, we choose $m = 10$ and therefore $M = \varepsilon^{-1/(26+6)}$ in hypotheses H3.

Remark 2.5. Note that for every fixed $\varepsilon$ we have one condition H3 for every $(k_0, l_0)$ such that $|(k_0, l_0)| < M_{BG}$, that depends explicitly on $(k_0, l_0)$. Hence, the number of non-degeneracy conditions H3 is finite but grows with $\varepsilon$.

Remark 2.6. Note that by the definition of $\tau^*(I, \varphi, s)$, the function $f(I, \varphi, s) = L(I, \varphi - I\tau^*(I, \varphi, s), s - \tau^*(I, \varphi, s))$ satisfies the equation

$$I\partial_{\varphi}f(I, \varphi, s) + \partial_s f(I, \varphi, s) = 0.$$ 

Therefore it is of the form $f(I, \varphi, s) = L^*(I, \varphi - Is)$, so we can alternatively define

$$L^*(I, \varphi - Is) = L(I, \varphi - I\tau^*(I, \varphi, s), s - \tau^*(I, \varphi, s)).$$

Remark 2.7. The main feature of theorem 2.1, as already said in the introduction, is that $h$ is not required to be a trigonometric polynomial in the variables $(\varphi, s)$, which is a non-generic assumption, as was the case in [DLS06a].

Before proving theorem 2.1 let us see that theorem 1.1 stated in the introduction is just a consequence of theorem 2.1. Indeed, for every fixed $\varepsilon$, conditions H1 and H2 are open and dense, that is they hold for an open and dense set of Hamiltonians in the $C^2$ topology.

For every fixed $\varepsilon$, the number of non-degeneracy conditions H3 is finite but grows with $\varepsilon$ (the number of conditions depends on $(k_0, l_0) \in \mathbb{Z}^2$ such that $\gcd(k_0, l_0) = 1$ and $|(k_0, l_0)| \leq \varepsilon^{-1/r}$). When $\varepsilon$ tends to 0 we have a countable number of conditions in terms of the functions

$$U_{\infty}^{k_0, l_0}(\theta) = \sum_{r \in \mathbb{Z} - \{0\}} h_{r k_0, r l_0}(0, 0, -l_0/k_0; 0)e^{i\theta},$$

which are the same as those in hypotheses H3 but without any truncation. This countable number of conditions involve only derivatives up to order 2 of the Hamiltonian. Hence the set of Hamiltonians satisfying them is a residual set in the $C^2$ topology, that is a countable intersection of open and dense sets in the $C^2$ topology.

Therefore the hypotheses of the theorem are $C^2$ generic in the set of $C^{r+2}$ Hamiltonians of the form (1). So, the short version of theorem 2.1 stated in theorem 1.1 in the introduction follows straightforwardly.
2.3. Proof of theorem 2.1

The proof of this theorem follows the geometric mechanism stated in [DLS06a] and it is organized in four parts that we first sketch now:

**Part 1.** The first part deals with the existence of a NHIM, which, jointly with its associated stable and unstable manifolds, organizes all the dynamics, and is a consequence of hypothesis H1. By hypothesis H2', its associated stable and unstable manifolds will intersect transversally, so we can associate with this object two types of dynamics: the inner and the outer.

**Part 2.** The outer dynamics, which is the one that takes into account the asymptotic motions to the NHIM, is studied in the second part. We will see that we can associate a scattering map with the NHIM and give formulae for the Hamiltonian function which determines the deformation of this scattering map.

**Part 3.** The third part of the proof consists of studying the inner dynamics, that is the one restricted to the NHIM. The goal is to show that, by hypotheses H3', there exists a discrete foliation of invariant tori, which are closely spaced. Among these tori, some of them are primary, so they are just a continuation of the ones that existed for the integrable system ($\varepsilon = 0$), and some of them are secondary, these ones are contractible to a periodic orbit, so they correspond to motions which were not present in the unperturbed system but they are created by the resonances. The method of proof will be a combination of an averaging procedure and a quantitative version of the KAM theorem, which requires the Hamiltonian to be differentiable enough.

**Part 4.** The last part of the proof consists of showing that the combination of both types of dynamics gives rise to a construction of a transition chain, that is a sequence of transition whiskered tori in which the stable manifold of one torus intersects transversally the unstable manifold of the next one. To this end, one needs to show that the discrete foliation of whiskered tori which are invariant under the (inner) flow is not invariant under the scattering map or outer map. This is ensured by hypotheses H2'', H3'' and H3'' in theorem 2.1, which indeed provide the transversality of this discrete foliation to the scattering map. Finally we prove, using a standard obstruction property, that there is an orbit that follows this transition chain.

Next we give a proof of theorem 2.1 organized in the four parts that we mentioned. The first two parts follow readily from [DLS06a] and theorems stated in [DLS06a] apply straightforwardly because hypotheses H1 and H2' required for the proof of the mentioned results are the same as in our case. Moreover, for the second part we use the symplectic properties developed in [DLS08] to generalize the computation of the scattering map using its Hamiltonian function. So, for these parts we only refer in sections 2.3.1 and 2.3.2, to the results in [DLS06a] and [DLS08] that we are using.

However, for the third part, the results obtained in [DLS06a] do not apply directly because in this paper we are not assuming that the perturbation has a finite number of harmonics. Therefore, it has been necessary to develop a new methodology in order to prove that when we have a $C^{r+2}$ perturbation $h$, with $r$ large enough, and hypothesis H3 is fulfilled, for every $\varepsilon$ we can truncate adequately its Fourier series and deal only with a finite number of harmonics and therefore a finite number of resonances to get a discrete foliation of tori closely spaced. Moreover, explicit approximate expressions for these tori are obtained as the level sets of a certain function. The mentioned results are stated and proved rigorously in section 3, giving rise to theorem 3.1 and they constitute the essential result of this paper. In section 2.3.3 we just refer to the results in section 3 needed to prove part 3 of theorem 2.1.
Once we have fixed in part 3, for every \( \varepsilon \), the number of resonances, part 4 follows readily from the finite hypotheses \( H_2^\prime, H_3^\prime \) and \( H_3^\prime\prime \) as in [DLS06a]. The main difference is that, in contrast to [DLS06a] and thanks to the new results about the symplectic properties of the scattering map obtained in [DLS08], we can use the Hamiltonian function generating the deformation of the scattering map instead of the scattering map itself, in order to compute the images of the leaves of a certain foliation under the scattering map. The results with their proof are stated in section 4. In section 2.3.4 we just refer to the results in section 4 needed to prove part 4 of theorem 2.1.

2.3.1. First part: existence of a NHIM and its associated stable and unstable manifolds.

The method of proof is based on the existence of an invariant object, a NHIM (see, for instance, [HPS77, Fen74, Fen77, Fen79, Lla00, Wig90] for the standard theory of NHIMs used in this paper), which, jointly with its associated stable and unstable manifolds, organizes all the dynamics around it.

We start by discussing the geometric features of the unperturbed case which will survive under the perturbation. For the case \( \varepsilon = 0 \), Hamiltonian (1) is integrable and consists of two uncoupled systems: a rotor and a pendulum. So, the Cartesian product of invariant objects of each of these subsystems will give an invariant object of the full system. Then, by hypothesis \( H_1 \), if we consider the product of the hyperbolic fixed point \((p, q) = (0, 0)\) of the pendulum \( P_s(p, q) \) in (5) with all the other variables, we have that, for the values \( I_-, I_+ \) given in theorem 2.1, the set

\[
\tilde{\Lambda} = \{ \tilde{x} = (p, q, I, \varphi, s) \in (\mathbb{R} \times \mathbb{T})^2 \times \mathbb{T} : p = q = 0, I \in [I_-, I_+] \}
\]

is a three-dimensional invariant manifold and normally hyperbolic for the flow of the Hamiltonian system (8) for \( \varepsilon = 0 \). The associated stable and unstable invariant manifolds of \( \tilde{\Lambda} \) are the ones inherited from the separatrices of the pendulum (stable and unstable manifolds of the hyperbolic fixed point) and they agree

\[
W^s \tilde{\Lambda} = W^u \tilde{\Lambda} = \{ (p_0(\tau), q_0(\tau), I, \varphi, s) : \tau \in \mathbb{R}, I \in [I_-, I_+], (\varphi, s) \in \mathbb{T}^2 \},
\]

where \((p_0(\tau), q_0(\tau))\) is the chosen orbit of the pendulum \( P_s \), provided by hypothesis \( H_1 \), which is homoclinic to the hyperbolic fixed point \((0, 0)\).

The Hamiltonian system (8) for \( \varepsilon = 0 \) restricted to the manifold \( \tilde{\Lambda} \) is given simply by

\[
\dot{I} = 0, \quad \dot{\varphi} = I, \quad \dot{s} = 1.
\]

The dynamics on this manifold is very simple: all the solutions lie on a two-dimensional invariant torus \( I = \text{constant} \). Therefore, the NHIM is foliated by a one-parameter family of two-dimensional invariant tori indexed by \( I \), with associated frequency \((I, 1)\).

For \( 0 < |\varepsilon| \ll 1 \), by the theory of NHIM (see the references above), the manifold \( \tilde{\Lambda} \) persists, giving rise to another manifold \( \tilde{\Lambda}_\varepsilon \) with associated local stable and unstable manifolds \( W^{s, \text{loc}} \tilde{\Lambda}_\varepsilon \) and \( W^{u, \text{loc}} \tilde{\Lambda}_\varepsilon \), which can be prolonged to \( W^s \tilde{\Lambda}_\varepsilon \) and \( W^u \tilde{\Lambda}_\varepsilon \), respectively. Both \( \tilde{\Lambda}_\varepsilon \) and its local stable and unstable manifolds, \( W^{s, \text{loc}} \tilde{\Lambda}_\varepsilon \) and \( W^{u, \text{loc}} \tilde{\Lambda}_\varepsilon \), are \( \varepsilon \)-close in the \( C^r \) sense to the unperturbed ones:

\[
\tilde{\Lambda}_\varepsilon = \tilde{\Lambda} + O_\varepsilon (\varepsilon); \quad W^{s, \text{loc}} \tilde{\Lambda}_\varepsilon = W^{s, \text{loc}} \tilde{\Lambda} + O_\varepsilon (\varepsilon); \quad W^{u, \text{loc}} \tilde{\Lambda}_\varepsilon = W^{u, \text{loc}} \tilde{\Lambda} + O_\varepsilon (\varepsilon).
\]

The result of the persistence of the NHIM \( \tilde{\Lambda}_\varepsilon \) and its stable and unstable manifolds is formulated in theorem 7.1 of [DLS06a], where the perturbation \( h \) in (1) was assumed to be a trigonometric polynomial. However, the only assumption required for the proof was the fact that the perturbation \( h \) and the potential \( V \) were \( C^{r+2} \), so theorem 7.1 can be applied straightforwardly in our case.
2.3.2. Second part: outer dynamics. The outer dynamics, which is the one that takes into account the asymptotic motion to the NHIM $\tilde{\Lambda}_\varepsilon$, is described by the scattering map. It is possible to construct a scattering map associated with the NHIM $\tilde{\Lambda}_\varepsilon$, as long as its stable and unstable manifolds intersect transversally.

In proposition 9.2 in [DLS06a] it is proved that if hypothesis $H_2'$ in theorem 2.1 is satisfied, then the stable and unstable manifolds $W^s \tilde{\Lambda}_\varepsilon$ and $W^u \tilde{\Lambda}_\varepsilon$ of the NHIM intersect transversally along a homoclinic manifold $\Gamma_\varepsilon$, which is also called a homoclinic channel (see [DLS08] for more details, in particular for the definition of the wave operators, needed for the construction of the scattering map). So, we will be able to locally define the scattering map associated with $\Gamma_\varepsilon$, and compute it in first order using the results in [DLS08]. Again, hypothesis $H_2'$ required for proposition 9.2 in [DLS08] does not depend on the number of harmonics of the perturbation $h$, so the results stated also hold for the case we are considering in this paper.

Therefore, the manifold $\tilde{\Lambda}_\varepsilon$ defined in (16) has a scattering map associated with a homoclinic manifold $\Gamma_\varepsilon$, defined in the following way:

$$S_\varepsilon : H \subset \tilde{\Lambda}_\varepsilon \to \tilde{\Lambda}_\varepsilon$$

$$x_+ \mapsto x_-$$

(19)

such that $x_+ = S(x_-)$ if and only if there exists $z \in \Gamma_\varepsilon$ such that

$$\text{dist}(\Phi_{t,\varepsilon}(z), \Phi_{t,\varepsilon}(x_+)) \to 0 \quad \text{for } t \to \pm \infty,$$

where $\Phi_{t,\varepsilon}$ is the flow of Hamiltonian (1). Indeed,

$$|\Phi_{t,\varepsilon}(z) - \Phi_{t,\varepsilon}(x_\pm)| \lesssim \text{const } e^{-\mu |t|/2} \quad \text{for } t \to \pm \infty,$$

where $\mu = \sqrt{-V''(0)} > 0$ is the characteristic exponent of the saddle point $(0, 0)$ of the pendulum $P_\pm(p, q)$ in (5).

Heuristically, the scattering map maps points of the manifold $\tilde{\Lambda}_\varepsilon$ to points of the manifold $\tilde{\Lambda}_\varepsilon$, such that the motion of $z$ synchronizes with that of $x_-$ (and $x_+$) in the past (and in the future).

Moreover, in proposition 9.2 in [DLS06a] is given a perturbative formula for the difference of the actions $I$ of the points $x_+ = S_\varepsilon(x_-)$ and $x_-$. Concretely, expressing the points $x_\pm$ in terms of the parametrization of $\tilde{\Lambda}_\varepsilon$, given in theorem 7.1 in [DLS06a], we have that

$$I(x_\pm) = I + O_C(\varepsilon), \quad \varphi(x_\pm) = \varphi + O_C(\varepsilon), \quad s(x_\pm) = s$$

and

$$I(x_+) - I(x_-) = \varepsilon \frac{\partial L^*}{\partial \tilde{\theta}}(I, \tilde{\theta}) + O_C(\varepsilon^2),$$

(20)

for $\tilde{\theta} = \varphi - Is$, where $L^*$ is the reduced Poincaré function defined in hypothesis $H_2''$.

Remark 2.8. Note that there is a wrong sign in formula (9.9) in [DLS06a].

The method used in [DLS06a], based on the fact that $I$ is a slow variable, allowed only to compute the leading term of the action component of the scattering map, but not the $\varphi$ component since it is not a slow variable.

In a more recent paper [DLS08] the authors showed that the scattering map is exact symplectic and introduced geometric methods that allow us to compute perturbatively an expression for both fast and slow variables.

Thus, using the method proposed in section 5 in [DLS08], we can give perturbative formulae for the Hamiltonian $S_\varepsilon$ generating the deformation of the scattering map $S_\varepsilon$. 
It follows straightforwardly from theorem 31 in [DLS08] that the reduced Poincaré function \( L^* \) introduced in (11) is equal to the Hamiltonian \(-S_0\), so that we obtain

\[
S_\varepsilon(I, \varphi, A, s) = -L^*(I, \tilde{\varphi}) + O(\varepsilon),
\]

with \( \tilde{\varphi} = \varphi - Is \).

Hence, the first order perturbative term of the scattering map is given by

\[
S_\varepsilon(I, \varphi, A, s) = (I, \varphi, A, s) + \varepsilon J \nabla S_0(I, \varphi, A, s) + O(\varepsilon^2),
\]

where \( J \) is the canonical matrix of the symplectic form \( \omega = dI \land d\varphi + dA \land ds \) and \( \nabla = (\partial I, \partial \varphi, \partial A, \partial s) \). The extra variable \( A \), conjugated to the angle \( s \), was introduced to make apparent the symplectic character of the scattering map.

Note that equation (20) is just the \( I \) component of equation (22).

We would like to remark that \( S_\varepsilon = \text{Id} + O(\varepsilon) \). In particular, one iteration of \( S_\varepsilon \) can only jump distances of order \( \varepsilon \) in the action direction \( I \).

**Remark 2.9.** For the mechanism of diffusion we are interested in comparing the inner dynamics in \( \tilde{\Lambda}_\varepsilon \) with the outer dynamics provided by the scattering map \( S_\varepsilon \). Although the computation up to first order of the scattering map for the \( I \) component is enough for our purposes, it is more natural to study the action of the scattering map in terms of the Hamiltonian \( S_\varepsilon \).

### 2.3.3. Third part: inner dynamics.

In this section we study the inner dynamics, that is the dynamics of the flow of Hamiltonian (1) restricted to the NHIM \( \tilde{\Lambda}_\varepsilon \). The main result is theorem 3.1, which states that there exists a discrete sequence of invariant tori \( T_i \) in the NHIM \( \tilde{\Lambda}_\varepsilon \), which are distributed along the actions in the interval \((I_-, I_+)\) introduced in theorem 2.1 and which are \( O(\varepsilon^{1+\eta}) \)-closely spaced in terms of the action variable, for some \( \eta > 0 \). Moreover, theorem 3.1 provides explicit approximate expressions for the invariant tori, which are of two types depending on the region of the phase of space where invariant tori lie: the big gaps region and the flat tori region.

The **big gaps region** is defined as

\[
D_{BG} = \{(I, \varphi, s) \in (I_-, I_+) \times T^2 : |I + l/k| \leq L/|k|, |(k, l)| < M_{BG}\},
\]

where \( L \) is defined in (56) and is going to be introduced precisely along this third part and \( M_{BG} \) was introduced in hypothesis \( H_3 \) of theorem 2.1. For the purpose of this exposition it is enough to know now that \( L = O(\varepsilon^{1/n}) \) and \( M_{BG} = O(\varepsilon^{-1/r}) \), where \( n \) is the regularity of the Hamiltonian required for the application of the KAM theorem \( (n = 26 \) will be enough, see hypotheses of theorem 3.1) and \( r \) \((r > n)\) is the regularity of the Hamiltonian required for theorem 2.1. The flat tori region is the complementary region of the big gaps region.

In the flat tori region, there exists a Cantorian foliation of primary KAM tori, which are just a continuation of invariant tori \( I = \text{constant} \) present in \( \Lambda_0 \) for the unperturbed Hamiltonian (1) for \( \varepsilon = 0 \).

The big gaps region is formed by gaps of size bigger than or equal to \( \varepsilon \) in the Cantorian foliation of primary KAM tori. These gaps are bigger than the size \( \varepsilon \) of the heteroclinic jumps provided by the scattering map (22). This is what is known in the literature as the large gap problem. Inside these regions, apart from the primary KAM tori which are bent, there appear other invariant objects, which were not present in the unperturbed case, like secondary KAM tori and lower dimensional tori, which are not detected by a direct application of the KAM theorem, but require a more careful analysis based on an averaging procedure.

In order to prove theorem 3.1 we will restrict Hamiltonian (1) to the NHIM \( \tilde{\Lambda}_\varepsilon \) and perform an averaging procedure before applying a quantitative version of the KAM theorem. The
fundamental difference with respect to [DLS06a] is that for every fixed \( \varepsilon \) it will be necessary to truncate adequately the perturbation in order to deal with a finite number of harmonics depending on \( \varepsilon \). The phase space of the truncated Hamiltonian possesses an heterogeneous sea of a finite number of big gaps of different sizes, depending on the size of the harmonics of the perturbation.

**Restriction to the NHIM \( \tilde{\Lambda}_\varepsilon \).** Following the same arguments given in sections 8.1 and 8.2 in [DLS06a], we have that the flow restricted to \( \tilde{\Lambda}_\varepsilon \) is Hamiltonian. More precisely, by proposition 8.2 of [DLS06a], we can construct a \( C^r \) system of coordinates \((J, \varphi, s)\) on \( \tilde{\Lambda}_\varepsilon \), where

\[
J = J(I, \varphi, s; \varepsilon) = I + O(\varepsilon^{-1}),
\]

such that the symplectic form on any \( \Lambda'_\varepsilon = \{ (J', \varphi', s') \in \tilde{\Lambda}_\varepsilon : s' = s \} \) has the standard expression \( \omega_{|\Lambda'_\varepsilon} = dJ \wedge d\varphi \). Since \( \tilde{\Lambda}_\varepsilon = \Lambda \) for \( \varepsilon = 0 \) according to equation (18), by proposition 8.4 in [DLS06a], the restriction of the Hamiltonian \( H_\varepsilon \) in (1) to \( \Lambda_\varepsilon \) expressed in these action angle coordinates \((J, \varphi, s)\) has the form

\[
k(J, \varphi, s; \varepsilon) = Z(J) + \varepsilon R(J, \varphi, s; \varepsilon)
\]

with

\[
Z(J) = J^2/2 \quad \text{and} \quad R(J, \varphi, s; 0) = h(0, 0, J, \varphi, s; 0),
\]

where \( h \) is the perturbation in \( H_\varepsilon \) given in (6) and \( R \) is \( C^0 \).

**Remark 2.10.** Note that, by expression (26), \( R_{k,l}(J; 0) = h_{k,l}(0, 0, J; 0) \), where \( h_{k,l} \) and \( R_{k,l} \) are the Fourier coefficients in the angle variables \((\varphi, s)\) of the perturbation \( h \) and its restriction \( R \) to \( \tilde{\Lambda}_\varepsilon \), respectively.

**Averaging procedure.** We start performing an averaging procedure to the restricted Hamiltonian (25), as was done in [DLS06a], which follows the argument used in the proof of the KAM theorem in [Arn63], but paying attention to the resonant regions. In [DLS06a] the perturbation was assumed to be a trigonometric polynomial, so there was only a finite number of resonances. However, in Hamiltonian (1) the perturbation \( h \) has an infinite number of harmonics, in the same way as \( R \) in equation (25), which give rise to an infinite number of resonances, so the results in [DLS06a] do not apply directly.

The main result for the implementation of an averaging procedure for a generic perturbation will be theorem 3.11 in section 3.2. This theorem makes precise the hypotheses required to truncate the Fourier series of the perturbation \( R \) in (25) with respect to the angle variables and develops a global averaging procedure that casts the Hamiltonian (25) into a global normal form that has different expressions in the non-resonant and resonant regions. The main property of the normal form is that it is almost ready to apply a quantitative version of the KAM theorem on it.

The precise statement and rigorous proof of theorem 3.11 are postponed to section 3.2. In the following we only describe its main features and the results needed to apply the KAM theorem.

There are three parameters that play an important role in the averaging procedure of theorem 3.11. One is the number of steps of averaging \( m \) to be performed, which imposes a restriction on the differentiability \( r \) of the perturbation: \( r > 2(m + 1)^2 \). This number of averaging steps is chosen later in the application of the KAM theorem. The other two are \( M \), which is of the order of truncation of the Fourier series and \( L \), which determines the size of the
resonant regions. Both of them are chosen to depend on $\varepsilon$ in the following way: $M \sim \varepsilon^{-\rho}$ and $L \sim \varepsilon^{\alpha}$ where $\rho, \alpha > 0$ are going to be chosen conveniently during this averaging procedure.

For every fixed $\varepsilon$, we truncate the Fourier series of the perturbation $R$ in equation (25) with respect to the angle variables $(\varphi, s)$ up to order $M$ in the following way:

$$R = R^{[\leq M]} + R^{[> M]},$$

where

$$R^{[\leq M]}(J, \varphi, s; \varepsilon) = \sum_{|k| + |l| \leq M} R_{k,l}(J; \varepsilon)e^{i(k \varphi + ls)}$$

(27)

and

$$R^{[> M]}(J, \varphi, s; \varepsilon) = \sum_{|k| + |l| > M} R_{k,l}(J; \varepsilon)e^{i(k \varphi + ls)},$$

(28)

and we deal only with $R^{[\leq M]}$, which is the trigonometric polynomial of degree $M$, as a perturbation. The error introduced in Hamiltonian (25) coming from the neglected tail of the Fourier series will have to be estimated later on.

Since the truncated Hamiltonian $R^{[\leq M]}$ has a finite number of harmonics, an averaging procedure of $m$ steps has to take into account a finite number of resonances, which are the set of rational numbers $J = -l/k$ with $|l| + |k| \leq mM$ (see definitions 3.6 and 3.4).

This averaging procedure divides the phase space $(J, \varphi, s)$ into two types of domains. On the one hand, the non-resonant regions up to order $m \mathcal{D}_{n\varepsilon}$, which are the set of points $(J, \varphi, s)$ such that its action variable $J$ is at a distance greater than $2L_k$ of any resonance $J = -l/k$, where $L_k = L/k$. On the other hand, the resonant regions up to order $m \mathcal{D}_{\varepsilon}$, which are the set of points $(J, \varphi, s)$ such that its action variable $J$ is at a distance smaller than $L_k$ of any resonance $J = -l/k$ (see definitions 3.7 and 3.9).

To avoid overlapping between all the resonant domains, the distance between a resonance $-l_0/k_0$ and any other $-l/k$ must be greater than $2(L'_{k_0} + L_k)$. Since the resonances considered satisfy $|k| \leq mM$ we need to impose $4L < 1/(mM)$, which requires $\rho \leq \alpha$ in terms of exponents of $\varepsilon$ and this corresponds to the left hand side inequality of hypothesis (57) in theorem 3.11.

Along the averaging procedure, one needs to control the $C^r$ norms of the averaged terms and the remainders, for $0 \leq \ell \leq n$ and $2m < n < r$, where $n$ is the regularity which will be needed for the KAM theorem and $r$ is the regularity of the perturbation $R$ in Hamiltonian (25). It turns out that the estimates for the $C^r$ norm blow up as a negative power of $L \sim \varepsilon^\alpha$. Since the averaged terms and the remainder contain a power of $\varepsilon$ in front of them, bounds for them can be kept small provided that $\alpha$ is small enough, that is for $\alpha < 1/n$. This corresponds to the right hand side inequality of hypothesis (57) in theorem 3.11 and also implies $\rho < 1/n$, which is formula (53) in the hypotheses of theorem 3.11.

In all these averaging procedures, there was an initial error coming from the neglected tail of the truncation of order $M$ of the perturbation $R$ in Hamiltonian (25), whose $C^r$ norm can be bounded by $\varepsilon/M^{\nu - \ell - 2}$, where $\nu$ is the regularity of the perturbation $R$. To keep it smaller than the $C^r$ norm of the remainder after $m$ steps of averaging, one has to impose a lower bound on $\rho$, which implies $r \geq (1/\rho - 2m + 2$ in order to make the lower and upper bounds compatible for $\rho$, and this is hypothesis (54) in theorem 3.11.

These conditions on $m, \rho, \alpha$ and $r$ are stated in the hypotheses of theorem 3.11. In it, it is proved that one can develop a global averaging procedure that casts the Hamiltonian (25) into a global normal form (58), that has different expressions in the non-resonant and resonant regions (these correspond to expressions (59) and (60) in theses of theorem 3.11). In the
non-resonant regions one can perform non-resonant averaging transformations in such a way that the averaged Hamiltonian is very close to a rotor. On the other hand, near the resonances, the resonant averaging transformations cast the system to a one d.o.f. Hamiltonian, which is close to an integrable pendulum, provided that the perturbation satisfies some non-degeneracy conditions like $H^3$.

Summing up, we end up with a Hamiltonian (58) that consists of an integrable part $\tilde{Z}^m$ (the averaged Hamiltonian) plus a perturbation $\epsilon^{m+1} \tilde{R}^m$ which is $O_{\ell}(\epsilon^{m+1-\alpha(\ell+2m)})$, for $\ell = 0, \ldots, n - 2m$, where $m$ is the number of steps of averaging performed. Recall that the integrable Hamiltonian $\tilde{Z}^m$ has different expressions in the resonant regions and the non-resonant regions.

The integrable part of Hamiltonian (58) gives us an approximate equation $\tilde{Z}^m = \text{constant}$ for the invariant tori. The next step is to show which tori survive and what is the distance between them when we add the perturbation term $\epsilon^{m+1} \tilde{R}^m$ in equation (58).

Quantitative version of KAM theorem. The main tool for this section will be KAM theorem 3.22, which is a result about the existence of invariant tori of a periodic perturbation of a Hamiltonian expressed in action-angle variables. It is a direct adaptation of theorem 8.12 in [DLS06a]. We will use theorem 3.22 to show that there exists a discrete foliation of invariant tori which are $O(\epsilon^1+\eta)$-closely spaced, for some $\eta > 0$, and give approximate explicit expressions for them.

Since the integrable Hamiltonian (58) after $m$ steps of averaging has different expressions in the resonant and non-resonant regions (up to order $m$) introduced along the averaging procedure, we perform this study separately. In the end, we will show that all these regions can be grouped into two according to the expressions for the invariant tori obtained in each one, which are the big gaps region (23) and its complement the flat tori region, already mentioned at the beginning of this subsection. Note that the big gaps region (23) is formed by the resonances $J = -l/k$ of order 1, such that $|(k, l)| \leq M_{BG}$, whereas the flat tori region is composed of the non-resonant regions up to order $m$ and the resonant regions up to order $m$ such that $J = -l/k$ and $|(k, l)| > M_{BG}$, where $M_{BG}$ is explicitly chosen in hypothesis $H^3$ as $M_{BG} = \varepsilon^{-1/\alpha(\varepsilon^{12})}$ for any $1/(r/6 - 1) < \alpha < 1/16$.

The non-resonant regions are studied in section 3.3.2. In proposition 3.24, we apply theorem 3.22 directly to Hamiltonian (58) and (59), which is already written in action-angle variables, and we conclude that for these regions there exist flat primary KAM tori given in (81) as the level sets of a flat function $F = I + O(\varepsilon^{1-k})$, for $\alpha < 1/n$, which are $O(\varepsilon^{1+\eta})$-closely spaced for some $\eta > 0$, provided that $m \geq 2$ and $n \geq 2m + 6$.

The resonant regions are studied in section 3.3.3. As we already said, for these regions Hamiltonian (58)–(60) is not written in action-angle variables but it is close to an integrable pendulum (60) provided that hypothesis $H^3$ is satisfied. The integrable pendulum has rotational and librational orbits as well as separatrices, which separate these two types of motion. Rotational orbits have the same topology as the primary tori in the integrable Hamiltonian $Z(J) = J^2/2$ in Hamiltonian (25) and librational orbits are contractible to a periodic orbit, so they correspond to motions which were not present in the unperturbed Hamiltonian $Z(J)$ and they are called secondary tori. Librational orbits cover all the region inside the separatrix loop of Hamiltonian (60), giving rise to a gap between the primary tori, and the size of this gap depends on the order of the corresponding resonance and the size of the Fourier coefficient associated with it.

When gaps are of size smaller than $\varepsilon$, which is the size of the heteroclinic jumps provided by the scattering map (19), they are called small gaps. In section 3.3.4, we study the resonant regions with small gaps $D_{SG}$ and in proposition 3.26 we show that we can apply the same
argument as in the case of non-resonant regions to conclude that for these regions there exist flat primary KAM tori given in (89) as the level sets of a flat function \( F = I + \mathcal{O}(\varepsilon^{1-4\alpha}) \), for \( \alpha < 1/n \), which is the same as in the non-resonant case, and which are \( \mathcal{O}(\varepsilon^{1+n\eta}) \)-closely spaced, for some \( \eta > 0 \), provided that \( m \geq 2 \) and \( n \geq 2m + 6 \).

Note that tori in the non-resonant regions and resonant regions with small gaps are given by the level sets of the same function \( F = I + \mathcal{O}(\varepsilon^{1-4\alpha}) \) and they are flat up to \( \mathcal{O}(\varepsilon^{1-4\alpha}) \), for \( \alpha < 1/n \). Both regions form the flat tori region.

The resonant regions with big gaps \( D_{BG} \) are studied in section 3.3.5. They correspond to resonances \( J = -l/k \) such that \( |(k, l)| < M_{BG} \), where \( M_{BG} = \varepsilon^{-(1+v)/r} \), for \( 1/(r/6 - 1) < v \leq 1/16 \). The size of the gap for these resonances is \( C\varepsilon^{1/2}|(k, l)|^{-r/2} \), where \( C \) is a constant independent of \( \varepsilon \) and \((k, l)\). Note that there is no uniform size of the gaps since it runs from order \( \varepsilon^{1/2} \) for resonances with low \(|(k, l)| \) to \( \varepsilon^{1+v/2} \) for resonances with \(|(k, l)| \sim M_{BG} \).

Our criterion for the choice of the big gaps has been motivated by the size of the heteroclinic jumps provided by the scattering map (22): small gaps are of size smaller than \( \varepsilon \), so they can always be traversed just connecting two primary tori by the scattering map, whereas this is not the case for big gaps. For these big gaps, we will show that we can find other invariant objects, like secondary tori, which fill the region inside the gaps and they get rather close to the frontier of the gaps among the primary KAM tori.

**Remark 2.11.** We would like to remark that our result about resonances that create big gaps is remarkably different from the one obtained in [DLS06a], where the case of a perturbation \( h \) with a finite number of harmonics was considered. In that case there was a uniform size for the gaps created by the resonances of order 1 which was \( C\varepsilon^{1/2} \). Moreover, for resonances of order 2 the uniform size of the associated gap was \( C\varepsilon \). Hence, both resonances of orders 1 and 2 were considered as big gaps.

In the case of resonances with big gaps, we will need to write the integrable pendulum \( \tilde{Z}^m \) given in (60) into action-angle variables before applying KAM theorem 3.22. Since this change of coordinates becomes singular on the separatrix of the pendulum, we will need to define different action-angle variables inside and outside the separatrix, and we will exclude a thin neighbourhood of the separatrix.

Moreover, since the behaviour of the tori outside is different depending on their distance to the separatrix (tori are flatter as they are further from the separatrix) we consider different regions in the outside part of the separatrix, where we perform different scalings. This strategy, which was already introduced in [DLS06a], has been improved introducing a new sequence of domains in theorem 3.30, which reduces the differentiability requirements.

The main result for the implementation of the above strategy for resonances with big gaps is theorem 3.30 jointly with corollary 3.31 which make explicit the relationship between the minimum distance between the surviving tori and the number \( m \) of steps of averaging performed.

In theorem 3.28 we use both theorem 3.30 and corollary 3.31 to show that many of the invariant tori (both primary and secondary) of the integrable averaged Hamiltonian persist under the perturbation forming a sequence of tori given in (96) as the level sets of a function \( F \), close to the averaged Hamiltonian with a distance between consecutive tori of order \( \varepsilon^{1+n\eta} \), for some \( \eta > 0 \), in terms of the action variable, provided that \( m \geq 10 \) and \( n \geq 2m + 6 \).

Propositions 3.24 and 3.26 and theorem 3.28 can be joined in a unique result about the existence of nearby invariant tori for the inner dynamics, which is theorem 3.1. This theorem also gives explicit approximate expressions for the invariant tori, which are of two types depending on the region of the phase of space where the invariant tori lie: the big gaps region and the flat tori region.
We refer to sections 3.2 and 3.3 for the referenced theorems where one can find the complete proof.

2.3.4. Fourth part: construction of a transition chain and obstruction property. In order to finish the proof, it remains to check that the finite sequence of KAM tori $T_i$ provided by theorem 3.1 form a transition chain along the NHIM $\tilde{\Lambda}_e$, traversing both the big gaps and the flat tori regions, and to show that there are orbits that follow it closely. These are the orbits claimed in theorem 2.1.

The scattering map $S_\varepsilon$ associated with the homoclinic channel $\Gamma_\varepsilon$, defined in (19), is the main tool to detect that there exist transverse heteroclinic connections between these tori, which are objects of different topology. Indeed, by lemma 10.4 in [DLS06a], we know that two submanifolds, like the invariant tori $T_i$ of a NHIM $\tilde{\Lambda}_e$, have a transverse heteroclinic intersection if they are transversal under the scattering map as submanifolds of $\tilde{\Lambda}_e$.

The main result of this section is proposition 4.1 where it is proved the existence of transition chains, that is chains of KAM tori $T_i$, both primary and secondary, such that their image under the scattering map $S_\varepsilon$ in (22) intersects transversally $T_{i+1}$ on $\tilde{\Lambda}_e$, that is

$$S_\varepsilon(T_i) \cap \tilde{\Lambda}_e T_{i+1}. \quad (29)$$

In section 2.3.2 we have obtained an explicit expression (22) up to first order for the scattering map $S_\varepsilon$ using the first order calculation of the Hamiltonian function $S_\varepsilon$. In section 2.3.3 we have shown that on the NHIM $\tilde{\Lambda}_e$, there exists a discrete foliation of KAM tori $T_i$ (primary and secondary) which are $O(\varepsilon^{1+\eta})$-closely spaced, for some $\eta > 0$. Moreover, we have obtained explicit expressions for tori $T_i$, both primary and secondary, and we have seen that these invariant objects are given approximately by the level sets of the averaged Hamiltonian.

In lemma 4.2 in section 4.1, we give an expression for the action of the scattering map $S_\varepsilon$ on a foliation given by the level sets of a given function $F$, using the expression for the Hamiltonian function $S_\varepsilon$ generating the deformation of the scattering map, introduced in section 2.3.2. Moreover, we give conditions to assure transversality between the foliation in $\tilde{\Lambda}_e$ and its image under the scattering map $S_\varepsilon$.

As we have seen in the previous section, the different types of tori that appear in our problem have different quantitative properties and therefore the dominant terms in the expression of these invariant objects as the level sets of a certain function are different whether they lie in a flat tori region or a big gaps region. Lemma 4.2 is applied in lemma 4.5 for the case of the flat tori region, and in lemma 4.7 for the case of the big gaps region. In these lemmas it is shown that the sufficient conditions on the perturbation of the Hamiltonian (1) for the transversality are hypotheses $H_{2''}$, $H_{3''}$ and $H_{3'''}$ in theorem 2.1.

Putting all these results together in proposition 4.1, we have that, by hypothesis $H_{2''}$ and the non-degeneracy conditions $H_{3''}$ and $H_{3'''},$ the scattering map $S_\varepsilon$ maps pieces of these tori transversally in $\Lambda_e$ to other tori at a distance $O(\varepsilon)$, that is $S_\varepsilon(T_i) \cap T_{i+1},$ where $T_i$ and $T_{i+1}$ are invariant tori at a distance smaller than $\varepsilon$. Therefore, by lemma 10.4 in [DLS06a] we have that $W^{u}_{\varepsilon,i} \cap W^{s}_{\varepsilon,i+1},$ and we have constructed a transition chain.

Finally, we use the well-known result that given a transition chain $\{T_i\}_{i=0}^N,$ we can find an orbit visiting all the elements of the chain. In our case, as was the case in [DLS06a], we have incorporated in the chain objects with different topologies, so applying lemma 11.1 in [DLS06a] to the transition chain obtained, we get that there is $\varepsilon^* > 0$ such that for $0 < |\varepsilon| < \varepsilon^*$, and for any interval $[I^-, I^+] \in (I^-, I^+), \tilde{x}(t)$ satisfies that, for some $T > 0$,

$$I(\tilde{x}(0)) \leq I^+_e; \quad I(\tilde{x}(T)) \geq I^+_e.$$
Note that this result, which proves the existence of a diffusing orbit whose action variable $I$ increases by order 1, is obtained when we assume in hypothesis $H_2''$ that there exists an open set $J_I^{++}$ where the function (13) is positive. By the same argument, if we assume that there exists an open set $J_I^{+-}$ where (13) is negative, the action variable of diffusing orbit $\tilde{x}(t)$ decreases, that is the orbit $\tilde{x}(t)$ satisfies that, for some $T > 0$,

$$I(\tilde{x}(0)) \geq I_0^+; \quad I(\tilde{x}(T)) \leq I_0^-.$$ 

### 3. Inner dynamics

The main goal of this section is to prove theorem 3.1 about the existence of a sequence of invariant tori $T_i$ in the NHIM $\tilde{\Lambda}_c$, which are distributed along all the actions in the interval $(I_-, I_+)$ and are $O(\epsilon^{1+\eta})$-closely spaced, for some $\eta > 0$. The method of proof will consist of the combination of two parts: averaging methods and the KAM theorem.

In section 3.2 we will consider the restricted Hamiltonian (25) and perform, in theorem 3.11, a global averaging procedure that casts the Hamiltonian into a global normal form, which has different expressions in the non-resonant and resonant regions. In the non-resonant regions, averaging transformations cast the system too close to a rotor and, in general, in the non-resonant regions too close to an integrable pendulum.

In section 3.3 we will use KAM theorem 3.22 to show that many of the invariant tori of the averaged Hamiltonian persist when we add the error terms of the normal form and they are close enough in terms of the action variables. For the flat tori region, which consists of non-resonant regions and resonant regions with small gaps, we can apply KAM theorem 3.22 almost straightforwardly and this is done in propositions 3.24 and 3.26, respectively. For the big gaps region, we will show in theorem 3.28 that we can apply the KAM theorem after we have written the Hamiltonian in action-angle coordinates.

#### 3.1. Main result

The main result for the inner dynamics shows that in the NHIM $\tilde{\Lambda}_c$, there exists a sequence of tori, which are distributed along all the actions in the interval $(I_-, I_+)$ and are $O(\epsilon^{1+\eta})$-closely spaced, for some $\eta > 0$. Moreover, it provides analytic expressions for the invariant tori. It is stated in the following theorem:

**Theorem 3.1.** Consider a Hamiltonian of the form (1) and assume that $r > 2(m + 1)^2$, with $m \geq 10$ and $n = 2m + 6$, as well as hypothesis $H_3'$. Choose $\eta = \min((m - 1 - \alpha n)/2, \nu/2 - 3(1 + \nu)/r)$, where $\alpha < 1/n$ and $1/(r/6 - 1) < \nu \leq 1/16$. Then, for $\epsilon$ small enough, there exists a finite sequence of invariant tori $\{T_i\}_{i=0}^N$ in $\tilde{\Lambda}_c$, which satisfy:

1. They are defined by the equation $F(I, \varphi, s; \epsilon) \equiv E_i$, where $F$ is a $C^4$-$\varphi$ function, for any $\varphi > 0$, which has the form (89) and (96) depending on the region where the invariant tori lie: the flat tori region or a connected component of the big gaps region defined in (84), respectively. In the flat tori region, the invariant tori are primary whereas in the big gaps region the invariant tori can be primary or secondary. In the big gaps region, for values of $E_i > 0$ equation (96) provides two primary tori $T_{E_{i}^\pm}$, whereas for $E_i < 0$ it gives a secondary torus $T_{E_{i}}$.

2. They can be also written as a graph of the variable $I$ over the angle variables $(\varphi, s)$: $I = \lambda_E(\varphi, s; \epsilon)$ with $\lambda_E$ given in (90) for the flat tori region. In the case of the big gaps region, the equations for them are given for two different invariant tori $T_{i}^\pm$ (two different components in the case of secondary KAM tori) in the form $I = \lambda_{E_{i}}^\pm(\varphi, s; \epsilon)$, with $\lambda_{E_{i}}^\pm$ given in (97).
3. These tori are $O(\varepsilon^{1+q})$-distributed along the actions in the interval $(I_-, I_+)$, that is, they are $O(\varepsilon^{1+q})$-closely spaced in terms of the action variable $I$. In the connected component (84) of the big gaps region associated with the resonance $-l_0/k_0$, they are $O(\varepsilon^{3/2+q}(l_0, k_0)|^{-7/2+1})$-closely spaced in terms of energies $E_i$, where $-l_0/k_0$ is the associated resonance.

4. $I_0$ and $I_N$ are $O(\varepsilon^{1+q})$-close to $I_-$ and $I_+$, respectively.

The proof of theorem 3.1 is a combination of an averaging procedure (section 3.2) and a KAM theorem (section 3.3). In section 3.4 we put the results obtained in the previous sections together to give a proof of theorem 3.1.

3.2. Averaging procedure

In this section we proceed to obtain a suitable global normal form of the restricted Hamiltonian (25), according to the procedure described in section 2.3.3. We use the standard formalism of Lie series, so we are considering canonical transformations obtained as the time-one map of Lie series, so we are considering canonical transformations obtained as the time-one map of the flow of a Hamiltonian. A very pedagogical treatment of this method can be found in [LM88]. As we have already mentioned, we consider a truncation of the Fourier series of the flow of a Hamiltonian. According to the procedure described in section 2.3.3, we use the standard formalism of Lie series, so we are considering canonical transformations obtained as the time-one map of Lie series, so we are considering canonical transformations obtained as the time-one map of the flow of a Hamiltonian. A very pedagogical treatment of this method can be found in [LM88].


We consider the space of functions defined on $\mathbb{T}^2$, $I \subset \mathbb{R}$ compact set, which consists of trigonometric polynomials of order $M$ on $(\varphi, s) \in \mathbb{T}^2$, and $C^r$ with respect to $J \in I \subset \mathbb{R}$. We denote this space $T_M(I \times \mathbb{T}^2)$. A function $u \in T_M(I \times \mathbb{T}^2)$ is of the form

$$u(J, \varphi, s) = \sum_{(k, l) \in \mathbb{Z}^2, \sum |k| + |l| \leq M} u_{k,l}(J) e^{ik\varphi + isl}. \tag{30}$$

Remark 3.2. Note that the product of two elements $u \in T_M(I \times \mathbb{T}^2)$ and $v \in T_N(I \times \mathbb{T}^2)$ is another trigonometric polynomial in the variables $(\varphi, s) \in \mathbb{T}^2$ of degree $M + N$, that is $uv \in T_{M+N}(I \times \mathbb{T}^2)$.

Clearly, the space $T_M(I \times \mathbb{T}^2)$ is a closed subset of $C^r(I \times \mathbb{T}^2)$. Therefore, $T_M(I \times \mathbb{T}^2)$ is a Banach space with the $C^r$ norm introduced in (2).

Moreover, since the functions $u$ are trigonometric polynomials in $(\varphi, s)$, we can consider expression (30) and deal with the Fourier norm:

$$\|u\|_{C^r(I \times \mathbb{T}^2)} := \sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^{\ell} \sum_{(k, l) \in \mathbb{Z}^2, \sum |k| + |l| \leq M} |u_{k,l}|_{C^r(I \times \mathbb{T}^2)} |(k, l)|^{m-n} \tag{31}$$

where $|u_{k,l}|_{C^r(I \times \mathbb{T}^2)}$ is defined in (2) and $|(k, l)| = \max(|k|, |l|)$, and $| \cdot |$ denotes the standard modulo. When there is no possibility of confusion about $M$ we abbreviate it as $\|\cdot\|_{C^r}$.

On the other hand, to understand better the behaviour of the function $u$ with respect to the variable $J$ when it gets closer to the resonances, we use the Fourier norm with a weight $L \leq 1$:

$$\|u\|_{C^r(I \times \mathbb{T}^2), L} := \sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^{\ell} \sum_{(k, l) \in \mathbb{Z}^2, \sum |k| + |l| \leq M} |u_{k,l}|_{C^r(I \times \mathbb{T}^2), L} |(k, l)|^{m-n} \tag{32}$$
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where \(|(k, l)|\) is as before and

\[ |u_{k,l}|_{C^n(I),L} := \sum_{i=0}^n L^i \frac{|Du_{k,l}(i)|_{C^n(I)}}{i!}. \]

As before, when there is no confusion about \(M\) we abbreviate these norms as \(||\cdot||_{C^n,L}\) and \(|\cdot|_{C^n,L}\), respectively.

Note that when \(L = 1\), we recover the Fourier norm (31).

The basic properties of these norms are collected in appendix B. In particular they are related by

\[ L^\ell |u|_{C^n,L} \leq \|u\|_{C^n,L} \leq C M^2 |u|_{C^n,L}, \quad \text{(33)} \]

where \(C\) is a constant that depends on \(\ell\) but it is independent of \(M\) and \(0 < L \leq 1\).

Moreover, given \(u \in T_M(I \times T^2)\) and \(v \in T_N(I \times T^2)\), we have that \(uv \in T_{M+N}(I \times T^2)\) and for \(0 < L \leq 1\) and \(0 \leq \ell \leq r\),

\[ \|uv\|_{C^n,L} \leq \|u\|_{C^n,L} \|v\|_{C^n,L}. \quad \text{(36)} \]

We will say that a function \(f\) is \(OC_{\ell,L}(\eta)\) when \(\|f\|_{C_{\ell,L}} \leq \eta\).

3.2.2. The homological equation. In this section, we will use the standard formalism of Lie series to perform a resonant averaging procedure. We first start discussing the infinitesimal equations for averaging, which will serve as a motivation for the phenomenon of resonances and therefore for the resonant averaging.

We begin with a Hamiltonian \(K(J, A, \varphi, s) = K_0(J, A) + \varepsilon K_1(J, A, \varphi, s)\), where \((J, A, \varphi, s) \in \mathbb{R}^2 \times T^2\) and \(K_0(J, A) = A + J^2/2\). We start looking for a canonical transformation \(g\), given by the time-one map of the flow of a Hamiltonian \(\varepsilon G(J, A, \varphi, s)\) (generating function), that eliminates, when it is possible, the dependence on the angle variables \((\varphi, s)\) up to order \(\varepsilon\). Therefore,

\[ K \circ g = K + \{K, \varepsilon G\} + \frac{1}{2!}\{\{K, \varepsilon G\}, \varepsilon G\} + \cdots \]

where \{,\} denotes the Poisson bracket in the canonical coordinates \((J, A, \varphi, s)\):

\[ \{f, g\} = \frac{\partial f}{\partial \varphi} \frac{\partial g}{\partial J} + \frac{\partial f}{\partial s} \frac{\partial g}{\partial A} - \frac{\partial f}{\partial J} \frac{\partial g}{\partial \varphi} - \frac{\partial f}{\partial A} \frac{\partial g}{\partial s}. \]

We seek for a solution \(G\) of the infinitesimal equation

\[ K_1 + \{K_0, G\} = \bar{K}, \]

which produces a \(\bar{K}\) as simple as possible. In Fourier coefficients this equation has the form

\[ K_{k,l}(J) - i(\omega(J) \cdot (k, l))G_{k,l}(J) = \bar{K}_{k,l}(J) \quad \text{(37)} \]
where \( K_{k,l}(J) \), \( G_{k,l}(J) \) and \( \tilde{K}_{k,l}(J) \) are the Fourier coefficients of \( K_1 \), \( G \) and \( \tilde{K} \), respectively, for \( (k,l) \in \mathbb{Z}^2 \), and \( \omega(J) \in \mathbb{R}^2 \) is of the form

\[
\omega(J) = \left( \frac{\partial K_0}{\partial J} - \frac{\partial K_0}{\partial A} \right) = (J, 1).
\]

This vector \( \omega(J) \) is called resonant when \( (J, 1) \cdot (k,l) = Jk + l = 0 \), for \( (k,l) \neq (0,0) \); and the values \( J = -l/k \), with \( k \neq 0 \), for which this equation vanishes and \( K_{k,l}(-l/k) \neq 0 \) are called resonances. Looking at equation (37) it is clear that these are the places where we cannot choose \( G_{k,l}(J) \) in order to have \( \tilde{K}_{k,l}(J) \equiv 0 \). So, for these values of \( J \) and, in order to keep smoothness, the ones in a neighbourhood around them, we choose \( \tilde{K}_{k,l}(J) \) to be the Fourier term \( K_{k,l}(-l/k) \). Note that we cannot have \( \tilde{K}_{0,0}(J) \equiv 0 \) for any \( J \) either, so we will also keep the Fourier coefficient \( K_{0,0}(J) \).

The precise result with the estimates for the functions is formulated in the following lemma:

**Lemma 3.3.** Let \( K(J, \varphi, s) \) be a Hamiltonian defined on \( \mathcal{I} \times \mathbb{T}^2 \), \( \mathcal{I} \subset \mathbb{R} \) compact set, which is a \( C^{r+1} \) function with respect to \( J \) and a trigonometric polynomial in \( (\varphi, s) \) of degree \( M \), so it can be expressed in the following way:

\[
K(J, \varphi, s) = \sum_{(k,l) \in \mathcal{N}} K_{k,l}(J)e^{i(k\varphi+l)} ,
\]

with \( \mathcal{N} = \{(k,l) \in \mathbb{Z}^2, |k| + |l| \leq M \} \). We refer to resonances as the elements of the finite set of rational numbers

\[
\mathcal{R} = \{-l/k \in \mathbb{Q} : (k,l) \in \mathcal{N}, k \neq 0, K_{k,l}(-l/k) \neq 0 \}.
\]

For any \( (k,l) \in \mathcal{N} \), we consider \( (\tilde{k}, \tilde{l}) \in \mathbb{Z}^2 \) such that \( -l/k = -\tilde{l}/\tilde{k} \) and \( \gcd(\tilde{k}, \tilde{l}) = 1 \) and we define \( L_k = L_{\tilde{k}} = L/|k| \), \( L \leq 1 \) being some constant small enough such that for all \( -1/k \in \mathcal{R} \), the real intervals \( [-1/k - 2L_k, -1/k + 2L_k] \) are all disjoint.

Then, there exists a function \( G = G^{\leq M} \) of class \( C^r \) with respect to \( J \) and \( \tilde{K} = \tilde{K}^{\leq M} \) of class \( C^{r+1} \), which are both trigonometric polynomials in \( (\varphi, s) \), such that they solve the homological equation

\[
K + \{K_0, G\} = \tilde{K} ,
\]

and verify the following:

1. If \(|J + l/k| \geq 2L_k \) for any \((k,l) \in \mathcal{N} \), then
   \[ \tilde{K}(J, \varphi, s) = K_{0,0}(J) . \]

2. If \(|J + l_0/k_0| \leq L_{k_0} \) for some \((k_0,l_0) \in \mathcal{N} \), then
   \[ \tilde{K}(J, \varphi, s) = K_{0,0}(J) + \sum_{t \in \mathbb{Z}, |t| \leq 0, |t| \leq M} \left( K_{t_0,k_0}(J) \frac{e^{it(k_0 \varphi + l_0 s)}}{k_0} \right) \]
   \[ =: K_{0,0}(J) + U_{k_0,l_0}(k_0 \varphi + l_0 s) . \]

3. The function \( \tilde{K} \) verifies
   \[ \|\tilde{K}\|_{C^r,L} \leq C_{r} \|K\|_{C^{r},L} . \]

4. The function \( G \) verifies
   \[ \|G\|_{C^{r},L} \leq C_{r} \|K\|_{C^{r+1},L} . \]

for \( \ell = 0, \ldots, r+1 \), where \( C_{r} \) is a constant independent of \( L, M \).
**Proof.** We want to solve for each \((k, l) \in \mathbb{N}\) equation (37)

\[
K_{k,l}(J) - i(Jk + l)G_{k,l}(J) = \bar{K}_{k,l}(J),
\]

(44)

where the unknowns are the Fourier coefficients of the generating function \(G\) and the averaged Hamiltonian \(\bar{K}\).

So, we first choose:

1. \(\bar{K}_{0,0}(J) = K_{0,0}(J)\),
2. if \((0, l) \in \mathcal{N}, l \neq 0\), \(\bar{K}_{0,l}(J) = 0\),
3. if \((k, l) \in \mathcal{N}, k \neq 0\), we choose \(\bar{K}_{k,l}(J)\) as

\[
\bar{K}_{k,l}(J) = K_{k,l}(-l/k)\psi\left(\frac{1}{L_k}(J + l/k)\right),
\]

(45)

where \(\psi(x)\) is a fixed \(C^\infty\) function such that: \(\psi(x) = 1\), if \(x \in [-1, 1]\), and \(\psi(x) = 0\), if \(x \notin [-2, 2]\). With this choice we have that \(\bar{K}\) verifies the following:

(a) If \(|J + l/k| \leq L_k\) then \(\bar{K}_{k,l}(J) = K_{k,l}(-l/k)\),

(b) if \(|J + l/k| \geq 2L_k\) then \(\bar{K}_{k,l}(J) = 0\).

Once we have defined \(\bar{K}\) as above, it is clear that it is a \(C^{r+1}\) function, and its Fourier coefficients satisfy

\[
\|\bar{K}\|_{C^{r+1},L} = \sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^\ell \sum_{(k,l) \in \mathbb{Z}^2, |k| + |l| \leq M} |\bar{K}_{k,l}|_{C^{r+1},L} |(k,l)|^{m-n}
\]

\[
\leq \sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^\ell \sum_{(k,l) \in \mathbb{Z}^2, |k| + |l| \leq M} |\psi|_{C^n} |K_{k,l}|_{C^0} |(k,l)|^{m-n}
\]

\[
\leq |\psi|_{C^n} \sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^\ell \sum_{(k,l) \in \mathbb{Z}^2, |k| + |l| \leq M} |K_{k,l}|_{C^0} |(k,l)|^m
\]

\[
\leq (\ell + 1)|\psi|_{C^n} \|K\|_{C^{r+1},L}
\]

for \(\ell = 0, \ldots, r + 1\), so choosing \(C_\ell = (\ell + 1)|\psi|_{C^n}\), which is independent of \(L\), we get the desired bound.

Now, we choose \(G\) to verify equation (44) so we get the following:

1. \(G_{0,0} = 0\) and \(G_{k,l}(J) = 0\) if \((k, l) \notin \mathcal{N}\),
2. if \((0, l) \in \mathcal{N}, l \neq 0\), \(G_{0,l}(J) = K_{0,l}(J)/l\).
3. if \((k, l) \in \mathbb{N}, k \neq 0\), we choose \(G_{k,l}(J)\) as

(a) If \(J \neq -l/k\) then
\[
G_{k,l}(J) = \bar{K}_{k,l}(J) - K_{k,l}(J) \frac{Jk + l}{i(Jk + l)}.
\]

(b) \(G_{k,l}(-l/k) = \lim_{j \to -l/k} \frac{K_{k,l}(J) - \bar{K}_{k,l}(J)}{i(Jk + l)} = \bar{K}_{k,l}(-l/k) \frac{1}{ik}.
\]

Then \(G(J, \psi, s)\) is a trigonometric polynomial in \((\psi, s)\) of degree \(M\), and of class \(C^r\) with respect to \(J\). To bound the function \(G\), we first need to bound its Fourier coefficients in terms of \(|\cdot|_{C^\ell,L}^\ast\) norm for \(0 \leq \ell \leq r\). Given a fixed \((k_0, l_0) \in \mathbb{N}\), by the definition of \(K\) and \(G\), we have

1. \(\forall J, |G_{0,l}|_{C^\ell,L} \leq \frac{|K_{0,l}|_{C^\ell,L}}{|l|}, \text{ for } \ell = 0, \ldots, r.\)

2. If \(|J + l_0/k_0| \leq Lk_0, \text{ then } |G_{k_0,l_0}|_{C^\ell,L} \leq (n + 1) \frac{|K_{k_0,l_0}|_{C^{n+1,L}}}{L |k_0|}, \text{ for } n = 0, \ldots, r.\)

This estimate comes from
\[
|G_{k_0,l_0}|_{C^\ell,L} = \sum_{i=0}^{n} L^i \left| \frac{D^i G_{k_0,l_0}}{i!} \right|^\ast
\leq \sum_{i=0}^{n} L^i \left| \frac{D^{i+1} K_{k_0,l_0}}{i!} \right|^\ast
\leq \frac{(n + 1)}{L |k_0|} \sum_{i=0}^{n} L^{i+1} \frac{(i + 1)!}{|D^{i+1} K_{k_0,l_0}|^\ast}
\leq (n + 1) \frac{|K_{k_0,l_0}|_{C^{n+1,L}}}{L |k_0|}.
\]

3. If \(|J + l_0/k_0| \geq 2Lk_0\) then \(|G_{k_0,l_0}|_{C^\ell,L} \leq \frac{n + 1}{L} \sum_{j=0}^{\ell} |K_{k_0,l_0}|_{C^\ell,L} |k_0|^{n-j}, \text{ for } n = 0, \ldots, r + 1.\)

This estimate is obtained using Leibniz rule for derivatives in the following way:
\[
|G_{k_0,l_0}|_{C^\ell,L} = \sum_{i=0}^{n} L^i \left| \frac{D^i G_{k_0,l_0}}{i!} \right|^\ast
\leq \sum_{i=0}^{n} L^i \sum_{j=0}^{i} \left| \frac{D^j K_{k_0,l_0}}{(2Lk_0)^{j+1} |k_0|^j} \right|^\ast
\leq \sum_{i=0}^{n} \frac{1}{L} \sum_{j=0}^{i} L^j \left| \frac{D^j K_{k_0,l_0}}{|k_0|^{j-i}} \right|^\ast
\leq \frac{n + 1}{L} \sum_{i=0}^{n} L^i \left| \frac{D^i K_{k_0,l_0}}{|k_0|^{n-i}} \right|^\ast
\leq \frac{n + 1}{L} \sum_{i=0}^{n} |K_{k_0,l_0}|_{C^\ell,L} |k_0|^{n-i}.
\]
Using their corresponding bounds we have
\[ G_{k_0,l_0} \leq \frac{n + 1}{L} \sum_{i=0}^{n} |K_{k_0,l_0}|^{n-i} + (n+1)|K_{l_0}|^n |\psi|^n, \]
for \( n = 0, \ldots, r \) and \( C \) is a constant independent of \( L \).

This estimate can be obtained in the same way as the previous one using the estimate obtained for \( K_{k,l} \) in (46), in the following way:
\[
|G_{k_0,l_0}|_{C^0,I} = \sum_{i=0}^{n} \frac{L^i}{i!} \left| D^i \left( i \frac{K_{k_0,l_0}}{Jk_0 + l_0} \right) \right|_{C^0} + \sum_{i=0}^{n} \frac{L^i}{i!} \left| D^i \left( -i \frac{K_{k_0,l_0}}{Jk_0 + l_0} \right) \right|_{C^0} \\
\leq \frac{(n + 1)^2}{L} \left| K_{k,l} \right|_0 |k|^n |\psi|^n + \frac{n + 1}{L} \sum_{i=0}^{n} \left| K_{k_0,l_0} \right|_{C^0,I} |k_0|^{n-i}. 
\]

In order to finish the proof, we will use these estimates for the Fourier coefficients of \( G \) to bound the function \( G \).

First we concentrate on the set \( \mathcal{I}' \subseteq \mathcal{I} \) formed by \( J \in \mathbb{R} \), such that \( |J + l_0/k_0| \leq L_0 \) for some \(-l_0/k_0 \in \mathbb{R}\). Note that if \( J \in \mathcal{I}' \), for any other \((k,l) \in \mathcal{N}\), such that \((k,l) \neq (k_0,l_0)\) for \( t \in \mathbb{Z} \), \( J \) satisfies that \( |J + l/k| \geq 2L_k \). Therefore, we will distinguish three types of Fourier coefficients \( G_{k,l} \) of \( G \), which are the ones described in points 1, 2 and 3 in this proof. Using their corresponding bounds we have
\[
\|G\|_{C^0(\mathcal{I}' \times \mathcal{J})} = \sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^\ell \left( \sum_{l=-M}^{M} \left| G_{0,l} \right|_{C^0,L} |l|^{m-n} + \sum_{(k,l) \in \mathcal{N}} \left| G_{k,l} \right|_{C^0,I} |\psi(k,l)|^{m-n} \right) \\
+ \sum_{(k,l) \notin \mathcal{N}} \left( (n + 1) \left| K_{k,l} \right|_0 |k|^n |\psi|^n \right)^{m-n} \\
+ \sum_{(k,l) \notin \mathcal{N}} \left( \left| K_{k,l} \right|_{C^0,I} |k|^{n-i} \right)^{m-n} \\
+ \sum_{(k,l) \notin \mathcal{N}} \left( \left| K_{k,l} \right|_{C^0,I} |k_0| |\psi(k,l)|^{m-n} \right)^{m-n} \\
+ \sum_{(k,l) \notin \mathcal{N}} \left( \left| K_{k,l} \right|_{C^0,I} |k_0| |\psi(k,l)|^{m-n} \right)^{m-n} \\
+ \sum_{(k,l) \notin \mathcal{N}} \left( \left| K_{k,l} \right|_{C^0,I} |k_0| |\psi(k,l)|^{m-n} \right)^{m-n}.
\]
for some $\ell$ for

$$\sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^l \left( \sum_{l=-M}^{M} |K_{0,l}|_{C^{\ell},L}^2 |\mathcal{R}|^{m-n} \right)$$

+ $\sum_{(k,l) \in \mathcal{N}} \frac{(n+1)}{L} \sum_{i=0}^{n} |K_{k,l}|_{C^{\ell},L} |(k,l)|^{m-i} \right)$$

+ $\sum_{\ell \in \mathbb{Z} \setminus \{0\}} \frac{(n+1)}{L} |K_{(k_0,t_0)}|_{C^{\ell},L} |t(k_0,t_0)|^{m-n-1} \right)$$

$$\leq \frac{n+1}{L} \sum_{(k,l) \in \mathcal{N}} \frac{(n+1)(m+1)}{L} \sum_{\ell \in \mathbb{Z} \setminus \{0\}} \frac{(n+1)}{L} |K_{(k_0,t_0)}|_{C^{\ell},L} |t(k_0,t_0)|^{m-n-1} \right)$$

$$\leq \frac{C_{\ell}}{L} \|K\|_{C^{\ell},L}$$

for $\ell = 0, \ldots, r$, where $C_{\ell} = 3(\ell + 1)^2$ is a constant independent of $L$.

Analogously, for the set $\mathcal{I}' \subset \mathcal{I}$ formed by $J \in \mathbb{R}$ such that $L_{k_0} \leq |J + l_0/k_0| \leq 2L_{k_0}$ for some $-l_0/l_0 \in \mathbb{R}$, we note that if $J \in \mathcal{I}'$ then for any other $(k,l) \in \mathcal{N}$ such that $(k,l) \neq (t(k_0,t_0), t \in \mathbb{Z}, J$ satisfies that $|J + l/k| \geq 2L_k$. In this case, we will distinguish three types of Fourier coefficients $G_{k,l}$ of $G$, which are the ones described in points 1, 3 and 4 in this proof. Using the same argument as in the previous case, jointly with the bounds for the Fourier coefficients, we have that

$$\|G\|_{C^{\ell}(\mathcal{I}' \times \mathcal{T}_\mathcal{I})} \leq \frac{(\ell + 1)^2}{L} \left( \|K\|_{C^{\ell},L} + \|\psi\|_{C^{\ell},L} \|K\|_{C^{\ell},L} \right)$$

$$\|G\|_{C^{\ell}(\mathcal{I}' \times \mathcal{T}_\mathcal{I})} \leq \frac{C_{\ell}}{L} \|K\|_{C^{\ell},L}$$

for $\ell = 0, \ldots, r$, where $C_{\ell} = 4(\ell + 1)^2$ is a constant independent of $L$.

And finally, for the remaining set $\mathcal{I}'' \subset \mathcal{I}$ formed by $J \in \mathbb{R}$, such that $|J + l/k| \geq 2L_k$ for any $(k,l) \in \mathcal{N}$, the Fourier coefficients $G_{k,l}$ of $G$ are just the ones described in points 1 and 3. Arguing as before we have

$$\|G\|_{C^{\ell}(\mathcal{I}'' \times \mathcal{T}_\mathcal{I})} \leq \frac{(\ell + 1)^2}{L} \|K\|_{C^{\ell},L}$$

for $\ell = 0, \ldots, r$, where $C_{\ell} = 2(\ell + 1)^2$ is a constant independent of $L$. 
So putting all these estimates together we get the desired bound (43) for the whole domain.

3.2.3. The main averaging result. In this section we apply repeatedly the procedure stated in the previous section to the truncated Fourier series of the perturbation $R^{q,M}$ in (27) to get a suitable normal form.

We start the averaging procedure with the Hamiltonian (25) truncated up to order $M$,

$$k_0(J,\varphi, s;\varepsilon) = Z^0(J,\varphi, s;\varepsilon) + \varepsilon^R(\varphi, s;\varepsilon),$$

where $Z^0(J,\varphi, s;\varepsilon) = J^2/2$ and $R^0(J,\varphi, s;\varepsilon) = R^{q,M}(J,\varphi, s;\varepsilon)$, which is a trigonometric polynomial of degree $M$ in the angle variables $(\varphi, s)$.

We will search for a canonical transformation $g_0$, given by the time-one map of the flow of Hamiltonian $\varepsilon G_0$ provided by lemma 3.3 that eliminates, when it is possible, the dependence on the angle variables $(\varphi, s)$ at order $\varepsilon$.

According to expression (38), we will refer to resonances of order 1 as the elements of

$$\mathcal{R}_1 = \{-l/k \in \mathbb{Q} \cap (I_-, I_+) : |l| + |k| \leq M, k \neq 0, R^0_{l,k}(-l/k; 0) \neq 0\},$$

where $R^0_{l,k}$ are the Fourier coefficients of $R^0$. For each resonance $-l/k \in \mathcal{R}_1$ we will define a strip of size $2L/|k|$, for $L \sim \varepsilon$ and $\alpha > 0$, centred on the resonance. We will call resonant region of order 1 the union of these strips, where the averaging transformation $g_0$ cannot eliminate the dependence on the angle variables, and non-resonant region up to order 1 the complementary region in $\tilde{\Lambda}_\varepsilon$, where $k_0 \circ g_0$ reduces to contain only the harmonic $R^0_{0,0}(J; 0)$ at order $\varepsilon$.

Hence, the Hamiltonian $k_1 = k_0 \circ g_0$ is now of the form

$$k_1(J,\varphi, s;\varepsilon) = Z^1(J,\varphi, s;\varepsilon) + \varepsilon^{R_1}(J,\varphi, s;\varepsilon),$$

where the normal form $Z^1$ is a $C^r$ function, which has different expressions in the resonant and non-resonant regions, and the remainder $\varepsilon^R$ is a $C^{r-2}$ function.

Proceeding by induction, we obtain a sequence of Hamiltonians $k_{q-1}$, for $q \geq 1$, which are normalized up to order $\varepsilon^{q-1}$, that is in adequate symplectic coordinates Hamiltonian $k_{q-1}$ takes the form

$$k_{q-1}(J,\varphi, s;\varepsilon) = Z^{q-1}(J,\varphi, s;\varepsilon) + \varepsilon^{R_{q-1}^{(q-1)}}(J,\varphi, s;\varepsilon),$$

where, as before, the normal form $Z^{q-1}$ is a $C^{r-2(q-2)}$ function, which has different expressions in the resonant and non-resonant regions up to order $q - 1$, and the remainder $\varepsilon^{R_{q-1}^{(q-1)}}$ is a $C^{r-2(q-1)}$ function.

The set of resonances of order $q$ and its associated resonant and non-resonant regions up to order $q$ are defined recursively in the following way:

**Definition 3.4.** The set of resonances of order $q \geq 1$ is the set of rational numbers $r \in \mathbb{Q} \setminus (\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_{q-1})$, where $\mathbb{Q}$ is the set of rational numbers $r \in \mathbb{Q} \cap (I_-, I_+)$ which admit a representation $r = -l/k$ for some integers $k, l$ satisfying $|l| + |k| \leq qM$, such that $R^q_{l,k}(-l/k; 0) \neq 0$; in symbols,

$$\mathcal{R}_q = \mathbb{Q} \setminus (\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_{q-1}) \setminus \{ -l/k \in \mathbb{Q} \cap (I_-, I_+) : |k| + |l| \leq qM, k \neq 0, R^q_{l,k}(-l/k; 0) \neq 0 \}. \quad (48)$$

where $R^q_{l,k}$ are the Fourier coefficients of the remainder $R^q$ in (47).

Roughly speaking, we call resonances of order $q$ the places in $J$ where the $q$th order averaging cannot eliminate the dependence on the angles at order $q$. 

![Image of a page from a document with text](image-url)
Remark 3.5. Note that, by hypothesis $H3'$ in theorem 2.1, for all $-l_0/k_0 \in \mathbb{Q} \cap (L_-, L_+)$ such that $|k_0, l_0| < M_{BG}$ there exists $t^* \in \mathbb{Z}^2$ such that $h_{r,k_0,t^*l_0}(0, 0, -l_0/k_0; 0) \neq 0$ and therefore, by equation (26), $R_{r,k_0,t^*l_0}(-l_0/k_0; 0) \neq 0$. Hence, by definition 3.4 for resonances of order 1, as long as $M_{BG} \leq M$, all the rational numbers $-l/k$ with $|k, l| < M_{BG}$ are resonant of order 1.

Definition 3.6. The set $\mathcal{R}_{\leq q}(M)$ of resonances up to order $q$ is the union of sets of resonances of order $i$, for $i = 1, \ldots, q$; in symbols,

$$\mathcal{R}_{\leq q} = \mathcal{R}_{\leq q}(M) = \bigcup_{i=1}^{q} \mathcal{R}_i(M) \subset \mathbb{Q}. \quad (49)$$

For this set of resonances we define different strips in $\tilde{\Lambda}_\epsilon$ of a width depending on a parameter $L$, which is $L \sim \epsilon^\alpha$, with $\alpha > 0$. This divides the phase space into two types of regions:

Definition 3.7. The non-resonant region up to order $q$ $\mathcal{D}_{ir}^q(M)$ is the set of points $(J, \varphi, s) \in \tilde{\Lambda}_\epsilon$ which are at a distance greater than $2L_k$ in terms of the $J$ variable of any resonance $-l/k \in \mathcal{R}_{\leq q}(M)$, where $L_k = L/|k|$; in symbols,

$$\mathcal{D}_{ir}^q = \mathcal{D}_{ir}^q(M, L) = \left\{ (J, \varphi, s) \in (J_-, J_+) \times \mathbb{T}^2 : \left| J + \frac{l}{k} \right| > 2L_k, \quad \text{for } -\frac{l}{k} \in \mathcal{R}_{\leq q}(M) \right\}. \quad (50)$$

Definition 3.8. The resonant region of order $q$ $\mathcal{D}_{r,q}(M, L)$ is the set of points $(J, \varphi, s) \in \tilde{\Lambda}_\epsilon$ which are at a distance smaller than $L_k = L/|k|$ in terms of the $J$ variable from any resonance $-l/k \in \mathcal{R}_q \setminus (\mathcal{R}_1 \cup \ldots \cup \mathcal{R}_{q-1})$; in symbols,

$$\mathcal{D}_{r,q} = \mathcal{D}_{r,q}(M, L) = \left\{ (J, \varphi, s) \in (J_-, J_+) \times \mathbb{T}^2 : \left| J + \frac{l}{k} \right| \leq L_k, \quad \text{for some } -\frac{l}{k} \in \mathcal{R}_q \setminus (\mathcal{R}_1 \cup \ldots \cup \mathcal{R}_{q-1}) \right\}. \quad (51)$$

The union of resonant regions of order $i$, for $i = 1, \ldots, q$, gives us the resonant region up to order $q$, which can be defined in the following way:

Definition 3.9. The resonant region up to order $q$ $\mathcal{D}_{i}^q(M, L)$ is the set of points $(J, \varphi, s) \in \tilde{\Lambda}_\epsilon$ which are at a distance smaller than $L_k = L/|k|$ in terms of the $J$ variable from any resonance $-l/k \in \mathcal{R}_{\leq q}(M)$; in symbols,

$$\mathcal{D}_{i}^q = \mathcal{D}_{i}^q(M, L) = \left\{ (J, \varphi, s) \in (J_-, J_+) \times \mathbb{T}^2 : \left| J + \frac{l}{k} \right| \leq L_k, \quad \text{for some } -\frac{l}{k} \in \mathcal{R}_{\leq q}(M) \right\} \quad (52)$$

The dependence of these domains on $M$ and $L$, $\mathcal{D}_{ir}^q = \mathcal{D}_{ir}^q(M, L)$, $\mathcal{D}_{r,q} = \mathcal{D}_{r,q}(M, L)$ and $\mathcal{D}_{i}^q = \mathcal{D}_{i}^q(M, L)$, will be suppressed to simplify notation.

Remark 3.10. Note that, by remark 3.5, the big gaps region $\mathcal{D}_{BG}$ introduced in (23) is contained in the resonant region of order $1 \mathcal{D}_{r,1}$.

The precise result to obtain a global normal form for the reduced Hamiltonian by applying repeatedly the averaging procedure, jointly with the estimates for the bounds of the normal form terms and the expression of the order of truncation $M$ and the constant $L$ as functions of $\epsilon$, is stated in the following theorem 3.11. It is worth remarking that we obtain a global normal form by a direct application of the global averaging procedure of lemma 3.3, thanks to the fact that the action variable $I$ is one-dimensional, so a standard smoothing procedure can be applied to the Fourier coefficients of the normal form. For $I$ multidimensional, one has to restrict to deal with local normal forms, which are nevertheless enough for the application of the KAM theorem and to get an approximate equation for the KAM tori.
Theorem 3.11. Let \( n, m \) be any given integers satisfying \( 1 \leq 2m \leq n \). Given \( \rho \) a real number satisfying
\[
\rho < \frac{1}{n},
\]
and \( r \) an integer verifying
\[
r \geq (1/\rho - 2)m + 2,
\]
consider a \( C^r \) Hamiltonian of the form (25):
\[
k(J, \varphi, s; \varepsilon) = \frac{J^2}{2} + \varepsilon R(J, \varphi, s; \varepsilon),
\]
satisfying \( \varepsilon R(J, \varphi, s; \varepsilon) = O_C(\varepsilon) \).
Introduce \( M \sim \varepsilon^{-\rho} \), for any \( -l/k \in \mathcal{R}_{\leq m}(M) \), introduced in (49), consider \( L_k = L/|k| \), where
\[
L = Ce^{\alpha}
\]
with
\[
\rho \leq \alpha < 1/n
\]
and \( C \) a constant independent of \( \varepsilon \), such that for \( -l/k \in \mathcal{R}_{\leq m} \), the real intervals \( \mathcal{I}_{-l/k} = [-l/k - 2L_k, l/k + 2L_k] \) are disjoint. Then, there exists a symplectic change of variables, depending on time, \( (J, \varphi, s) = g(B, \phi, s) \), periodic in \( \phi \) and \( s \), and of class \( C^{r-2m} \), which is \( \varepsilon \)-close to the identity in the \( C^{n-2m-1} \) sense, such that transforms the Hamiltonian system associated with \( k(J, \varphi, s; \varepsilon) \) into a Hamiltonian system of Hamiltonian
\[
\tilde{k}^m(B, \phi, s; \varepsilon) = \tilde{Z}^m(B, \phi, s; \varepsilon) + \varepsilon^{m+1} \tilde{R}^m(B, \phi, s; \varepsilon),
\]
where the function \( \tilde{Z}^m \) is of class \( C^{r-2m+2} \) and \( \tilde{R}^m \) is of class \( C^{r-2m} \) and they verify the following:

1. If \( B \notin \bigcup_{-l/k \in \mathcal{R}_{\leq m}} \mathcal{I}_{-l/k} \), then
\[
\tilde{Z}^m(B, \phi, s; \varepsilon) = \frac{1}{2} B^2 + \varepsilon \tilde{Z}^m(B; \varepsilon),
\]
for any \( (B, \phi, s) \in D^m_{\varepsilon^q} \) (\( D^m_{\varepsilon^q} \) was introduced in (50)).

2. If \( B \in \mathcal{I}_{-l/k} \) for some \( -l_0/k_0 \in \mathcal{R}_i \setminus (\mathcal{R}_i \cup \ldots \mathcal{R}_{i-1}) \), for some \( 1 \leq i \leq m \), then
\[
\tilde{Z}^m(B, \phi, s; \varepsilon) = \frac{1}{2} B^2 + \varepsilon \tilde{Z}^m(B; \varepsilon) + \varepsilon^i U^i_{m,l_0}(k_0\phi + l_0s; \varepsilon),
\]
for any \( (B, \phi, s) \in D_{\varepsilon^q} \) (\( D_{\varepsilon^q} \) was introduced in (51)).

In a particular case of a resonance \( -l_0/k_0 \) of order \( l \), \( U^i_{m,l_0}(k_0\phi + l_0s; 0) \) does not depend on \( m \) and is given by
\[
U^i_{m,l_0}(\theta; 0) = U^i_{1,l_0}(\theta) = \sum_{|r| \leq k_0 l_0} R_{k_0 l_0}(-l_0/k_0; 0)e^{ir\theta}
\]
where \( \theta = k_0\phi + l_0s \) and \( R_{k_0 l_0}(J; \varepsilon) \) are the Fourier coefficients of the perturbation \( R(J, \varphi, s; 0) \) with respect to \( (\varphi, s) \).

3. The function \( \varepsilon \tilde{Z}^m(B; \varepsilon) \) in (59) and (60) is a polynomial of degree \( m \) in \( \varepsilon \), whose term of order \( q + 1 \) is of class \( C^{r-2q} \) and of size \( O_C(\varepsilon^{q+1-a(\varepsilon^{2q})}) \), for \( \ell = 0, \ldots, n - 2q \) and \( q = 0, \ldots, m - 1 \). The function \( \varepsilon^i U^i_{m,l_0}(k_0\phi + l_0s; \varepsilon) \) in (60) is a polynomial of degree \( m \) in \( \varepsilon \) and a trigonometric polynomial in \( \theta = k_0\phi + l_0s \), which is \( O_C(\varepsilon^{(q+1-a(\varepsilon^{2q}))}) \), for \( \ell = 0, \ldots, n - 2(i - 1) \). The function \( \varepsilon^{m+1} \tilde{R}(B, \phi, s; \varepsilon) \) in (58) is \( O_C(\varepsilon^{m+1-a(\varepsilon^{2m})}) \), for \( \ell = 0, \ldots, n - 2m \). Finally, the change of variables \( (J, \varphi, s) = g(B, \phi, s) \) satisfies
\[
g - \text{Id} = O_C(\varepsilon^{1-a(\varepsilon^{2})}), \quad \text{for } \ell = 0, \ldots, n - 2m.\]
3.2.4. Proof of theorem 3.11.

This proof of theorem will follow by the repeated application of the inductive lemma 3.18 and finally theorem 3.11.

Lemma 3.15. Let \( G(J, \varphi, s) \) a Hamiltonian and assume that \( G \) is \( C^r \) trigonometric polynomial of order \( M \) defined in a compact domain \( I \times \mathbb{T}^2 \), with \( I \subset \mathbb{R} \), such that \( \sup_{x \in I \times \mathbb{T}^2} |x| \leq D \). Consider the \( C^{r-1} \) change of variables defined on \( I \times \mathbb{T}^2 \),

\[
(J, \varphi, s) = g_t(B, \phi, \varepsilon),
\]
given by the time-\( t \) map of the flow of Hamiltonian \( \varepsilon^p G(J, \varphi, s) \), for some \( p \in \mathbb{N} \). Assume that \( G \) is \( O_{C^r}(\varepsilon^n) \), \( \eta_1 \) being some positive number. Then,

\[
\max_{0 \leq t \leq 1} |g_t|_{C^r} \leq D_1, \quad \max_{0 \leq t \leq 1} |g_t - \text{Id}|_{C^r} \leq D'_1 \varepsilon^\eta_1 \tag{63}
\]

for \( \ell = 0, \ldots, r - 1, D_1 \) and \( D'_1 \) being some constants, which depend on the domain and \( \ell \), but not on \( \varepsilon \). In terms of the notation introduced in section 2.1, the above inequalities read \( g_t = O_{C^r}(1) \) and \( g_t - \text{Id} = O_{C^r}(\varepsilon^\eta_1) \), for \( \ell = 0, \ldots, r - 1 \) and \( 0 \leq t \leq 1 \).

Proof. By the fundamental theorem of calculus we can write

\[
g_t(x) = x + \int_0^t \frac{\partial g_\tau}{\partial \tau}(x) \, d\tau = x + \int_0^t J \nabla G \circ g_\tau(x) \, d\tau,
\]

where \( x = (B, \phi, s) \in I \times \mathbb{T}^2 \) and \( J \) is the canonical matrix of the symplectic form \( \omega = dJ \land d\phi + dA \land ds \). The extra variable \( A \), conjugated to the angle \( s \), was introduced to make apparent the symplectic character of the change of variables.

Using formula (C.5) in appendix C we obtain

\[
|g_t|_{C^r} \leq |\text{Id}|_{C^r} + \int_0^1 |J \nabla G \circ g_\tau|_{C^r} \, d\tau
\]

\[
\leq |\text{Id}|_{C^r} + C_1 \int_0^1 \left( |J \nabla G|_{C^r} |g_\tau|_{C^r} + |J \nabla G|_{C^r} |g_\tau|_{C^{r-\ell}} \right) \, d\tau \tag{64}
\]

for \( \ell = 2, \ldots, r - 1 \), where \( C_1 \) is a constant depending on \( \ell \); and

\[
|g_t|_{C^r} \leq |\text{Id}|_{C^r} + \int_0^1 |J \nabla G|_{C^r} |g_\tau|_{C^r} \, d\tau.
\]
Let us define $a_\ell = \max_{0 \leq \ell \leq 1} |g_{\ell}|_{C^\gamma}$. Then,

$$a_1 \leq D + \delta_1 a_1,$$

and

$$a_\ell \leq D + \delta_1 a_\ell + C_\ell \delta_\ell a_{\ell-1}^\ell,$$

with $\delta_\ell = |G|_{C^{\gamma+1}}$. Hence,

$$a_\ell \leq \frac{D + \delta_1 a_{\ell-1}^\ell}{1 - \delta_1} \quad \text{for } \ell \geq 2,$$

$$\text{since } \delta_1 \sim \varepsilon^{\mu_2} \ll 1 \text{ and } \delta_\ell \sim \varepsilon^{\mu_{\ell+1}} \ll 1,$$ it is easy to check by induction that $a_\ell \leq D_\ell$, for $\ell \geq 1$, $D_\ell$ being some constant independent of $\varepsilon$.

Denoting by $b_\ell = \max_{0 \leq \ell \leq 1} |g_{\ell} - \text{Id}|_{C^\gamma}$, one has

$$b_1 \leq \delta_1 a_1,$$

and

$$b_\ell \leq \delta_1 a_\ell + C_\ell \delta_\ell a_{\ell-1}^\ell, \quad \text{for } \ell \geq 2.$$

So that,

$$b_\ell \leq D_\ell \delta_1 + C_\ell D_{\ell-1}^\ell \delta_\ell \leq D_\ell \delta_\ell^\ell = D_\ell \varepsilon^{\mu_{\ell+1}},$$

for $\ell \geq 1$, $D_\ell$ being some constant independent of $\varepsilon$. □

Since the averaging procedure is based on the method of Lie transforms, the transformed Hamiltonian will be expressed in terms of Poisson brackets. In the following lemma 3.16 we give an estimate for the bound of the Poisson bracket of two functions, where the second one is a generating function, in terms of the bounds on the norm (32) of each one.

**Lemma 3.16.** Let $\rho, \alpha$ be two positive real numbers, such that $\rho \leq \alpha$ and $M \sim \varepsilon^{-\rho}$ and $L = C \varepsilon^\alpha$. Given $F^p(J, \varphi, s)$ and $G^q(J, \varphi, s)$ two trigonometric polynomials in $(\varphi, s)$, assume that $F^p(J, \varphi, s)$ is a $C^\alpha$, $n > 0$, function in $J$ and a trigonometric polynomial of degree $M_p = (p + 1)M$ and $G^q(J, \varphi, s)$ is a $C^m$, $m > 0$, function in $J$ and a trigonometric polynomial of degree $M_q = (q + 1)M$, that satisfy $\|\varepsilon^{p+1} F^p\|_{C^\gamma, L} \leq \varepsilon^{p+1-\alpha(2\rho)}$ and $\|\varepsilon^{q+1} G^q\|_{C^\gamma, L} \leq \varepsilon^{q+1-\alpha(2\rho)}$, for $\ell = 0, \ldots, n$, with $\varepsilon > 0$. Then $\{F^p, G^q\}$ is a $C^\gamma$ function in $J$, for $r = \min(n, m) - 1$ and a trigonometric polynomial of degree $M_p = (\tilde{p} + 1)M$ in $(\varphi, s)$, where $\tilde{p} = p + q + 1$, and $\varepsilon^{\tilde{p}+1} F^\tilde{p} = \{\varepsilon^{p+1} F^p, \varepsilon^{q+1} G^q\}$ satisfies

$$\|\varepsilon^{\tilde{p}+1} F^\tilde{p}\|_{C^\gamma, L} \leq \varepsilon^{\tilde{p}+1-\alpha(2\tilde{p})},$$

for $\ell = 0, \ldots, r$.

**Proof.** From

$$[F^p, G^q] = \frac{\partial F^p}{\partial \varphi} \frac{\partial G^q}{\partial J} - \frac{\partial F^p}{\partial J} \frac{\partial G^q}{\partial \varphi},$$

we have

$$[F^p, G^q] = \sum_{(k, l) \in \mathbb{Z}^2} i k F^p_{k,l}(J) e^{i(k \varphi + l s)} \sum_{|k| + |l| \leq M_p} \frac{\partial G^q_{k,l}(J)}{\partial J} e^{i(k \varphi + l s)}$$

$$= \sum_{(k, l) \in \mathbb{Z}^2} \frac{\partial F^p_{k,l}(J)}{\partial J} e^{i(k \varphi + l s)} \sum_{|k| + |l| \leq M_p} i k G^p_{k,l}(J) e^{i(k \varphi + l s)}.$$
It is clear from this expression that \( \{ F^p, G^q \} \) is a trigonometric polynomial of degree \( M_p + M_q = (p + q + 2)M \).

On the other hand, using equation (36), it follows that
\[
\| \varepsilon^{p+1} F^p, \varepsilon^{q+1} G^q \|_{C^1, L} \leq \varepsilon^{p+1} \sum_{(k, l) \in \mathbb{Z}^2, |k| + |l| \leq M_p} i k F^p_{k,l}(J) e^{ik\varphi + i\epsilon s} \| \varepsilon^{q+1} \sum_{(k, l) \in \mathbb{Z}^2, |k| + |l| \leq M_q} \frac{\partial G^q_{k,l}(J)}{\partial J} e^{ik\varphi + i\epsilon s} \|_{C^1, L} + \frac{1}{L} \varepsilon^{q+1} \sum_{(k, l) \in \mathbb{Z}^2, |k| + |l| \leq M_q} G^q_{k,l}(J) e^{ik\varphi + i\epsilon s} \| \varepsilon^{q+1} \sum_{(k, l) \in \mathbb{Z}^2, |k| + |l| \leq M_p} G^q_{k,l}(J) e^{ik\varphi + i\epsilon s} \|_{C^1, L} \leq \frac{2}{L} \| \varepsilon^{p+1} F^p \|_{C^1, L} \| \varepsilon^{q+1} G^q \|_{C^1, L}.
\]

Using now the hypotheses on \( \varepsilon^{q+1} F^{p} \) and \( \varepsilon^{p+1} G^{p} \) in this lemma and the fact that \( L = C \alpha \), where \( C \) is a constant independent of \( \varepsilon \), we have
\[
\| \{ \varepsilon^{p+1} F^p, \varepsilon^{q+1} G^q \} \|_{C^1, L} \leq \varepsilon^{-\alpha} \varepsilon^{p+1-\alpha(2p)} \varepsilon^{q+1-\alpha(2q+1)} = \varepsilon^{p+q+2-\alpha(2p+q+1)} = \varepsilon^{p+1-\alpha(2\tilde{p})}. \]

**Remark 3.17.** This lemma will be applied a certain number of times and expresses the fact that given two functions \( \varepsilon^{p+1} F^p \) and \( \varepsilon^{q+1} G_{q} \), which are trigonometric polynomials in \((\varphi, s)\) of degree \( M_p = (p + 1)M \) and \( M_q = (q + 1)M \), respectively, with bounds \( \| \varepsilon^{p+1} F^p \|_{C^1, L} \leq \varepsilon^{p+1-\alpha(2p)} \) and \( \| \varepsilon^{q+1} G_{q} \|_{C^1, L} \leq \varepsilon^{q+1-\alpha(2q+1)} \), its Poisson bracket is a function \( \varepsilon^{p+1} F^p \), with \( \tilde{p} = p + q + 1 \), that is, \( \varepsilon^{p+1} F^p = \{ \varepsilon^{p+1} F^p, \varepsilon^{q+1} G_{q} \} \) is a trigonometric polynomial in \((\varphi, s)\) of degree \( M_{\tilde{p}} = (\tilde{p} + 1)M \) with a bound \( \| \varepsilon^{p+1} F^p \|_{C^1, L} \leq \varepsilon^{\tilde{p}+1-\alpha(2\tilde{p})} \).

Note that this process of ‘\( \varepsilon^{q+1} G_{q} \) Poisson-bracketing’ can be iterated: \( \varepsilon^{p+1} F^p \), \( \varepsilon^{q+1} G_{q} \), with \( \tilde{p} = p + q + 1 \), is a trigonometric polynomial in \((\varphi, s)\) of degree \( M_{\tilde{p}} = (\tilde{p} + 1)M \) with a bound \( \| \varepsilon^{p+1} F^p \|_{C^1, L} \leq \varepsilon^{\tilde{p}+1-\alpha(2\tilde{p})} \).

We state and prove now the iterative lemma 3.18 for averaging, which will be applied a finite number of times \( q = 1, \ldots, m \) in the proof of theorem 3.11 and \( m \) will be chosen \( m \leq 10 \) in theorem 3.28. It basically tells us that given a Hamiltonian already in normal form up to some order \( \varepsilon^q \), we can produce another Hamiltonian which is normalized up to order \( \varepsilon^{q+1} \). The averaged Hamiltonian is given rather explicitly both in the resonant regions and in the non-resonant ones, which are redefined at every step according to the new resonances that will come up.
Lemma 3.18. Let $r > n > 1$ and $0 < 2q < n$ be any given integers. Consider a Hamiltonian of the form
\[
 k_q(J, \varphi, s; \varepsilon) = Z^q(J, \varphi, s; \varepsilon) + \varepsilon^{q+1} R^q(J, \varphi, s; \varepsilon),
\]
satisfying the following hypotheses:

1. $Z^q(J, \varphi, s; \varepsilon) = \frac{j^2}{2}$ and, for $q > 1$, $Z^q(J, \varphi, s; \varepsilon)$ is a $C^{r-2q+2}$ function that verifies the following:

There exist finite sets $R_i \subset \mathbb{Q}$, $i = 1, \ldots, q$, depending on $M \sim \varepsilon^{-\rho}$, where $\rho$ is a positive number satisfying $\rho < 1/2n$, and a number $L = C\varepsilon^r > 0$, which satisfy hypothesis (57), that is, $\rho \leq \alpha < 1/n$ and $C$ a constant independent of $\varepsilon$, such that

1a For a resonance $-l/2k$ up to order $q$, that is $-l/k \in R_{\varepsilon} \equiv \bigcup_{i=1}^{q} R_i$, see (49), the intervals $\mathcal{I}_{-l/k} \equiv [-l/k - 2L_k, -l/k + 2L_k]$, with $L_k = L/|k|$, are disjoint.

1b If $J \notin \bigcup_{l/k \in R_{\varepsilon}} \mathcal{I}_{-l/k}$, then
\[
 Z^q(J, \varphi, s; \varepsilon) = \frac{j^2}{2} + \varepsilon \tilde{Z}^q(J; \varepsilon),
\]
for any $(J, \varphi, s) \in D_{\varepsilon}^q$, $D_{\varepsilon}^q$ was introduced in (50), where $\varepsilon \tilde{Z}^q(J; \varepsilon)$ is a polynomial of degree $q$ in $\varepsilon$ whose term of order $p + 1$ is $O_{\varepsilon^{-1}}(\varepsilon^{p+1-\alpha(2q)})$, for $\ell = 0, \ldots, r-2p$ and $p = 0, \ldots, q - 1$.

1c If $J \in \mathcal{I}_{-l/k}$, for some resonance $-l_0/|k_0|$ of order $q$, that is $-l_0/|k_0| \in R_0 \cap \bigcup_{i=1}^{q} R_i$, then
\[
 Z^q(J, \varphi, s; \varepsilon) = \frac{j^2}{2} + \varepsilon \tilde{Z}^q(J; \varepsilon) + \varepsilon^j U^j_k(k_0 \varphi + l_0s; \varepsilon)
\]
for any $(J, \varphi, s) \in D_{\varepsilon}^q$, $D_{\varepsilon}^q$ was introduced in (51), where $\varepsilon \tilde{Z}^q(J; \varepsilon)$ is a polynomial of degree $q$ in $\varepsilon$ and $U^j_k(\varepsilon \varphi + l_0s; \varepsilon)$ is a polynomial of degree $q-i$ in $\varepsilon$ and a trigonometric polynomial in $\theta = k_0 \varphi + l_0s$. The term of order $p + 1$ in $\varepsilon$ of $Z^q$ is $O_{\varepsilon^{-1}}(\varepsilon^{p+1-\alpha(2q)})$, for $\ell = 0, \ldots, r-2p$ and $p = 0, \ldots, q - 1$.

2. $\varepsilon^{q+1} R^q(J, \varphi, s; \varepsilon)$ is a $C^{r-2q}$ function and is $O_{\varepsilon^{-1}}(\varepsilon^{q+1-\alpha(q+2q)})$, for $\ell = 0, \ldots, n - 2q$. For the particular case of the first iteration $(q = 0)$, $\varepsilon R^0$ is $O_{\varepsilon}$, for $\ell = 0, \ldots, n$.

The term of order $i+1$ of the Taylor expansion with respect to $\varepsilon$ of $\varepsilon^{q+1} R^q(J, \varphi, s; \varepsilon)$ is a trigonometric polynomial in $(\varphi, s)$ of degree $M_i = (i+1)M_i$ and is $O_{\varepsilon^{-1}}(\varepsilon^{q+1-\alpha(2i)})$, for $\ell = 0, \ldots, r - q - i$ and for $i = q, \ldots, r - q$.

Denote $K = R^q(J, \varphi, s; 0)$, which is the term of the perturbation of order exactly $q + 1$ in $\varepsilon$. Following definition 3.4, introduce the set
\[
 R_{q+1} = \{-l/k \in \mathbb{Q} \cap (I_-, I_+), |k| + |l| \leq M_q, k \neq 0, R^q_{q+1}(-l/k; 0) \neq 0\}.
\]
(65)

where $M_q = (q+1)M$ and $R^q_{q+1}$ are the Fourier coefficients of $R^q$.

Choose a new value of $C$, independent of $\varepsilon$, in $L = Ce^\rho$, such that the intervals $\mathcal{I}_{-l/k} \equiv [-l/k - 2L_k, -l/k + 2L_k]$, with $L_k = L/|k|$, are disjoint for $-l/k \in R_{\varepsilon}$.

Let $G(J, \varphi, s)$ be $G_q(J, \varphi, s)$ be the $C^{r-2q+1}$ trigonometric polynomial of order $M_q$ given by lemma 3.3, verifying (39) with $K = R^q(J, \varphi, s; 0)$.

Then, the $C^{r-2q-2}$ change of variables
\[
 (J, \varphi, s) = \varphi_q(\mathcal{B}, \phi, s),
\]
given by the time-one map of the flow of Hamiltonian $\varepsilon^{q+1}G_q(\mathcal{B}, \phi, s)$, transforms the Hamiltonian $k_q(J, \varphi, s; \varepsilon)$ into a Hamiltonian $k_{q+1} = k_q \circ \varphi_q$ of the form
\[
 k_{q+1}(\mathcal{B}, \phi, s; \varepsilon) = Z^{q+1}(\mathcal{B}, \phi, s; \varepsilon) + \varepsilon^{q+2} R^{q+1}(\mathcal{B}, \phi, s; \varepsilon),
\]
with
\[ Z^{q+1}(B, \phi, s; \varepsilon) = Z^q(B, \phi, s; \varepsilon) + \varepsilon^{q+1} \tilde{R}^q(B, \phi, s; 0) \]
where \( \tilde{R}^q(B, \phi, s; 0) = \tilde{K}(B, \phi, s) \) given in lemma 3.3, is a \( C^{r-2q} \) function, such that

i. If \( B \notin \bigcup_{-l/k \leq \ell \leq l/k} I_{-l/k} \), then
\[ \tilde{R}^q(B, \phi, s; 0) = R^q_{0,0}(B; 0), \]
for any \( (B, \phi, s) \in \mathcal{D}_{r+1} \) and \( \varepsilon^{q+1} \tilde{R}^q \) is \( O_{C^r,L}(\varepsilon^{q+1-\alpha(2q)}) \), for \( \ell = 0, \ldots, r - 2q \).

ii. If \( B \in I_{-l/k} \), for some \(-l_0/k_0 \in R_i \setminus R_{r+1}, \ldots \cup R_{l-1} \) for some \( 1 \leq i \leq q + 1 \), then
\[ \tilde{R}^q(B, \phi, s) = R^q_{0,0}(B; 0) + \sum_{m \in \mathbb{Z} \setminus \{0\}} R^q_{m,k,m}(-l_0/k_0; 0) \varepsilon^q. \]  \( (66) \)
for any \( (B, \phi, s) \in \mathcal{D}_{r+1} \), where \( R^q_{m,k,m}(J; \phi, s) \) are the Fourier coefficients of the function \( \tilde{R}^q(J, \phi, s; \varepsilon) \) with respect to \( (\phi, s) \). Moreover, \( \varepsilon^{q+1} \tilde{R}^q \) is \( O_{C^r,L}(\varepsilon^{q+1-\alpha(2q)}) \), for \( \ell = 0, \ldots, r - 2q \).

Moreover, the Hamiltonian \( Z^{q+1}(B, \phi, s; \varepsilon) \) verifies properties 1b and 1c up to order \( q + 1 \), and \( R^{q+1}(B, \phi, s; \varepsilon) \) verifies property 2 replacing \( q \) by \( q + 1 \).

Remark 3.19. Note that all the terms of order \( p + 1 \), for \( p \geq 0 \), in the Taylor expansion in \( \varepsilon \) that appear in lemma 3.18 are \( C^{r-2p} \) functions in \( J \) and trigonometric polynomials in the variables \( (\phi, s) \) and they are bounded independently of \( \varepsilon \) in the Fourier weighted norm \( \| \cdot \|_{C^r,L} \) defined in (32) for \( \ell \) up to \( r - 2p \). However, the whole remainder term \( \varepsilon^{q+2} R^{q+1} \) is not a trigonometric polynomial in the variables \( (\phi, s) \), so we cannot use the Fourier weighted norm. In this case we estimate their supremum norm \( \| \cdot \|_{C^r} \) defined in (2), but only for \( \ell \) up to \( n - 2q \), as in theorem 3.11 (see remark 3.14).

Proof. We will apply lemma 3.3 with \( K = R^q(J, \phi, s; 0) \), which is a \( C^{r-2q} \) function, as well as a trigonometric polynomial in \( \phi, s \) of degree \( M_q = (q + 1)M \). Accordingly, by definition 3.4, resonances of order \( q + 1 \) correspond to the set of rational numbers \( r \in R_{q+1} \setminus (R_1 \cup \ldots \cup R_q) \).

Let us first see that taking \( L = Ce^\alpha \), with \( \alpha \) satisfying \( \alpha < 1/n \) and \( C = C_q \) chosen adequately, the real intervals \( I_{-l/k} = [-l/k - 2L_k, -l/k + 2L_k] \), with \( L_k = L/|k| \), for \(-l/k \in R_{\leq q+1} \) are disjoint. Indeed, the distance \( d_{k,k_0} \) between any two resonances \(-l_0/k_0 \in R_{\leq q+1} \) is greater than or equal to \( 1/|k_0||k| \). In order to avoid overlapping between all these intervals, the distance \( d_{k,k_0} \) must be greater than \( 2L_k + 2L_{k_0} \). Taking into account that we only consider resonances with denominators \( |k|, |k_0| \leq (q + 1)M \), the condition that ensures that these intervals are separated is \( 1/((q + 1)M) > 4L \), which requires \( \rho \leq \alpha \) in terms of exponents of \( \varepsilon \). This is guaranteed by the hypothesis on \( \alpha \) and \( \rho \) in this lemma.

Hence, we can apply lemma 3.3, obtaining a \( C^{r-2q-1} \) function \( G_q(J, \phi, s) \) and a \( C^{r-2q} \) function \( K = R^q(J, \phi, s) \), which are also trigonometric polynomials in \( \phi, s \) of degree \( M_q \).

Under the canonical change of variables \( (J, \phi, s) = g_q(B, \phi, s) \), where \( g_q \) is the time-one map of the flow of Hamiltonian \( \varepsilon^{q+1} G_q \), the extended autonomous Hamiltonian \( A + k_q \) becomes
\[ A + k_{q+1} = (A + k_q) \circ g_q \]
\[ = (A + Z^q + \varepsilon^{q+1} R^q) \circ g_q \]
\[ = A + Z^q + \varepsilon^{q+1} (\{A + Z^q, G_q\} + R^q (\cdot, 0)) \]
\[ + (Z^q - Z^0) \circ g_q - (Z^q - Z^0) \]
\[ + (A + Z^0) \circ g_q - (A + Z^0) - \{A + Z^q, \varepsilon^{q+1} G_q\} \]
\[ + \varepsilon^{q+1} (R^q \circ g_q - R^q) + \varepsilon^{q+1} (R^q - R^q (\cdot, 0)) \]
\[ := A + Z^q + \varepsilon^{q+1} \tilde{R}^q + \varepsilon^{q+2} R^{q+1}, \]
\[ R^q = \{ A + Z^0, G_q \} + R^q (\cdot, 0), \tag{67} \]
and
\[ \varepsilon \varepsilon^2 R^{q+1} = (Z^q - Z^0) \circ g_q - (Z^q - Z^0) + (A + Z^0) \circ g_q - (A + Z^0) - \{ A + Z^0, \varepsilon \varepsilon^2 G_q \} + \varepsilon \varepsilon^2 (R^q \circ g_q - R^q) + \varepsilon \varepsilon^2 (R^q - R^q (; 0)) \tag{68} \]

We first see that the normal form term \( \varepsilon \varepsilon^2 R^q \) is bounded in the \( || \cdot ||_{C^1} \) norm by \( \varepsilon \varepsilon^2 (2^q) \), for \( \ell = 0, \ldots, n - 2q \).

Indeed, using (40) and (41) from lemma 3.3 we have

i. If \( B \notin \bigcup_{i/k \in \mathcal{R}_i} I_{-i/k}, \) then
\[ R^q (B, \phi, s) = R^q_{0,0}(B; 0) \tag{69} \]
for any \((B, \phi, s) \in D_{n+1}^{\ell,L}\) and, by formula (42) and the second part of hypothesis 2 for \( i = q \) of lemma 3.18, we have
\[ \| \varepsilon \varepsilon^2 \varepsilon^i R^q \|_{C^1} \leq \| \varepsilon \varepsilon^2 R^q (: 0) \|_{C^1} \leq \varepsilon \varepsilon^{q+1-\alpha(2q)}, \tag{70} \]
for \( \ell = 0, \ldots, r - 2q \).

ii. If \( B \in \mathcal{I}_{-i/k} \), for some \(-l_0/k_0 \in \bigcup_{i} R_i \cup \cdots \cup R_i_{-1} \) for some \( 1 \leq i \leq q + 1 \), then, by equation (41) in lemma 3.3,
\[ R^q (B, \phi, s) = R^q_{0,0}(B; 0) + \sum_{I \in \mathcal{D}_{n+1}^\ell} R^q_{I_{-l_0}^n t_0} (-l_0/k_0; 0) e^{it_0} \tag{71} \]
for any \((B, \phi, s) \in D_{n+1}^{\ell,L}\), where \( R^q_{I_{-l_0}^n t_0} (J; 0) \) are the Fourier coefficients of the function \( R^q (J, \phi, s; 0) \) with respect to \((\phi, s)\).

As before, by formula (42) from lemma 3.3 and the second part of hypothesis 2 of this lemma for \( i = q \), we have
\[ \| \varepsilon \varepsilon^2 \varepsilon^i R^q \|_{C^1} \leq \| \varepsilon \varepsilon^2 R^q (: 0) \|_{C^1} \leq \varepsilon \varepsilon^{q+1-\alpha(2q)} \tag{72} \]
for \( \ell = 0, \ldots, r - 2q \).

Note that, since \( \alpha < 1/n \) and \( 2q < n \), the power of \( \varepsilon \) in the bounds obtained in (70) and (72) is a positive number greater than \( q \).

To finish the proof, we only need to estimate the remainder term \( \varepsilon \varepsilon^2 R^{q+1} \) in (68) and its Taylor expansion coefficients with respect to \( \varepsilon \).

We will first estimate the remainder term \( \varepsilon \varepsilon^2 R^{q+1} \) in (68). Since it is not a trigonometric polynomial we will estimate it in terms of the supremum norm \( || \cdot ||_{C^1} \). Using the integral bound for the Taylor remainder and definitions (67) and (68) of \( R^q \) and \( \varepsilon \varepsilon^2 R^{q+1} \), respectively, we have
\[
\left| \varepsilon \varepsilon^2 R^{q+1} \right|_{C^1} \leq \int_0^1 \left| [Z^q - Z^0, \varepsilon \varepsilon^2 G_q] \circ g_q, t \right|_{C^1} \, dt \\
+ \int_0^1 \left| (1 - t) ([A + Z^0, \varepsilon \varepsilon^2 G_q], \varepsilon \varepsilon^2 G_q] \circ g_q, t \right|_{C^1} \, dt \\
+ \int_0^1 \left| \varepsilon \varepsilon^2 (R^q , \varepsilon \varepsilon^2 G_q] \circ g_q, t \right|_{C^1} \, dt \\
+ \int_0^1 \left| \varepsilon \varepsilon^2 (R^q - R^q (: 0)) \right|_{C^1} \, dt
\]
\[
= \int_0^t \left| [Z^q - Z^0, \varepsilon^{\varphi + 1} G_q] \circ g_{q,t} \right|_{\mathcal{C}^1} dt \\
+ \int_0^t \left| (1 - t) [\varepsilon^{\varphi + 1} (\tilde{R}^q - R^q(; 0)), \varepsilon^{\varphi + 1} G_q] \circ g_{q,t} \right|_{\mathcal{C}^1} dt \\
+ \int_0^t \left| [\varepsilon^{\varphi + 1} R^q, \varepsilon^{\varphi + 1} G_q] \circ g_{q,t} \right|_{\mathcal{C}^1} dt \\
+ \left| \varepsilon^{\varphi + 1} (R^q - R^q(0)) \right|_{\mathcal{C}^1},
\]
for \( t = 0, \ldots, n - 2(q + 1) \).

Using Faa–di Bruno formulae (C.4) we obtain
\[
\left| \varepsilon^{\varphi + 2} R^{\varphi + 1} \right|_{\mathcal{C}^1} \leq \left| [Z^q - Z^0, \varepsilon^{\varphi + 1} G_q] \right|_{\mathcal{C}^1} \int_0^t \left| g_{q,t} \right|_{\mathcal{C}^1}^\ell dt \\
+ \left| \varepsilon^{\varphi + 1} (\tilde{R}^q - R^q(; 0)), \varepsilon^{\varphi + 1} G_q \right|_{\mathcal{C}^1} \int_0^t (1 - t) \left| g_{q,t} \right|_{\mathcal{C}^1}^\ell dt \\
+ \left| [\varepsilon^{\varphi + 1} R^q, \varepsilon^{\varphi + 1} G_q] \right|_{\mathcal{C}^1} \int_0^t \left| g_{q,t} \right|_{\mathcal{C}^1}^\ell dt \\
+ \left| \varepsilon^{\varphi + 1} (R^q - R^q(0)) \right|_{\mathcal{C}^1},
\]
for \( t = 0, \ldots, n - 2(q + 1) \).

By formula (43) from lemma 3.3, the second part of hypothesis 2 for \( i = q \) of this lemma, and using that \( L \sim \varepsilon^m \), we get that
\[
\varepsilon^{\varphi + 1} G_q \left|_{\mathcal{C}^1} \right| \leq \frac{C}{L} \varepsilon^{\varphi + 1} R^q(0) \left|_{\mathcal{C}^1} \right| \leq \varepsilon^{\varphi + 1 - \alpha(2q + 1)},
\]
for \( t = 0, \ldots, r - 2q - 1 \) and the power of \( \varepsilon, \eta \), \( q + 1 - \alpha(\ell + 2q + 1) > q + 1 - \alpha \alpha \) in equation (74) is positive. So, we can apply lemma 3.15 with \( G = \varepsilon^{\varphi + 1} G_q \) in \( D = (I, \ldots, I_s) \times T^2 \), and we have that \( g_{q,t} = O_{\mathcal{C}^1}(1) \) and \( g_{q,t} = \text{Id} = O_{\mathcal{C}^1}(\varepsilon^{\varphi + 1 - \alpha(2q + 1)}) \) for \( t \in [0, 1) \) and \( \ell = 0, \ldots, n - 2(q + 1) \).

In expression (73), the terms \( Z^q - Z^0, G_q, \tilde{R}^q \) and \( R^q(; 0) \) are trigonometric polynomials in the variables \((\varphi, s)\). In order to bound their corresponding Poisson brackets in the \( \left| \cdot \right|_{\mathcal{C}^1} \) norm, we will first estimate their \( \left\| \cdot \right\|_{\mathcal{C}^1, L} \) norm and apply lemma 3.16. Finally, using the equivalence relation (33) between \( \left| \cdot \right|_{\mathcal{C}^1} \) and \( \left\| \cdot \right\|_{\mathcal{C}^1, L} \) norms, \( \varepsilon^{\varphi + 1} G_q \) satisfies
\[
\left| \varepsilon^{\varphi + 1} G_q \right|_{\mathcal{C}^1} \leq \varepsilon^{\varphi + 1 - \alpha(2q + 1)},
\]
for \( t = 0, \ldots, n - 2q - 1 \), and the power of \( \varepsilon, \eta \), \( q + 1 - \alpha(\ell + 2q + 1) > q + 1 - \alpha \alpha \) in equation (74) is positive. So, we can apply lemma 3.15 with \( G = \varepsilon^{\varphi + 1} G_q \) in \( D = (I, \ldots, I_s) \times T^2 \), and we have that \( g_{q,t} = O_{\mathcal{C}^1}(1) \) and \( g_{q,t} = \text{Id} = O_{\mathcal{C}^1}(\varepsilon^{\varphi + 1 - \alpha(2q + 1)}) \) for \( t \in [0, 1) \) and \( \ell = 0, \ldots, n - 2(q + 1) \).

The terms \( \varepsilon^{\varphi + 1} R^q(; 0) \) and \( \varepsilon^{\varphi + 1} \tilde{R}^q \) in (73) are both bounded in the \( \left\| \cdot \right\|_{\mathcal{C}^1, L} \) norm by \( \varepsilon^{\varphi + 1 - \alpha(2q)} \), for \( t = 0, \ldots, r - 2q \), because of the second part of hypothesis 2 for \( i = q \) and points (i) and (ii) already proved, respectively. Note that both terms are of type \( \varepsilon^{\varphi + 1} F^q \), according to remark 3.17.

The term \( Z^q - Z^0 = \varepsilon \tilde{R}^0 + \varepsilon^2 \tilde{R}^1 + \ldots \) is a polynomial with respect to \( \varepsilon \), so it can be bounded by its main term \( \varepsilon \tilde{R}^0 \). Hence, using the bound for the term of order 1 \((p = 0)\) of \( Z^q \)
given in hypotheses 1b and 1c, we have
\[ \| Z^g - Z^0 \|_{c', L} \leq \| \tilde{R}^0 \|_{c', L} \leq \varepsilon, \]  
for \( \ell = 0, \ldots, r - 2(q - 1) \). Note that \( \varepsilon \tilde{R}^0 \) is of type \( \varepsilon F^0 \), according to remark 3.17.

The estimate for the \( |.|_{c'} \) norm of the term \( (R^g - R^0(\cdot; 0)) \) can be obtained from the bound for the Taylor remainder and the first part of hypothesis 2. More precisely,
\[ |\varepsilon^{q+1}(R^g - R^0(\cdot; 0))|_{c'} \leq \varepsilon^{q+2}|R^g|_{c'\times 1} \leq \varepsilon^{q+2-\alpha(\ell+1+2q)}, \]  
for \( \ell = 0, \ldots, n - 2q - 1 \).

Moreover using the bounds for \( \varepsilon^{q+1}R^g \) and \( \varepsilon^{q+1}G_q \) in the \( |.|_{c'} \) norm, and Leibniz rule for derivatives we have
\[ \left| \varepsilon^{q+1}R^g, \varepsilon^{q+1}G_q \right|_{c'} \leq \sum_{i=0}^{\ell} \binom{\ell}{i} \left( |\varepsilon^{q+1} \frac{\partial R^g}{\partial \varphi}|_c |\varepsilon^{q+1} \frac{\partial G_q}{\partial \varphi}|_c + |\varepsilon^{q+1} \frac{\partial R^g}{\partial J}|_c |\varepsilon^{q+1} \frac{\partial G_q}{\partial J}|_c \right). \]

Hence, using that \( |R^0|_{c'} \leq 1 \) and \( |G_0|_{c'} \leq \varepsilon^{-\alpha(2+q)} \) from (74), we have
\[ \left| \varepsilon^{q+1}R^g, \varepsilon^{q+1}G_q \right|_{c'} \leq \varepsilon^{1-\alpha(2+q)} \leq \varepsilon^{2-\alpha(2+q)}, \]
otherwise,
\[ \left| \varepsilon^{q+1}R^g, \varepsilon^{q+1}G_q \right|_{c'} \leq \sum_{i=0}^{\ell} \binom{\ell}{i} \varepsilon^{\alpha(i+i+2q) \varepsilon^{q+1-\alpha(\ell+i+2q+1)}} \leq \varepsilon^{2(q+1)-\alpha(i+2(q+1)+1)}, \]
for \( \ell = 0, \ldots, n - 2(q + 1) \).

Putting together in (73) the estimates in (74), (75) and (76), as well as the estimate for \( \varepsilon^{q+1}R^g, \varepsilon^{q+1}G_q \) and \( \varepsilon^{q+1}\tilde{R}^0 \) (these last two not relevant for \( q \neq 0 \)), and using lemma 3.16 and the equivalence relation (33) one gets the following bound for (68):
\[ \left| \varepsilon^{q+2}R^{q+1} \right|_{c'} \leq \varepsilon^{q+2-\alpha(2(q+1))}, \]
for \( \ell = 0, \ldots, n - 2(q + 1) \).

Finally, all the terms in the Taylor expansion of \( \varepsilon^{q+2}R^{q+1}(B, \phi, s, \varepsilon) \) with respect to \( \varepsilon \) are obtained from a finite number of algebraic operations and a process of "\( \varepsilon^{p+1}G_q \) Poisson bracketing", as stated in remark 3.17, to the Taylor coefficients in \( \varepsilon \) of \( Z^g \) and of \( \varepsilon^{q+1}R^g \), all of which are of the form \( \varepsilon^{p+1}F^p \). Applying lemma 3.16 iteratively, we conclude that the Taylor expansion coefficient of order \( i + 1 \) of \( \varepsilon^{q+2}R^{q+1}(B, \phi, s, \varepsilon) \) with respect to \( \varepsilon \) is of the type \( \varepsilon^{q+1}F^i \) according to remark 3.17, that is a trigonometric polynomial of order \( M_i = (i+1)M \) in the angle variables, satisfying \( C_{c', L}(\varepsilon^{q+1-\alpha(2i)}) \) for \( \ell = 0, \ldots, r - q - i \) and for \( i = q, \ldots, r - q \). Again, by condition \( \alpha < 1/n \), the power of \( \varepsilon \) is a positive number greater than \( i \).

**Proof of theorem 3.11.** The proof is by induction in \( q \). To begin the induction process, we consider \( R^{[\leq M]} \), which is the truncated Fourier series of the perturbation \( R \) up to some order \( M_0 = M \) as in (27). The order of truncation \( M = M \sim \varepsilon^{-\rho} \), with \( \rho \) satisfying hypothesis (53).

We want to apply lemma 3.18 for \( q = 0 \) to the Hamiltonian
\[ k_0(J, \varphi, s; \varepsilon) = Z^0(J, \varphi, s; \varepsilon) + \varepsilon R^0(J, \varphi, s; \varepsilon), \]
where \( Z^0(J, \varphi, s; \varepsilon) = J^2/2 \) and \( R^0(J, \varphi, s; \varepsilon) = R^{[\leq M]}(J, \varphi, s; \varepsilon) \).
We introduce the finite set
\[ \mathcal{R}_1 = \{ -l/k \in \mathbb{Q} \cap (I_n, I_\epsilon), |k| + |l| \leq M, k \neq 0, R_{ij}^0(-l/k; 0) \neq 0 \}, \]
where \( R_{ij}^0 \) are the Fourier coefficients of \( R^0 \). According to definition 3.4 we will refer to resonances of order 1 the elements of the set \( \mathcal{R}_1 \).

Since \( Z^0 = J^2/2 \) and \( R^0 \) satisfy trivially hypotheses 1 and 2 of lemma 3.18 and hypothesis (57) holds, we can apply lemma 3.18 for \( q = 0 \), which provides a symplectic change of variables \( (\mathcal{B}, \phi, s) \mapsto (J, \varphi, s) = g_0(\mathcal{B}, \phi, s) \) of class \( C^{-2} \), and we get a Hamiltonian of the form
\[ k_1(J, \varphi, s; \varepsilon) = Z^1(J, \varphi, s; \varepsilon) + \varepsilon^2 R^1(J, \varphi, s; \varepsilon), \]
where \( Z^1 \) is a \( C^\infty \) function and \( \varepsilon^2 R^1 \) is a \( C^{-2} \) function, verifying properties 1b,1c and 2 of lemma 3.18 with \( q = 1 \).

In particular, in the resonant regions of order 1 \( \mathcal{D}_{\mathcal{R}_1} \) defined in (51), expression (66) in lemma 3.18 for \( q = 0 \) provides that \( Z^1 \) has the form (60) for \( i = m = 1 \), that is
\[ Z^1(\mathcal{B}, \phi, s; \varepsilon) = \frac{1}{2} \mathcal{B}^2 + \varepsilon Z^1(B) + \varepsilon U^{h_0,l_0}_1(k_0 \phi + l_0 s; \varepsilon), \]
where \( U^{h_0,l_0}_1 \) is given by expression (61).

Proceeding by induction, we assume that we have applied lemma 3.18 up to order \( q \), for \( 0 < q < m \), so that in adequate symplectic coordinates, the Hamiltonian \( k_q \) of this theorem takes the form
\[ k_q(J, \varphi, s; \varepsilon) = Z^q(J, \varphi, s; \varepsilon) + \varepsilon^q R^q(J, \varphi, s; \varepsilon), \]
and satisfies hypotheses 1 and 2 of lemma 3.18, so that it can be applied again to the Hamiltonian \( k_q \), providing a Hamiltonian
\[ k_{q+1}(J, \varphi, s; \varepsilon) = Z^{q+1}(J, \varphi, s; \varepsilon) + \varepsilon^{q+1} R^{q+1}(J, \varphi, s; \varepsilon) \]
satisfying properties 1 and 2 of lemma 3.18 replacing \( q \) by \( q + 1 \) and a new constant \( C = C_q \) in \( L = C\varepsilon^\ell \) to accommodate new resonances.

Applying the inductive lemma \( m \) times, we get a Hamiltonian \( k_m \)
\[ k_m(J, \varphi, s; \varepsilon) = Z^m(J, \varphi, s; \varepsilon) + \varepsilon^{m+1} R^m(J, \varphi, s; \varepsilon), \]
that consists of an integrable Hamiltonian \( Z^m \), which already satisfies theses 1 and 2 of theorem 3.11 for \( \tilde{Z}^m = Z^m \), plus a perturbation \( \varepsilon^{m+1} R^m \) of order \( O_v(\varepsilon^{m+1+\alpha(2\ell+m)}) \), \( 0 \leq \ell \leq n - 2m \).

Moreover, lemma 3.18 gives us estimates for the terms of the integrable part \( \tilde{Z}^m \) of the Hamiltonian \( k_m \) in the Fourier weighted norm \( || \cdot ||_{\mathcal{C}^\ell} \) defined in (32). More precisely, we know that \( \tilde{Z}^m \) is a polynomial of degree \( m \) in \( \varepsilon \), whose term of order \( q+1 \) is \( O_v(\varepsilon^{m+1+\alpha(2q)}) \), for \( \ell = 0, \ldots, r - 2q \) and \( q = 0, \ldots, m - 1 \). By the equivalence relation (33) we immediately also have that this term of order \( q+1 \) is \( O_v(\varepsilon^{m+1+\alpha(2q+1)}) \), for \( \ell = 0, \ldots, n - 2q \) and \( q = 0, \ldots, m - 1 \).

It remains to prove the estimates of thesis 3 of theorem 3.11 on \( \tilde{Z}^m \) and \( \tilde{R}^m \) in the supremum norm \( || \cdot ||_{\mathcal{C}^\ell} \).

The estimation for \( \tilde{Z}^m \) follows from the ones obtained for \( \tilde{Z}^m \) and we will concentrate on the ones for \( U^{h_0,l_0}_m \).

In particular, in \( \mathcal{D}_{\mathcal{R}_1} \), we can obtain a better estimate for the \( || \cdot ||_{\mathcal{C}^\ell} \) norm of the term \( \varepsilon^i U^{h_0,l_0}_m(\theta; \varepsilon) \) in expression (66), which is the one claimed in point 3 of the theorem. In order to check this, we first note that the function \( U^{h_0,l_0}_m(\theta; \varepsilon) \) in expression (60) is a polynomial of degree \( m - i \) in \( \varepsilon \) and a trigonometric polynomial in \( \theta = k_0 \phi + l_0 s \). So, \( \varepsilon^i U^{h_0,l_0}_m(\theta; \varepsilon) \) can be
bounded by its main term $\varepsilon^i U_m^{k_0,l_0}(\theta; 0)$, which is a trigonometric polynomial in the variable $\theta = k_0\phi + l_0s$ and independent of the action variable $B$. Using that 
\[
\|\varepsilon^i U_m^{k_0,l_0}(\cdot; 0)\|_{C^{\gamma-2u-1},L} \leq e^{i-\alpha(2(i-1))},
\]
and the definition of the Fourier weighted norm in (32), we have 
\[
\|\varepsilon^i U_m^{k_0,l_0}(\cdot; 0)\|_{C^{\gamma-2u-1},L} \leq e^{i-\alpha(2(i-1))},
\]
for all $\ell = 0, \ldots, n - 2(i - 1)$, as claimed in point 3 of theorem 3.11.

Finally, it remains to prove that the tail $e R^{[\gamma]}$ of the Fourier series of the perturbation $\varepsilon R$ that we have truncated at order $M \sim e^{-\rho}$ at the beginning of this proof is $O_{\varepsilon^p} \left(e^{m+1-\alpha(\ell+2m)}\right)$, for $0 \leq \ell \leq n - 2m$. Since the perturbation $R$ in Hamiltonian (55) of theorem 3.11 is a $O_{\varepsilon^p}(1)$ function, the Fourier coefficients $R_{k,l}(J,\varepsilon)$ of $R(J, \varphi, s, \varepsilon)$ decrease at a rate of order $1/|\langle k, l \rangle|^\rho$, for $\langle k, l \rangle \to \infty$. So, by equation (A.2) in proposition A.2 we have the following bound for $e R^{[\gamma]}$:
\[
|e R^{[\gamma]}|_{C^p} \leq \frac{e^{1+\rho(\ell-2)}}{M^{\ell-1}} \leq e^{1+\rho(\ell-2)},
\]
for $\ell = 0, \ldots, n - 2m$.

From lemma 3.18 and equation (63), we know that the changes of coordinates $g_\ell$ satisfy, for $q = 0, \ldots, m - 1, g_\ell = Q_\ell(1)$ and $g_\ell - \mathrm{Id} = Q_\ell(e^{q+1-\alpha(\ell+2m)})$, for $\ell = 0, \ldots, n - 2(q + 1)$. Therefore, the total change of coordinates of theorem 3.11 ($J, \varphi, s = g(B, \phi, s)$ where $g = g_{m-1} \circ \cdots \circ g_0$, satisfies $g = Q_\ell(1)$ and $g - \mathrm{Id} = Q_\ell(e^{1-\alpha(\ell+2m)})$, for $\ell = 0, \ldots, n - 2m$. Then, using this fact and formula (77), by Faa–di Bruno formula (C.4) we have
\[
|R^{[\gamma]}|_{C^p} \leq e^{1+\rho(\ell-2)}.  
\]
To get $|e R^{[\gamma]}|_{C^p} \leq e^{m+1-\alpha(\ell+2m)}$, we need $e^{1+\rho(\ell-2)} \leq e^{m+1-\alpha(\ell+2m)}$, that is
\[
\rho \geq \frac{m - \alpha(\ell + 2m)}{(r - \ell - 2)}.  
\]
for $\ell = 0, \ldots, n - 2m$. In order that bounds (57) and (78) are compatible, we need to choose $r \geq \left(\frac{1}{\rho} - 2\right)m + 2$, which is condition (54) in the hypotheses of this theorem.
Finally the choice \( \tilde{Z} = Z^m \) and \( \tilde{R} = R^m + R^{(r-M)} \circ g \), with \( g = g_m \circ \cdots \circ g_0 \), gives the desired averaged Hamiltonian (58) which satisfies theses 1, 2 and 3.

**Remark 3.20.** Choosing \( \rho = 1/(n+\delta) \), with \( 0 < \delta < 1/m \), so that condition (57) is fulfilled for any \( \alpha \) between \( \rho \) and \( 1/n \), we have that \( r \) must satisfy

\[
    r \geq (n - 2 + \delta)m + 2,
\]

where \( m \) is the number of steps of averaging performed. So, as long as the regularity \( r \) of the Hamiltonian satisfies

\[
    r > r_{\min} := (n-2)m + 2,
\]

(79) there exist \( \rho, \alpha \) satisfying condition (57) and therefore (53) of theorem 3.11 and henceforth, \( m \) steps of averaging can be performed to provide estimates of class \( C^{n-2m} \), contained in the theses of theorem 3.11.

**Remark 3.21.** It is important to note that the averaging procedure is valid in the full domain \( (I_c, I_s) \times \mathbb{T}^2 \subset \tilde{\Lambda}_c \). Indeed, we have performed an averaging procedure to the Hamiltonian \( k(J, \varphi, s; \varepsilon) \) in all \( (I_c, I_s) \times \mathbb{T}^2 \), except at the subsets \( \mathcal{D}_i(L) \), where

\[
    \mathcal{D}_i(L) = \{(J, \varphi, s) \in (I_c, I_s) \times \mathbb{T}^2; L \leq |J + l/k| \leq 2L, \quad \text{for} \quad -l/k \in \mathcal{R}_{[\ell, m]}\}.
\]

To provide an averaging procedure in the full domain \( (I_c, I_s) \times \mathbb{T}^2 \), we apply again theorem 3.11 with \( \tilde{L} = L/|k| \), where \( L = L/2 \). The region \( \mathcal{D}_i(L) \) is now contained in the non-resonant region corresponding to \( \tilde{L}_k, \mathcal{D}_i^m(M, \tilde{L}) \) defined in definition 3.7. So the averaged Hamiltonian in \( \mathcal{D}_i \) is also given by theorem 3.11, with slightly different constants.

### 3.3. KAM theorem

Up to this point, once we choose \( m \), by theorem 3.11 we can perform \( m \) steps of averaging and we obtain a \( C^{r-2m} \) Hamiltonian (58) which consists of an integrable Hamiltonian \( Z^m \) plus a perturbation \( \varepsilon^{m+1} \tilde{R}^m \) which is \( C^{n-2m} \) small, more precisely it is \( O(\varepsilon^{\ell m+1-a(\varepsilon^{2m})}) \), for \( \ell = 0, \ldots, n - 2m \). Note that \( n \geq 2m \) is required as well as a large \( r \) and that the integrable Hamiltonian has different expressions in the resonant regions and non-resonant regions as specified in theorem 3.11.

The integrable part of the Hamiltonian gives us an approximate equation \( \tilde{Z}^m = \text{constant} \) for the invariant tori in \( \tilde{\Lambda}_c \). To finish the proof of theorem 3.1 it remains to determine which tori survive and what is the distance between them when we add the perturbation term \( \varepsilon^{m+1} \tilde{R}^m \). By choosing an adequate \( m \) large enough the goal is to show that we can cover the whole region \( (I_c, I_s) \times \mathbb{T}^2 \subset \tilde{\Lambda}_c \) with invariant tori which are \( O(\varepsilon^{\ell \eta}) \)-closely spaced, for some \( \eta > 0 \), and obtain an approximate expression for them.

To that end, we will use KAM theorem 3.22 stated in section 3.3.1, which is a result about the existence of invariant tori for a periodic perturbation of a Hamiltonian expressed in action-angle variables. It is a direct adaptation of theorem 8.12 in [DLS06a].

Since the integrable Hamiltonian (58) has different expressions in the resonant and non-resonant regions, we perform this study separately.

The non-resonant regions are studied in section 3.3.2. In proposition 3.24, we apply theorem 3.22 directly to Hamiltonian (58) for \( m \geq 2 \) and we conclude that for these regions there exist primary KAM tori which are \( O(\varepsilon^{\ell \eta}) \)-closely spaced, for some \( \eta > 0 \).

The resonant regions are studied in section 3.3.3. As has been described in section 2.3.3, we will see that for these regions, gaps of different sizes are created in the foliation of primary KAM tori. According to the size of the gaps, we will distinguish two types of resonant
regions: the resonant regions with big gaps, where gaps are of size greater than or equal to $\varepsilon$, which is the size of the heteroclinic jumps provided by the scattering map, and the resonant regions with small gaps, where gaps are of size smaller than $\varepsilon$.

In the referred section 3.3.3, we will see that the resonant regions with big gaps introduced in (23) correspond to the resonances $J = -l/k$ of order $1$ such that $|(k, l)| < M_{BG} = \varepsilon^{-(1+\eta)/\epsilon}$, for $1/(r/6 - 1) < \eta \leq 1/16$, whereas the resonant regions with small gaps correspond to the rest of the resonances.

The case of resonant regions with small gaps is studied in section 3.3.4. It will not be different from the non-resonant case and it will be enough to apply KAM theorem 3.22 to Hamiltonian (58) for $m \geq 2$ to obtain primary tori $O(\varepsilon^{1+\eta})$-closely spaced, for some $\eta > 0$. This is done in proposition 3.26. The resonant regions with small gaps constitute, jointly with the non-resonant regions, what we call the flat tori region introduced in section 2.3.3.

The case of resonant regions with big gaps is significantly different and it will be studied in section 3.3.5. In this case the integrable Hamiltonian $\tilde{Z}^m$ is like a pendulum, and we will need to write it first in action-angle variables before applying KAM theorem 3.22 to Hamiltonian (58) for $m \geq 10$. We will see that in these regions we can find other invariant objects, the secondary tori, which fill the region inside the gaps and they get rather close to the frontier of the gaps among the primary KAM tori. The precise result, jointly with the approximate equations for the invariant tori, is given in proposition 3.28.

Finally, theorem 3.1 follows directly from propositions 3.24, 3.26 and theorem 3.28.

3.3.1. The KAM theorem. The following result is about the existence of invariant tori for a periodic perturbation $2\pi k_0$-periodic in the variable $\varphi$ and $2\pi$-periodic in the variable $s$, of a Hamiltonian system expressed in action-angle variables and it is standard in the KAM theory (see [Lla01] for a tutorial on this theory). We skip its proof since it is simply an adaptation of theorem 8.12 in [DLS06a], where the explicit dependence of the constants on $k_0$ is given, since $k_0$ will be chosen depending on $\varepsilon$. It relies on a quantitative KAM theorem of Herman [Her83, theorem 5.4, p 198] for exact symplectic mappings of the annulus.

**Theorem 3.22.** Let $K(I, \varphi, s ; \varepsilon)$ be Hamiltonian of the form

$$K(I, \varphi, s ; \varepsilon) = K_0(I ; \varepsilon) + K_1(I, \varphi, s ; \varepsilon),$$

for $(I, \varphi, s) \in S \times (\mathbb{R}/2\pi k_0 \mathbb{Z}) \times \mathbb{T}$, for some $k_0 \in \mathbb{N}$. Assume that

i. $K$ is a $C^{\alpha+\beta}$ function of the variables $(I, \varphi, s)$, with $n_0 \geq 2$ and $0 < \beta < 1$,

ii. For any $s \in \mathbb{T}$, $|K_1(\cdot, \varphi, s ; \varepsilon)|_{C^\alpha} \leq \delta$ and $|K_0(\cdot ; \varepsilon)|_{C^\alpha} \geq M > 0$, where $\delta = \delta(\varepsilon)$ and $M = M(\varepsilon)$ depend on $\varepsilon$.

Then, for $\varepsilon$ sufficiently small and fixed, there exists a constant $C(k_0) = \text{const} \cdot |k_0|^{(n_0+\beta)/2}$ and a finite set of values $I_i \in S$, such that the Hamiltonian $K(I, \varphi, s ; \varepsilon)$ has invariant tori $T_{I_i}$, such that

a. The torus $T_i$ can be written as a graph of the variable $I$ over the angle variables $(\varphi, s)$:

$$T_i = \{ (I, \varphi, s) \in S \times \mathbb{T}^2 : I = I_i + \Psi_i(\varphi, s ; \varepsilon) \},$$

where $\Psi_i(\varphi, s ; \varepsilon)$ is a $C^{n_0-2+\beta}$ function and $|\Psi_i(\cdot, \varphi ; \varepsilon)|_{C^{n_0-2+\beta}} \leq C(k_0)M^{-1}\delta^{1/2}$.

b. The motion on the torus is $C^{n_0-2+\beta}$ conjugate to a rigid translation of frequency $(\omega(I_i), 1)$, where $\omega(I_i)$ is a Diophantine number of constant type and Markov constant $\kappa = C(k_0)\delta^{1/2}$, that is

$$|\omega(I_i) k - l|^{-1} \leq C\kappa^{-1}|(k, l)| \quad \forall (k, l) \in \mathbb{Z}^2 \setminus \{(0, 0)\}.$$
c. The union of neighbourhoods of size $C(k_0)M^{-1}q^{1/2}$ of these tori cover all the region $I \times (\mathbb{R}/2\pi k\mathbb{Z}) \times \mathbb{T}$.

**Remark 3.23.** This version of the KAM theorem requires to have the system written in action-angle variables. We would like to mention that recently some quantitative results have appeared on the KAM theory without action-angle variables (see [LGJV05, FLS09]) for analytic maps, which could be adapted but some extra work is required.

### 3.3.2. Non-resonant region.

In this section we apply directly theorem 3.22 to the averaged Hamiltonian (58) in the non-resonant region up to order $m$ $\mathcal{D}^m_{\text{nr}}$ introduced in (50). According to remark 3.21, we use $L/2$ instead of $L$, so that

$$
\mathcal{D}^m_{\text{nr}} = \mathcal{D}^m_{\text{nr}}(M, L/2) = \{(J, \varphi, s) \in (I_-, I_+) \times \mathbb{T}^2 : |J + l/k| \geq L_\ell, \text{ for } -l/k \in \mathcal{R}[\leq m]\},
$$

where $L_\ell = L/|k|$, with $L = C\varepsilon^m$ and $\alpha < 1/n$, as required in theorem 3.11.

Going back to the original variables $(I, \varphi, s)$, using the changes given by theorem 3.11 and equation (24), we obtain the following result about the existence of invariant tori of Hamiltonian (I):

**Proposition 3.24 (Invariant tori in the non-resonant region).** Assume that $m \geq 2$, $n \geq 2m + 6$ and $r > (n - 2)m + 2$. Choose any $0 < \eta \leq (m - 1 - \alpha n)/2$, where $\alpha < 1/n$ as required in theorem 3.11. Then, for $\varepsilon$ small enough, in any connected component of the non-resonant region up to order $m$, $\mathcal{D}^m_{\text{nr}}$, there exists a finite set of values $E_i$ such that

i. For any $E_i$ there exists a torus $\mathcal{T}_i$ invariant by the flow of Hamiltonian (1) contained in $\tilde{\Lambda}_\varepsilon$, which is given in $\Lambda_\varepsilon$ by the equation $F(I, \varphi, s; \varepsilon) = E_i$, where $F$ is a $C^{n-2m-2}$ function, for any $q > 0$, of the form

$$
F(I, \varphi, s; \varepsilon) = I + O_{\varepsilon}(\varepsilon^{1-4q}).
$$

ii. The torus $\mathcal{T}_i$ contained in $\tilde{\Lambda}_\varepsilon$ can also be written as a graph of the variable $I$ over the angle variables $(\varphi, s)$:

$$
\mathcal{T}_i = \{(I, \varphi, s) \in \mathcal{D}^m_{\text{nr}}, I = \lambda_{E_i}(\varphi, s; \varepsilon)\},
$$

with

$$
\lambda_{E_i}(\varphi, s; \varepsilon) = E + U_E(\varphi, s; \varepsilon);
$$

where $U_E(\varphi, s; \varepsilon)$ is a $C^{n-2m-2}$ function, for any $q > 0$, and $U_E = O_{\varepsilon}(\varepsilon^{1-4q})$.

iii. These tori are $O(\varepsilon^{1+\eta})$-closely spaced in terms of the variable $I$.

**Proof.** By equations (58) and (59) in theorem 3.11, in one connected component of the non-resonant region $\mathcal{D}^m_{\text{nr}}$, the Hamiltonian (25) expressed in the averaged variables $(B, \phi, s)$ has the following expression:

$$
k_m(B, \phi, s; \varepsilon) = \frac{B^2}{2} + \varepsilon Z_m(B, \varepsilon) + \varepsilon^{m+1} \tilde{R}_m(B, \phi, s; \varepsilon),
$$

where $\varepsilon Z_m(B; \varepsilon)$ is a polynomial of degree $m$ in $\varepsilon$, whose coefficient in terms of $\varepsilon$ of order $q + 1$, for $q = 0, \ldots, m - 1$, is a $C^\ell$ function and is $O_{\varepsilon}(\varepsilon^{q+1-\ell q(2q)})$, for $\ell = 0, \ldots, n - 2q$. Moreover, $\varepsilon^{m+1} \tilde{R}_m(B, \phi, s; \varepsilon)$ is a $C^{\ell}$ function, which is $O_{\varepsilon}(\varepsilon^{m+1-\ell q(2m)})$ for $\ell = 0, \ldots, n - 2m$.

Our next step is to apply KAM theorem 3.22 to the Hamiltonian (83), which is of the form (80), for $K_0 = B^2/2 + \varepsilon Z_m(B, \varepsilon)$ and $K_1 = \varepsilon^{m+1} \tilde{R}_m(B, \phi, s; \varepsilon)$ and $2\pi$-periodic in $\phi$ and $s$, so that $k_0 = 1$. Assuming that $n \geq 2m + 6$, it satisfies properties (i) and (ii) with $n_0 = n - 2m - 1$,
\( \beta = 1 - \varrho \), for any \( \varrho > 0, \delta = \varepsilon^{m+1-an} \) and \( M \) independent of \( \varepsilon \). Therefore we can apply KAM theorem 3.22 and we conclude that the non-resonant region \( D^m_{\varepsilon} \) contains KAM tori given by

\[
B = B_\ell + \Psi_\ell (\theta, s; \varepsilon),
\]

where \( \Psi_\ell \) is a \( C^{m-2m-2-\varepsilon} \) function, for any \( \varrho > 0 \), and \( |\Psi_\ell| \leq \varepsilon^{m+1-an}/2 \). These tori are \( O(\varepsilon^{m+1-an}/2) \)-closely spaced in terms of the averaged variable \( B \).

For a fixed value of \( \varepsilon \ll 1 \), we have that \( \varepsilon^{m+1-an}/2 \leq \varepsilon^{1+\eta} \), where \( \eta = \frac{1}{2}(m - 1 - an) \) is positive by hypotheses \( m \geq 2 \) and \( \alpha < 1/n \) for \( n \geq 2m + 6 \).

After applying the KAM theorem to Hamiltonian (83), we can go back to the original variables \((I, \varphi, s)\). Using that the change \((J, \varphi, s) \mapsto (B, \varphi, s)\) is \( \varepsilon^{1-\alpha(m+2)} \)-close to the identity in the \( C^\alpha \) sense for \( \ell = 1, \ldots, n - 2m \) by theorem 3.11 and \((J, \varphi, s) \mapsto (J, \varphi, s)\) is \( \varepsilon \)-close to the identity in the \( C^{\alpha-1} \) sense by equation (24), the invariant tori obtained in the region \( D^m_{\varepsilon} \) are given by

\[
I = I_\ell + U_\ell (\varphi, s; \varepsilon)
\]

where \( U_\ell \) is a \( C^{m-2m-2-\varepsilon} \) function, for any \( \varrho > 0 \) and \( U_\ell = O_{\varepsilon^{1-\alpha}} \), and they are \( O(\varepsilon^{1+\eta}) \)-closely spaced in terms of the variable \( I \). We get the results claimed for \( E_\ell = I_\ell \).

3.3.3. Resonant region. In this section, we analyse Hamiltonian (25) in the resonant region up to order \( mD^m_{\varepsilon} \) defined in (52).

We will perform an accurate study in this resonant region \( D^m_{\varepsilon} \) and we will estimate the size of the gaps created in the foliation of primary KAM tori. We will see that this size depends on the order \( j \) of the resonance, for \( 1 \leq j \leq m \), and on the size of the harmonic associated with the corresponding resonance. According to this, we define two types of regions: the small gaps regions \( D_{BG} \) where the size of the gap is smaller than \( \varepsilon \) and the big gaps regions \( D_{BG} \) where the size of the gap is bigger than or equal to \( \varepsilon \).

We will work in one connected component of the resonant domain \( D^m_{\varepsilon} \), which, according to (52), is of the form

\[
\{(J, \varphi, s) \in [-l_0/k_0 - L_{k_0}, -l_0/k_0 + L_{k_0}] \times \mathbb{T}^2\},
\]

for some \(-l_0/k_0 \in \mathcal{R}_1 \setminus (\mathcal{R}_1 \cup \ldots \cup \mathcal{R}_{j-1})\), for \( 1 \leq j \leq m \), where \( L_{k_0} = L/|k_0| \), with \( L = C\varepsilon^{a} \) and \( \alpha < 1/n \), as required in theorem 3.11.

By formulae (58) and (60) of theorem 3.11, in component (84), Hamiltonian (25) expressed in the averaged variables \((B, \varphi, s)\), can be written as

\[
k_m (B, \varphi, s; \varepsilon) = \frac{1}{2}B^2 + \varepsilon \tilde{Z}^m (B, \varepsilon) + \varepsilon^j U_{m}^{j_0} (k_0 \phi + l_0 s; \varepsilon) + \varepsilon^{m+1} \tilde{R}^m (B, \varphi, s; \varepsilon),
\]

where \( \tilde{Z}^m (B, \varphi, s) \) and \( U_{m}^{j_0} (k_0 \phi + l_0 s; \varepsilon) \) are polynomials of degree \( m - 1 \) and \( m - j \) in \( \varepsilon \), respectively, and \( U_{m}^{j_0} (k_0 \phi + l_0 s; \varepsilon) \) is a trigonometric polynomial in \( \theta = k_0 \phi + l_0 s \).

For \( q = 0, \ldots, m - 1 \), the coefficient of order \( q + 1 \) in \( \varepsilon \) of \( \tilde{Z}^m \) is a \( C^{\alpha-2} \) function which is \( O_{\varepsilon^{1-\alpha}} \) for \( \ell = 0, \ldots, n - 2q \). The function \( \varepsilon^j U_{m}^{j_0} (\theta; \varepsilon) \), for \( \theta = k_0 \phi + l_0 s \), satisfies

\[
|\varepsilon^j U_{m}^{j_0} (\theta; \varepsilon)| \leq \varepsilon^{1-2\alpha(j-1)} |(k_0, l_0)|^{-(r+2)(j-1)},
\]

for \( \ell = 0, \ldots, n - 2(j - 1) \) and \((k_0, l_0) = \max(|k_0|, |l_0|)\).

Moreover, \( \varepsilon^{m+1} \tilde{R}^m \) is a \( C^{\alpha-2} \) function which is \( O_{\varepsilon^{1-\alpha}} \) for \( \ell = 0 \ldots n - 2m \).

From expression (85) it is clear that the integrable part \( \tilde{Z}^m \) is like a pendulum. The integrable pendulum has rotational and librational orbits as well as separatrices, which separate these two types of motion. It is straightforward to see that the size of the gap, created by the
separatrix loop, associated with the resonance \(-l_0/k_0 \in \mathcal{R}_j \setminus \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_{j-1}\), in terms of the \(J\) variables, can be bounded from above by \(\sqrt{2e^{1/2}} |U_m^{(k_0-l_0)}(\cdot; \varepsilon)|^{1/2}\).

From expression (86), we have that the size of the gap for a resonance \(-l_0/k_0\) of order \(j\) is

\[
O(\varepsilon^{(j-2\alpha(j-1))/2}|(k_0, l_0)|^{(r+2(j-1))/2}).
\]

Expression (87) shows that the gaps form a heterogeneous sea since their size depends on the order \(j\) of the resonance and the size of the harmonic \(|(k_0, l_0)|\). Among them, the biggest gaps are those of order \(j = 1\) and harmonic \(|(k_0, l_0)| \leq M_{BG}\), where \(M_{BG} = \varepsilon^{-(1+\nu)/r}\) was introduced in theorem 2.1 and satisfies \(M_{BG} > M\), where \(M\) is the order of truncation.

Indeed, in the particular case of a resonance \(-l_0/k_0\) of order 1, the size of the gap is

\[
O(\varepsilon^{1/2}|(k_0, l_0)|^{-r/2}),
\]

so that for any \(\nu > 0\), the resonances of order 1 such that \(|(k_0, l_0)| \geq M_{BG} = \varepsilon^{-(1+\nu)/r}\), create gaps of size \(O(\varepsilon^{1+\nu/2})\), that is, smaller than \(\varepsilon\).

On the other hand, we know that resonances \(-l_0/k_0\) of order greater than 1 satisfy \(M_{BG} \leq |(k_0, l_0)| \leq mM\) (see remark 3.5). Hence, according to (87) the size of the gap created by a resonance \(-l_0/k_0\) of order \(j\), for \(j = 2, \ldots, m\) is

\[
O(\varepsilon^{(j+1+r^{-\alpha}(1+\nu)/r)(2j-1)/2}).
\]

Using the condition \(\alpha < 1/r\), with \(r > n \geq 2m\) and \(m \geq 2\), the size of the gap is

\[
O(\varepsilon^{(j+1+4\alpha(j-1))/2}).
\]

For \(j \geq 2\) the size of the gaps is smaller than \(\varepsilon^{1+\eta}\), for \(\eta = (1 - 4\alpha)/2\). Note that \(\eta > 0\) thanks to the condition on \(\alpha\).

As we already said, we will distinguish between two types of resonant regions depending on whether the size of the gaps created in the foliation of primary KAM tori is bigger or smaller than the size \(\varepsilon\) of the heteroclinic jumps provided by the scattering map (19).

- **Resonant regions with big gaps** \(\mathcal{D}_{BG}\). Gaps of size of order equal or greater than \(\varepsilon\) are created in the foliation of the primary invariant tori. According to (88) they correspond to resonances \(-l_0/k_0\) of order 1 with \(\gcd(k_0, l_0) = 1\), satisfying \(|(k_0, l_0)| < M_{BG}\), where \(M_{BG} = \varepsilon^{-(1+\nu)/r}\), for \(1/6 < \nu < 1/16\). See definition (23).

- **Resonant regions with small gaps** \(\mathcal{D}_{SG}\). Gaps between primary tori are smaller than \(\varepsilon\). They correspond to the resonant regions of resonances \(-l_0/k_0\) of order 1 such that \(|(k_0, l_0)| \geq M_{BG}\), and resonances of order greater than or equal to 2 (which also satisfy \(|(k_0, l_0)| \geq M_{BG}\), see remark 3.5).

**Remark 3.25.** We would like to emphasize that our result about resonances is remarkably different from the one obtained in [DLS06a], where the case of a perturbation \(h\) with a finite number of harmonics was considered. In that case there was a uniform size for the gaps created by the resonances. For instance, the gaps created by the resonances of orders 1 and 2 were \(C_\epsilon^{1/2}\) and \(C_\varepsilon\), respectively. In our case we have a heterogeneous sea of gaps of different sizes.

Moreover, in our case the resonances that create big gaps are just the resonances of order 1 up to some order \(M_{BG}\), whereas in [DLS06a], both resonances of orders 1 and 2 created big gaps.

**3.3.4. Resonant regions with small gaps.** In this section, we will study the resonant regions with small gaps \(\mathcal{D}_{SG}\), which correspond to resonances \(-l_0/k_0\) such that \(|(k_0, l_0)| \geq M_{BG}\), where \(M_{BG}\) was introduced in theorem 2.1, of order \(j\) greater than or equal to 1.
We will work in one connected component, and we will apply directly theorem 3.22 to Hamiltonian (85) in order to prove that this component is covered by the primary tori which are $O(\epsilon^{1+\nu})$-closely spaced, for some $\eta > 0$.

Going back to the original variables $(I, \varphi, s)$ using the changes given by theorem 3.11 and equation (24), we obtain the following result about the existence of invariant primary KAM tori of Hamiltonian (1):

**Proposition 3.26 (Invariant tori in the small gaps region).** Assume that $m \geq 2$, $n \geq 2m + 6$ and $r > (n - 2)m + 2$. Choose any $0 < \eta \leq \frac{1}{2} \min(\nu - 6(1 + \nu)/r, m - 1 - \alpha(6 + 2m))$, for $\nu > 1/(r/6 - 1)$. Then, for $\epsilon$ small enough, in any connected component of $D_{SG}$, which is of the form (84) for some $-l_0/k_0 \in \mathbb{R}^{(\leq m)}$ with $|(k_0, l_0)| \geq M_{BG}$ and $L_{k_0} = L/|k_0|$ with $L = C\epsilon^\alpha$ and $\alpha < 1/n$, as required in theorem 3.11, there exists a finite set of values $E_i$ such that

i. For any $E_i$, there exists a torus $T_i$ invariant by the flow of Hamiltonian (1) contained in $\Lambda_x$, which is given in $\Lambda_x$ by the equation $F(I, \varphi, s; \epsilon) \equiv E_i$, where $F$ is a $C^{m-2m-2-\nu}$ function, for any $\nu > 0$, of the form

$$ F(I, \varphi, s; \epsilon) = I + O_{\mathbb{C}^\nu}(\epsilon^{1-4\nu}). \quad (89) $$

ii. The torus $T_i$ can be written as a graph of the variable $I$ over the angle variables $(\varphi, s)$:

$$ T_i = \{ (I, \varphi, s) \in D_{SG}; I = \lambda_{E_i}(\varphi, s; \epsilon) \}, $$

with

$$ \lambda_{E_i}(\varphi, s; \epsilon) = E + U_E(\varphi, s; \epsilon) \quad (90) $$

where $U_E$ is a $C^{4-\nu}$ function, for any $\nu > 0$, and $U_E = O_{\mathbb{C}^\nu}(\epsilon^{1-4\nu})$.

iii. These tori are $O(\epsilon^{1+\nu})$-closely spaced in terms of the variable $I$.

**Proof.** By theorem 3.11, in any connected component of $D_{SG}$, Hamiltonian (25) expressed in the averaged variables $(B, \phi, s)$ has expression (85).

Hamiltonian (85) is of the form (80), with $K_0(B; \epsilon) = \frac{1}{2} B^2 + \epsilon Z^m(B; \epsilon)$, which is a $C^{t-2m+2}$ function and

$$ K_1(B, \phi, s; \epsilon) = \epsilon^j (t_m^{k_0,l_0}(k_0 \phi + l_0 s; \epsilon) + \epsilon^{m+1-j} R^m(B, \phi, s; \epsilon)), \quad (91) $$

which is a $C^{t-2m}$ function and $2\pi$-periodic in both angle variables $\phi$ and $s$.

Our aim is to apply KAM theorem 3.22. It is clear that $|K_0^\nu(\cdot; \epsilon)| \geq M > 0$, for $M$ independent of $\epsilon$. We will now see that $K_1$ in (91) satisfies $|K_1(\cdot, \cdot; \cdot; \epsilon)|_{C^\nu} \leq \delta$, for $\delta = \epsilon^{2+2\nu}$, for $\eta > 0$.

Recall from theorem 3.11 that $t_m^{k_0,l_0}(k_0 \phi + l_0 s; \epsilon)$ is a polynomial in $\epsilon$ of degree $m - j$ and a trigonometric polynomial in $\theta = k_0 \phi + l_0 s$, which has the following bound with respect to $\theta$:

$$ |\epsilon^j U_m^{k_0,l_0}(\cdot; \epsilon)|_{C^t, \theta} \leq \epsilon^{j-2m(j-1)} |(k_0, l_0)|^{-r+2(j-1)}, \quad (92) $$

and therefore

$$ |\epsilon^j U_m^{k_0,l_0}(\cdot, \epsilon)|_{C^t, (\phi, s)} \leq \epsilon^{j-2m(j-1)} |(k_0, l_0)|^{-r+2(j-1)+\epsilon} \quad (93) $$

for $\epsilon = 0, \ldots, n - 2m$. Moreover, $\epsilon^{m+1} R^m$ is a $C^{t-2m}$ function with a bounded $C^t$ norm up to $\epsilon = n - 2m$ given by

$$ |\epsilon^{m+1} R^m (\cdot; \epsilon)|_{C^t} \leq \epsilon^{m+1-a(t+2m)}. \quad (94) $$
Hence, from the estimates for the $C^l$ norm of functions $\varepsilon^{j}U_{m}^{k_{0},l_{0}}$ in (93) and $\varepsilon^{n+1}\tilde{R}_{m}$ in (94) with $l = 6$, one gets

$$|K_{i}(\cdot, \cdot, s; \varepsilon)|_{C^l} \leq \varepsilon^{j-2\alpha(j-1)}|\langle k_{0}, l_{0} \rangle|^{-r+2(j-1)+6} + \varepsilon^{m+1-\alpha(6+2m)},$$

for any $1 \leq j \leq m$. Taking into account that $|\langle k_{0}, l_{0} \rangle| \geq M_{BG} = \varepsilon^{-(1+\nu)/r}$ and that the worse estimate comes from $j = 1$, one gets

$$|K_{i}(\cdot, \cdot, s; \varepsilon)|_{C^l} \leq \varepsilon^{\frac{1}{r}(r-6)} + \varepsilon^{m+1-\alpha(6+2m)} = \varepsilon^{2\eta_{1} + \varepsilon^{2\eta_{2}},}$$

where $\eta_{1} = \nu - 6(1 + \nu)/r$ and $\eta_{2} = m - 1 - \alpha(6 + 2m)$ are both positive. Indeed, by hypotheses $m \geq 2$ and $\alpha < 1/n \leq 1/(2m + 6)$, we have $\eta_{2} > 0$ and $\eta_{1} > 0$ is equivalent to $\nu > 1/(r/6 - 1)$.

So, for any $\eta \leq \frac{1}{2}\min(\eta_{1}, \eta_{2})$ we have $|K_{i}(\cdot, \cdot, s; \varepsilon)|_{C^l} \leq \varepsilon^{2+2\eta}$ and we can apply KAM theorem 3.22 with $n_{0} = 5$, $\beta = 1 - \varepsilon$, for any $\varepsilon > 0$, $\delta = \varepsilon^{2+2\eta}$ and $M$ independent of $\varepsilon$. Therefore, we conclude that for a constant $C(k_{0})$ because $k_{0} = 1$, regions $D_{BG}$ contain the KAM tori given by

$$B = B_{\varepsilon} + \Psi_{i}(\phi, s; \varepsilon),$$

where $\Psi_{i}(\phi, s; \varepsilon)$ is a $C^{l+\varrho}$ function, for any $\varrho > 0$, and

$$|\Psi_{i}|_{C^l} \leq \varepsilon^{1+\eta}.$$

These tori are $O(\varepsilon^{1+\eta})$-closely spaced in terms of the variable $B$.

As in the non-resonant regions we can go back to the original variables $(I, \varphi, s)$. Using that the change $(J, \varphi, s) \mapsto (B, \phi, s)$ is $\varepsilon^{1-\alpha(\nu+2)}$-close to the identity in the $C^l$ sense for $l = 0, \ldots, n - 2m$ by theorem 3.11 and $(I, \varphi, s) \mapsto (J, \varphi, s)$ is $\varepsilon$-close to the identity in the $C^{r-1}$ sense by equation (24), the invariant tori obtained in the region $D_{BG}^{m}$ are given by

$$I = I_{i} + U_{i}(\varphi, s; \varepsilon),$$

where the function $U_{i}$ is a $C^{l+\varrho}$ function, for any $\varrho > 0$, and $U_{i} = O_{C^{l}}(\varepsilon^{1-4\alpha})$, and they are $O(\varepsilon^{1+\eta})$-closely spaced in terms of the variable $I$. We get the results claimed for $E_{i} = I_{i}$.

\[ \square \]

Remark 3.27. Note that invariant tori in the small gaps region $D_{BG}$ are given by a certain function $F$ in (89) that, as in the case of non-resonant regions (see (81)), is of the form

$$F(I, \varphi, s; \varepsilon) = I + O_{C^{l}}(\varepsilon^{1-4\alpha}).$$

3.3.5. Resonant regions with big gaps. In this section, we will see that the resonant regions with big gaps $D_{BG}$ which correspond to resonances of order 1 such that $|\langle k_{0}, l_{0} \rangle| < M_{BG}$ are covered with invariant objects (either primary tori or secondary tori) which are $O(\varepsilon^{1+\eta})$-closely spaced in terms of the action variable $I$, for some $\eta > 0$.

To that end, we will apply theorem 3.22 to Hamiltonian (83) as we did in the previous cases. The main difference is that in this case the integrable Hamiltonian is not written down into action-angle variables, so we will need to perform a change of coordinates before applying the KAM theorem. Furthermore, we will perform two useful changes of coordinates, which are not symplectic but conformally symplectic, that is, the new Hamiltonian is just the old one multiplied by a constant.
Theorem 3.28 (Invariant tori in the big gaps region). Assume that \( m \geq 10, n \geq 2m + 6 \) and \( r > (n-2)m+2 \). Assume that the function \( U_{m_0}^{k_0 l_0}(k_0 \varphi + l_0 s; 0) \) in Hamiltonian (85) has a global maximum which is non-degenerate (this assumption corresponds to the hypothesis \( H3' \) on \((k_0, l_0)\) in theorem 2.1). Choose any \( 0 < \eta \leq v/2 \) and assume \( v \leq 1/16 \).

Then, for \( \varepsilon \) small enough, in any connected component of \( D_{BG} \), which is of the form (84), for some \(-l_0/k_0\) of order 1 such that \(|(k_0, l_0)| < M_{BG} \), \( L_{k_0} = L/k_0 \) with \( L = C \varepsilon^n \) and \( \alpha < 1/n \), as required in theorem 3.11, there exists a finite set of values \( E_i \) in some range of energies \(-\varepsilon |(k_0, l_0)|^{-r+2} \leq E \leq L^2 \) such that

i. For any \( E_i \) there exist invariant objects by the flow of Hamiltonian (1) contained in \( \Lambda_{E_i} \), which are given in \( \Lambda_{E_i} \) by the equation \( F(I, \varphi, s; \varepsilon) \equiv E_i \), where \( F \) is a \( C^4 \)-\( \varepsilon \) function, for any \( \varphi > 0 \), of the form

\[
F(I, \varphi, s; \varepsilon) = \frac{(k_0 I + l_0 + O_2(|k_0|^{1-4\alpha}))^2}{2} (1 + \varepsilon k_0^2 \tilde{h}(k_0 I + l_0 + O_2(|k_0|^{1-4\alpha}); \varepsilon))
\]

\[+ \varepsilon k_0^2 U_{m_0}^{k_0 l_0}(k_0 \varphi + l_0 s; \varepsilon) + O_{C}(k_0^{4}(k_0, l_0))^{-r/2} \varepsilon^{3/2+\eta}, \tag{96}\]

where \( \tilde{h} \) satisfies (108). For values of \( E_i > 0 \), equation \( F \equiv E_i \) consists of two invariant objects that are primary KAM tori \( T_{E_i}^\pm \), whereas for \( E_i < 0 \) it consists of an invariant object which is a secondary KAM torus \( T_{E_i} \). In this case we denote by \( T_{E_i}^\pm \) each of the components of

\[
T_{E_i} \cap \{ (I, \varphi, s) \in D_{BG}; \rho \leq k_0 \varphi + l_0 s \leq 2\pi - \rho \},
\]

for some \( 0 < \rho < 2\pi \).

ii. There exists \( \rho \geq 0 \), such that the two primary KAM tori (components of the secondary tori) \( T_{E_i}^\pm \) contained in \( \Lambda_{E_i} \) can be written as graphs of the variable \( I \) over the angle variables \((\varphi, s)\):

\[
T_{E_i}^\pm = \{ (I, \varphi, s) \in [-l_0/k_0 - L_{k_0}, -l_0/k_0 + L_{k_0}] \times [\rho, 2\pi - \rho] \times \mathbb{T}; \quad I = \lambda_{E_i}^\pm(\varphi, s; \varepsilon) \},
\]

where

\[
\lambda_{E_i}^\pm(\varphi, s; \varepsilon) = -\frac{l_0}{k_0} + \frac{1}{k_0} \mathcal{Y}_{\pm}(\theta, E) + O_{C}(\varepsilon^{1-4\alpha}), \tag{97}\]

for \( \rho \leq \theta = k_0 \varphi + l_0 s \leq 2\pi - \rho \), where

\[
\mathcal{Y}_{\pm}(x, E) = \pm(1 + 6b)\ell(\theta, E) + \varepsilon \mathcal{Y}_{\pm}^{\prime}(\ell(\theta, E)), \tag{98}\]

\( \ell(\theta, E) = \sqrt{2(E - \varepsilon k_0^2 U_{m_0}^{k_0 l_0}(\theta; \varepsilon))} \) and \( \mathcal{Y}_{\pm}^{\prime} \) satisfies (120).

iii. These invariant tori are \( O(\varepsilon^{1+\eta}) \)-closely spaced in terms of the variable \( I \) and \( \mathcal{O}(\varepsilon^{3/2+\eta}(k_0, l_0))^{-r/2+1} \) in terms of energies \( E_i \).

Remark 3.29. In remark 3.12 we already pointed out that the function \( U_{m_0}^{k_0 l_0}(k_0 \varphi + l_0 s; 0) \) given explicitly in (61) is the function \( U_{m_0}^{k_0 l_0}(\theta) \) for \( \theta = k_0 \varphi + l_0 s \) in hypothesis \( H3 \) on \((k_0, l_0)\) in theorem 2.1.

3.3.6. Proof of theorem 3.28. The proof of this theorem is organized in three parts.
Invariant tori given by the averaged Hamiltonian. By theorem 3.11, in any connected component of the resonant domain $D_{ig}$, which is of the form (84), Hamiltonian (25) expressed in the averaged variables $(B, \phi, s)$ is of the form (85) with $j = 1$, so it can be written as

$$k_m(B, \phi, s; \varepsilon) = \frac{1}{2} \dot{B}^2 + \varepsilon \dot{Z}^m(B; \varepsilon) + \varepsilon U^m_{k_m}(k_0 \phi + l_0 s; \varepsilon) + \varepsilon^{m+1} \tilde{R}^m(B, \phi, s; \varepsilon)$$

(99)

on the domain

$$\{ (B, \phi, s) \in \mathbb{R} \times \mathbb{T}^2; |B + l_0/k_0| \leq \tilde{L}_{k_0} \},$$

(100)

where $|L_{k_0} - \tilde{L}_{k_0}| \leq \text{const} \varepsilon$.

In this domain, $\varepsilon \dot{Z}^m(B; \varepsilon)$ is a $C^r-2m+2$ function in the variable $B$ and it is a polynomial of degree $m$ in $\varepsilon$, whose coefficient of order $q + 1$, for $q = 1, \ldots, m - 1$ is a $C^r-2q$ function and $\mathcal{O}_r(\varepsilon^{q+1-\ell(q+2/3)})$, for $\ell = 0, \ldots, n - 2q$, so that $\dot{Z}^m$ has a bounded norm up to $\ell = n - 2m + 2$. Its main term $\dot{Z}^m(B; 0)$ is equal to $h_{0,0}(0, 0; B; 0)$ by remark 3.12.

Moreover $U^m_{k_m}(k_0 \phi + l_0 s; \varepsilon)$ is a polynomial of degree $m - 1$ in $\varepsilon$ and a trigonometric polynomial in $\theta = k_0 \phi + l_0 s$, satisfying $\varepsilon U^m_{k_m}(\theta; \varepsilon) = \mathcal{O}_r(\varepsilon(\theta_0, l_0)^{-r})$, for $\ell = 0, \ldots, n$. Its main term $U^m_{k_m}(\theta; 0)$ is given in expression (62) in remark 3.12.

Finally, $\varepsilon^{m+1} \tilde{R}^m$ is a $C^r-2m$ function in the variables $(B, \phi, s)$ with a bounded $C^r$ norm up to $\ell = n - 2m$, which is

$$|\varepsilon^{m+1} \tilde{R}^m(\cdot; \varepsilon)|_{C^r} \leq \varepsilon^{m+1-a(3+2m)}.$$  

(101)

By the hypothesis in theorem 3.28, the function $U^m_{k_m}(\theta, 0)$ (the first order term in $\varepsilon$ of the function $U^m_{k_m}(\theta, \varepsilon)$) has a global maximum which is non-degenerate and this implies that the integrable part $Z^m$ of the Hamiltonian (99) is like an integrable pendulum.

As has been done in section 8.5.2 in [DLS06a], we perform two useful changes of coordinates which are not symplectic but conformally symplectic. The first one depends on the time $s$ and the resonance $(k_0, l_0)$ and is given by

$$b = k_0(B + l_0/k_0), \quad \theta = k_0 \phi + l_0 s, \quad s = s,$$

(102)

hence the system of equations verified by $(b, \theta, s)$ is also Hamiltonian of Hamiltonian:

$$\dot{K}(b, \theta, s; \varepsilon) = \dot{K}_0(b; \varepsilon) + \varepsilon \tilde{V}(\theta; \varepsilon) + \varepsilon^{m+1} \tilde{K}_1(b, \theta, s; \varepsilon),$$

(103)

with

$$\dot{K}_0(b, \varepsilon) = b^2/2 + \varepsilon b^2 Z^m(-l_0/k_0 + b/k_0; \varepsilon),$$

$$\tilde{V}(\theta; \varepsilon) = k_0^2 U^m_{k_m}(\theta; \varepsilon),$$

$$\dot{K}_1(b, \theta, s; \varepsilon) = k_0^2 \tilde{R}^m(-l_0/k_0 + b/k_0, \theta - l_0 s/k_0; s, \varepsilon).$$

(104)

Note that $\dot{K}_0$ is of class $C^r-2m+2$ with a bounded $C^r$ norm up to $\ell = n - 2m + 2$ and $\tilde{V}$ is analytic because it is a trigonometric polynomial in $\theta$ and a polynomial of degree $m - 1$ in $\varepsilon$. $\dot{K}_1$ is a function of class $C^r-2m$ with a bounded $C^r$ norm up to $\ell = n - 2m$, which is $2\pi k_0$-periodic in $\theta$ and $2\pi$-periodic in $s$. Note that $\tilde{V}$ is $2\pi$-periodic in $\theta$, whereas $\dot{K}_1$ is $2\pi k_0$-periodic in $\theta$.

The integrable part $\dot{K}_0(b; \varepsilon) + \varepsilon \tilde{V}(\theta; \varepsilon)$ of the Hamiltonian (103) is a one degree of freedom Hamiltonian close to a pendulum-like Hamiltonian

$$\frac{b^2}{2} + \varepsilon \tilde{V}(\theta; 0) = \frac{b^2}{2} + \varepsilon b^2 U^m_{k_m}(\theta; 0),$$

where $U^m_{k_m}(\theta; 0)$ is given in (62). By hypothesis H3' on $(k_0, l_0)$ this pendulum-like Hamiltonian has a hyperbolic saddle at $(0, \theta_1)$ and by the implicit function theorem the whole integrable Hamiltonian $\dot{K}_0(b; \varepsilon) + \varepsilon \tilde{V}(\theta; \varepsilon)$ also has a saddle at $(b(\varepsilon), \theta_1(\varepsilon))$. Since
$\tilde{Z}^m(B; 0) = h(0, 0, B; 0)$ does not depend on $\epsilon$, the function $b(\epsilon)$ is of class $C^{r-2m+1}$ in $\epsilon$ and of the form $b(\epsilon) = O(|k_0|\epsilon)$ whereas $\theta_1(\epsilon)$ is analytic in $\epsilon$ and of the form $\theta_1(\epsilon) = \theta_1 + O(\epsilon)$.

To make the analysis of this system easier we perform a second change of variables, which depends on $\epsilon$ and consists of the following translation:

$$y = b - b(\epsilon), \quad x = \theta - \theta_1(\epsilon), \quad s = s,$$

(105)

in such a way that the integrable part of the Hamiltonian expressed in these new variables has a saddle point at $(0, 0)$ and the energy of the saddle and the separatrices is 0. More precisely, we obtain the $C^{r-2m}$ Hamiltonian with respect to $(y, x, s)$ with a bounded $C^r$ norm up to $\ell = n - 2m$

$$K(y, x, s; \epsilon) = \tilde{h}^0(y; \epsilon) + \epsilon U(x; \epsilon) + \epsilon^{m+1} S(y, x, s; \epsilon)$$

(106)

which consists of an integrable part corresponding to the terms up to order $\epsilon^m$, which is the following $C^{r-2m+2}$ function with a bounded $C^r$ norm up to $\ell = n - 2m + 2$,

$$K_0(y, x; \epsilon) = \tilde{h}^0(y; \epsilon) + \epsilon U_{j_0}(x; \epsilon),$$

(107)

and a perturbation $\epsilon^{m+1} S(y, x, s; \epsilon)$, which is a $C^{r-2m}$ function with a bounded $C^r$ norm up to $\ell = n - 2m$.

The function $\tilde{h}^0(y; \epsilon)$ in the integrable part $K_0$ is a $C^{r-2m+2}$ function in $y$ with a bounded $C^r$ norm up to $\ell = n - 2m + 2$ of the form

$$\tilde{h}^0(y; \epsilon) = \frac{y^2}{2} \tilde{h}(y; \epsilon) = \frac{y^2}{2}(1 + \epsilon k_0^2 \tilde{h}(y; \epsilon)),$$

(108)

for some $C^{r-2m}$ function in $(y, \epsilon)$, $\tilde{h}(y; \epsilon)$, with a bounded $C^r$ norm up to $\ell = n - 2m$ in $y$. The function $U$ in $K_0$ is given by

$$U(x; \epsilon) = k_0^2 \tilde{U}_m(x + \theta_1(\epsilon); \epsilon) - U_{m, k_0}(\theta_1(\epsilon); \epsilon),$$

(109)

and it satisfies

$$[\epsilon U(\cdot; \epsilon)]_{C^r} \leq \epsilon |k_0|^2 |(k_0, l_0)|^{-r}$$

(110)

for $\ell = 0, \ldots, n$.

We also note that the following conditions are satisfied:

$$\frac{\partial \tilde{h}^0}{\partial y}(0; \epsilon) = 0, \quad U(0; \epsilon) = \frac{\partial U}{\partial x}(0; \epsilon) = 0, \quad \frac{\partial^2 U}{\partial x^2}(0; \epsilon) < 0,$$

as well as that $x = 0$ is a global maximum of $U$.

The perturbation term $\epsilon^{m+1} S(y, x, s; \epsilon)$ is given by

$$S(y, x, s; \epsilon) = k_0^2 \tilde{R}_m \left( - \frac{l_0}{k_0} + \frac{y + b(\epsilon)}{k_0} \frac{x + \theta_1(\epsilon) - l_0 s}{k_0}, s; \epsilon \right)$$

and by equation (101) it can be bounded in the variables $(y, x)$ by

$$|\epsilon^{m+1} S(\cdot, s; \epsilon)|_{C^r} \leq |k_0|^{2-\ell} \epsilon^{m+1-\gamma(\ell+2m)}$$

(111)

for $\ell = 0, \ldots, n - 2m$.

Since we want to apply some of the results in [DLS06a], it will be convenient for us to have $K_0$ written in another way adapted to the notation in [DLS06a]. Motivated by the size $\epsilon |k_0|^2 |(k_0, l_0)|^{-r}$ of $\epsilon U$ estimated in formula (110), we introduce here the parameter $\gamma \in \mathbb{R}$, $2 > \gamma \geq 1$, depending on $(k_0, l_0)$ and $\epsilon$, such that

$$\epsilon^\gamma = \epsilon |k_0|^2 |(k_0, l_0)|^{-r},$$

(112)

in such a way that $\epsilon U(\cdot; \epsilon) = O_{C^r}(\epsilon^\gamma)$, for $\ell = 0, \ldots, n$. 

Note that \( \gamma = 1 \) for small values of \((k_0, l_0)\), that is \(|(k_0, l_0)| \sim 1\), and in general \(1 < \gamma < 2 + \nu\) for \(|(k_0, l_0)| \sim \varepsilon^{-\varrho}\), for any \(0 < \varrho < (1 + \nu)/r\), where \(0 < \nu \leq 1/16\).

With this choice of \(\gamma\), we will denote \(K_0\) the one degree of freedom \(C^{r-2m+2}\) Hamiltonian in \((y, x)\)
\[
K_0(y, x; \varepsilon) = h^0(y; \varepsilon) + \varepsilon^{\gamma} \hat{U}(x; \varepsilon),
\]
(113)
where
\[
\varepsilon^{\gamma} \hat{U}(x; \varepsilon) = \varepsilon U(x; \varepsilon),
\]
(114)
with \(2 + \nu > \gamma \geq 1\) and \(\hat{U}(\cdot; \varepsilon) = C_{\mathbb{Z}}^0(1)\), for \(\ell = 0, \ldots, n\).

The energy level \(K_0(y, x; \varepsilon) = 0\) consists of the saddle \((0, 0)\) and its separatrices.

The Hamiltonian \(K(y, x, s; \varepsilon)\) introduced in (106) is \(2\pi k_0\)-periodic in \(x\) and \(2\pi\)-periodic in \(s\) and is defined in the domain \(D_{k_0}\) given by
\[
D_{k_0} = \{(y, x, s) \in \mathbb{R} \times \mathbb{R}/(2\pi k_0 \mathbb{Z}) \times \mathbb{T}, |y| \leq \bar{L}\},
\]
(115)
where \(\bar{L} = k_0 \tilde{L}_{k_0}\), whereas the integrable part \(K_0(y, x; \varepsilon)\) in (113) is \(2\pi\)-periodic in \(x\) and independent of \(s\), therefore the region \(D_{k_0}\) can be seen as \(k_0\) copies of the region
\[
D = \{(y, x, s) \in \mathbb{R} \times \mathbb{T}^2, |y| \leq \bar{L}\}.
\]
This effect is colloquially described as saying that the resonance \(-l_0/k_0\) has \(k_0\) eyes. As \(k_0\) increases, these eyes form long necklaces.

The region \(D\) (and also \(D_{k_0}\)) is filled by the energy surfaces of the Hamiltonian \(K_0\),
\[
T^0_E = \{(y, x, s) \in [-\bar{L}, \bar{L}] \times \mathbb{T}^2 : K_0(y, x; \varepsilon) = E\}
\]
which are invariant under the flow of Hamiltonian \(K_0\).

As we already said, the energy surface \(T^0_E\) corresponding to \(E = 0\) consists of the saddle \((0, 0)\) and its separatrices forming a separatrix loop. Therefore, this separatrix loop \(T^0_E\) separates two types of topological invariant objects. The energy surfaces corresponding to the values \(E > 0\) are primary tori and the ones corresponding to the values \(E < 0\) are called secondary tori, which are tori of different topology than the primary ones because they are contractible to points. Secondary tori cover all the region inside the separatrix loop \(T^0_E\). In the next section we will discuss the persistence of primary and secondary tori when we add the perturbation term.

**KAM theorem.** In this section, we will show that many of the invariant tori \(T^0_E\) of the Hamiltonian \(K_0(y, x; \varepsilon)\) in (113), inside the region \(D_{k_0}\) given in (115), both primary and secondary, survive when we add the perturbation term \(\varepsilon^{m+1}S(y, x, s; \varepsilon)\) to consider the Hamiltonian \(K\) given in equation (106). Moreover, we will estimate the number of steps of averaging \(m\) required to get invariant tori with a distance of \(O(\varepsilon^{1+n})\) between them, for some \(n > 0\), in terms of the original variables \((I, \varphi, s)\).

To establish this we will write the Hamiltonian (113) into action-angle variables and apply KAM theorem 3.22. Since the unperturbed Hamiltonian \(K_0(y, x; \varepsilon)\) is a pendulum, we cannot define global action-angle variables because the change of coordinates becomes singular on the separatrix. Therefore, we will define different action-angle variables inside and outside the separatrix and we will exclude a thin neighbourhood around it.

We find it convenient to consider different regions in the domain \(D_{k_0}\) in terms of the values of the energy \(E\), in which the behaviour of the tori is different.

Recall that tori \(T^0_E\) in \(D_{k_0}\) are given approximately by the energy surfaces of Hamiltonian \(K_0\), that is
\[
K_0(y, x; \varepsilon) = E,
\]
and we see that by excluding a small interval they can be seen as a graph of the action variable \( y \) over the angle variables \((x, s)\).

Introducing \( \delta = \varepsilon^\gamma \), we consider the foliation given by the level sets

\[
h^0(y; \varepsilon) + \delta \bar{U}(x; \varepsilon) = E,
\]

where \( h^0(y; \varepsilon) \) is of the form \((108)\) and \( \bar{U}(\cdot; \varepsilon) = O(\varepsilon) \) for \( \ell = 0, \ldots, n \) satisfies also that on \( x = 0 \) there is a non-degenerate global maximum of \( \bar{U}(x; \varepsilon) \), which verifies \(-c \leq \bar{U}(\cdot; \varepsilon) \leq 0\) and \( \bar{U}(\cdot; \varepsilon) \simeq -ax^2 \) as \( x \to 0 \), with \( a > 0 \).

Since \( h^0(y; \varepsilon) + \delta \bar{U}(x; \varepsilon) \simeq \frac{\varepsilon^2}{2} + \delta \bar{U}(x; \varepsilon) \), the main term in the solution of \((116)\) is

\[
y = \pm \ell(x, E),
\]

where

\[
\ell(x, E) = \sqrt{2(E - \delta \bar{U}(x; \varepsilon))}.
\]

Writing \( y \) in \((116)\) as a function of \((117)\), we can apply the implicit function theorem to equation \((116)\) and we get a solution \( y = \bar{Y}_\pm(x, E) \) given by

\[
\bar{Y}_\pm(x, E) = \pm(1 + \varepsilon b) \ell(x, E) + \varepsilon \bar{Y}_\pm(\ell(x, E)),
\]

where

\[
\begin{align*}
i. & \quad b = O(|k_0|\varepsilon) \text{ and independent of } \delta. \text{ Moreover, } \bar{Y}_\pm(0) = \bar{Y}'_\pm(0) = 0. \\
ii. & \quad \varepsilon \bar{Y}_\pm \text{ is a } C^{r-2m+2} \text{ function and}
\end{align*}
\]

\[
\begin{align*}
&\left| \varepsilon \bar{Y}_\pm \circ \ell \right|_{L^s(I_{E_0})} \leq |k_0|\varepsilon, \quad s = 0, 1, \\
&\left| \varepsilon \bar{Y}_\pm \circ \ell \right|_{L^s(I_{E_0})} \leq |k_0|\varepsilon E_0^{-s+1/2}, \quad 2 \leq s \leq n - 2m + 2,
\end{align*}
\]

where \( I_{E_0} = ((x, E), x \in T, E \geq E_0 > 0) \).

This result is stated explicitly in lemma 8.34 in [DLS06a]. For more details and a rigorous proof we refer the reader to it.

From expression \((118)\) it is clear that the size of the energy determines the dominant terms in \( \ell(x, E) \). Thus, if \( E \gg \delta = \varepsilon^\gamma \) the tori \( T_{E_0}^0 \) are rather flat because the term \( \varepsilon^\gamma \bar{U}(x; \varepsilon) \) is very small compared with \( E \), whereas if \( E \leq \varepsilon^\gamma \), the term \( \sqrt{E - \varepsilon^\gamma \bar{U}(x; 0)} \) and therefore the size of \( y \) oscillates between \( E \) and \( \varepsilon \) and it has the effect of bending the tori up to the point that they are bunched near the critical point (see figure 1).

Hence \( D_h \) will be divided into three regions in a similar way as in [DLS06a]: \( D_t \) is the region far from the separatrix, \( D_o \) close to the separatrix but outside the region bounded by the separatrix loop and \( D_m \) close to the separatrix but inside the separatrix loop, in the following way:

\[
\begin{align*}
D_t &= \{(y, x, s) \in D_h : K_0(y, x; \varepsilon) = E, \varepsilon^\beta \leq E \leq E^2 \} \quad (121) \\
D_o &= \{(y, x, s) \in D_h : K_0(y, x; \varepsilon) = F, \varepsilon^\beta \leq F \leq \varepsilon^\gamma \} \quad (122) \\
D_m &= \{(y, x, s) \in D_h : K_0(y, x; \varepsilon) = G, -\varepsilon^\gamma \leq G \leq -\varepsilon^\beta \} \quad (123)
\end{align*}
\]

where \( 1 \leq \gamma < 2 + \nu \) as in \((112)\) and \( \beta \) is arbitrary provided that \( \beta > \gamma \) (see figure 1).

Theorem 3.30 establishes the existence of primary tori in \( D_t \cup D_o \) and secondary tori in \( D_m \) at a certain distance between them that depends on the number \( m \) of averaging steps and close to the level sets of the averaged Hamiltonian \( K_0(y, x; \varepsilon) \).
Theorem 3.30 (KAM theorem in the big gaps region). Consider the $C^{r-2m}$ reduced Hamiltonian $K(y,x,s;\varepsilon)$ given in (106) inside the region $D_{k_0}$ defined in (115). Consider $\beta > \gamma$, with $\gamma$ as in (112) and assume that $r > (n-2)m+2$, $n \geq 2m+6$ and $m \geq 14(\beta - \gamma) + 3\gamma/2$. Then, for $|\varepsilon|$ small enough, one has the following:

1. Primary tori far from resonance. There exists a set of values $E_1 < \cdots < E_{l_F}$ verifying $\varepsilon^\gamma \leq E_i \leq \bar{L}^2 \sim \varepsilon^{2\alpha}$ and $\alpha < 1/n$, such that

(a) The frequencies $\omega(E_i)$ are Diophantine numbers of constant type and Markov constant $E_i^{1/4}/E_i^{1/4}[k_0]$.

(b) For any value $E_i$, there exist two primary invariant tori $T^\pm_{E_i}$ of Hamiltonian (106) contained in $D_t$.

(c) The motion of the tori $T^\pm_{E_i}$ is $C^1$-conjugated to a rigid translation of frequencies $(\omega(E_i), 1)$.

(d) These tori can be written as

$$T^+_E = \{(y, x, s) \in D_t, K_{E_i}(y, x, s; \varepsilon) = E_i, y > 0\}$$
$$T^-_E = \{(y, x, s) \in D_t, K_{E_i}(y, x, s; \varepsilon) = E_i, y < 0\}$$

where $K_{E_i}(y, x, s; \varepsilon)$ is a $C^4-\varepsilon$ function, for any $q > 0$, given by

$$K_{E_i}(y, x, s; \varepsilon) = K_0(y, x; \varepsilon) + O_C\left(\varepsilon^{\frac{1}{2} + \min\{2m, 2\gamma\}}E_i^{1/4}[k_0]\right)$$

(124)

(e) $D_t \subset \bigcup B(T^\pm_{E_i}, \varepsilon^{\frac{1}{2} + \min\{2m, 2\gamma\}}E_i^{1/4}[k_0])$, where

$$B(T^\pm_{E_i}, \delta) = \{(y, x, s) \in D_{k_0}, |K_0(y, x; \varepsilon) - E_i| \leq \delta\}$$

2. Primary tori close to resonance. There exists a set of values $F_1 < \cdots < F_{l_F}$ verifying $\varepsilon^\beta \leq F_i \leq \varepsilon^\gamma$, such that

(a) The frequencies $\omega(F_i)$ are Diophantine numbers of constant type and Markov constant $\varepsilon^{\frac{1}{2} + \min\{2m, 2\gamma\}}F_i^{1/4}[k_0]$.

(b) For any value $F_i$, there exist two primary invariant tori $T^\pm_{F_i}$ of Hamiltonian (106) contained in $D_o$. 
(c) The motion of the tori $T_{E_i}^\pm$ is $C^1$-conjugated to a rigid translation of frequencies $(\omega(F_i), 1)$.

(d) These tori can be written as

\[ T_{E_i}^- = \{ (y, x, s) \in D_o, K_{E_i}(y, x, s; \varepsilon) = F_i, y < 0 \} \]

\[ T_{E_i}^+ = \{ (y, x, s) \in D_o, K_{E_i}(y, x, s; \varepsilon) = F_i, y > 0 \} \]

where $K_{E_i}(y, x, s; \varepsilon)$ is a $C^{q}$ function, for any $q > 0$, given by

\[ K_{E_i}(y, x, s; \varepsilon) = \frac{K_0(y, x; \varepsilon) + O_{C^{2}} \left( \frac{\varepsilon^{1+2m(\nu+1)} E_i^{-\gamma}}{2^{4\nu}} \right)}{F_i^{-\gamma}} \]

(125)

(e) $D_o \subset \bigcup_i B(T_{E_i}^\varepsilon, \varepsilon) \subset \bigcap_i K_{E_i}(y, x, s; \varepsilon), \varepsilon) \subset D_m$.

3. Secondary tori close to resonance. There exists a set of values $G_1 < \cdots < G_{\bar{\kappa}}$ verifying $-\varepsilon^2 \leq G_i \leq -\varepsilon^\beta$, such that

(a) The frequencies $\omega(G_i)$ are Diophantine numbers of constant type and Markov constant $\varepsilon^{1+2m(\nu+1)}(G_i)^{-\gamma}k_0$.

(b) For any value $G_i$, there exists a secondary invariant torus $T_{G_i}^\pm$ of Hamiltonian (106) contained in $D_m$, contractible to the set

\[ \{(0, \alpha, s), \alpha \in \mathbb{R}, s \in \mathbb{R}/(2\pi k_0 \mathbb{Z}) \} \subset D_m \]

(c) The motion on the torus $T_{G_i}$ is $C^{1}$-conjugated to a rigid translation of frequencies $(\omega(G_i), 1)$.

(d) This torus can be written as

\[ T_{G_i} = \{ (y, x, s) \in D_{m}, K_{G_i}(y, x, s; \varepsilon) = G_i \} \]

where $K_{G_i}(y, x, s; \varepsilon)$ is a $C^{q}$ function, for any $q > 0$, given by

\[ K_{G_i}(y, x, s; \varepsilon) = \frac{K_0(y, x; \varepsilon) + O_{C^{2}} \left( \frac{\varepsilon^{1+2m(\nu+1)} G_i^{-\gamma}}{2^{4\nu}} \right)}{G_i^{-\gamma}} \]

(126)

(e) $D_m \subset \bigcup_i B(T_{G_i}^\varepsilon, \varepsilon) \subset \bigcap_i K_{G_i}(y, x, s; \varepsilon), \varepsilon) \subset D_m$.

The following corollary makes more explicit the assertions about the proximity of these tori as a function of $m$, and it also gives properties of the KAM tori when expressed as graphs of the action $y$ in terms of the angle variables $(x, s)$.

**Corollary 3.31.** Consider the $C^\gamma$ reduced Hamiltonian $K(y, x, s; \varepsilon)$ given in (106) inside the region $D_0$, defined in (115). Consider $\beta = \gamma/2 + 1 + \nu/2$, with $1 \leq \gamma < 2 + \nu$ as in (112) and $\nu \leq 1/16$. Assume that $r > (n - 2m + 2)/n, n \geq 2m + 6$ and $m \geq 10$. Then, the tori obtained in theorem 3.30 verify the following:

1. For any value $E_i$, the primary tori $T_{E_i}^\pm$ can be written as graphs of the action $y$ over the angles $(x, s)$:

\[ T_{E_i}^\pm = \{ (y, x, s) \in D_o, y = f_{E_i}^\pm(x, s; \varepsilon) \} \]

2. For any value $F_i$, the primary tori $T_{F_i}^\pm$ can be written as graphs of the action $y$ over the angles $(x, s)$:

\[ T_{F_i}^\pm = \{ (y, x, s) \in D_o, y = f_{F_i}^\pm(x, s; \varepsilon) \} \]

3. There exists $\rho_0 > 0$ such that for any $0 < \rho_0 \leq \rho \leq \pi$, and for any value $G_i$, each of the components of

\[ T_{G_i} \cap \{ (y, x, s) : x \in I_{\rho} \}, \quad I_{\rho} = \bigcup_{l=0}^{k_0-1} [2\pi l + \rho, 2\pi (l + 1) - \rho] \]
that we denote by $T^{\pm}_{Gi}$, can be written as a graph of the action $y$ over the angles $(x, s)$:

$$T^{\pm}_{Gi} = \{(x, y, s) \in D_i, x \in I_p, y = f^\pm_{Gi}(x, s; \epsilon)\}
$$

4. All these functions $f_\epsilon = f^\pm$ are at least of class $C^2$ with respect to $(x, s)$, and, denoting by $D$ the derivatives with respect to $x$ and $s$, for $v = E_i$, $i = 1, \ldots, I_E$, $v = F_i$, $i = 1, \ldots, I_F$, and $v = G_i$, $i = 1, \ldots, I_G$, they verify the following:

(a) There exists a function $\mathcal{Y}(x, E)$ given explicitly in (119) such that

$$|f_\epsilon - \mathcal{Y}(x, v)|_{1} \leq |k_0|\epsilon^{1+\nu}/2
$$

(b) $|Df_\epsilon| \leq \epsilon^{\nu}/2$, $|D^2f_\epsilon| \leq \epsilon^{\nu}/2$.

(c) For any two consecutive values $v$ and $\bar{v}$ we have

$$|v - \bar{v}| \leq |k_0|\epsilon^\beta,
$$

and

$$|f_\epsilon - f_\epsilon|_{1} \leq \frac{|v - \bar{v}|}{\epsilon^{\nu/2}} \leq |k_0|\epsilon^{1+\nu}/2.
$$

**Proof of theorem 3.30.** The proof follows the strategy established in [DLS06a], with the same scaling in the domains $D_0$ and $D_{in}$. The main difference is that we will perform a sequence of scalings in the far domain $D_1$, whereas in [DLS06a] there was no scaling in this region. This sequence of scalings in $D_1$ will reduce the number of averaging steps $m$ needed to get tori close enough in the region $D_1$, and therefore the required differentiability $r$.

We will first give a detailed proof of part (1) of this theorem. Note that in $D_1$ defined in (121), the energy $E$ ranges from $\epsilon^\nu$ to $L^2 \sim \epsilon^{2\nu}$. Hence, we consider a value of $E$, let us say $E_i$, in the interval $[\epsilon^\nu, \epsilon^{2\nu}]$ and a small neighbourhood around it of the form $[c_a E_i, c_b E_i] \subseteq [\epsilon^\nu, \epsilon^{2\nu}]$, where $c_a$, $c_b$ are constants independent of $\epsilon$ and $E_i$, such that $c_a < 1$ and $c_b > 1$. Thus, we introduce the following domain contained in $D_1$:

$$D_{E_i} = \{(y, x, s) \in D_1 : K_0(y, x; \epsilon) = E_i, c_a E_i \leq E \leq c_b E_i\}.
$$

By the equation of $K_0$ in (113) and the expression of $h^\nu$ in (108), the main term in $y$ is given in (118). Therefore, in $D_{E_i}$ the coordinate $y$ is of size $O(\sqrt{E_i})$ and it is natural to perform the scaling

$$y = \sqrt{E_i}Y
$$

which transforms the Hamiltonian system of Hamiltonian $K(y, x, s; \epsilon)$ given in (106), which is $C^{\nu-2m}$ with respect to the variables $(y, x, s)$ with a bounded $C^\nu$ norm up to $\ell = n - 2m$, into a Hamiltonian system of $C^{\nu-2m}$ Hamiltonian with respect to $(Y, x, s)$ with a bounded $C^\nu$ norm up to $\ell = n - 2m$,

$$K(Y, x, s; \sqrt{E_i}, \epsilon) = \frac{1}{\sqrt{E_i}}K(\sqrt{E_i}Y, x, s; \epsilon)
$$

$$= \sqrt{E_i}K_0(Y, x; \sqrt{E_i}; \epsilon) + \frac{\epsilon^{m+1}}{\sqrt{E_i}}S(\sqrt{E_i}Y, x, s; \epsilon),
$$

with

$$K_0(Y, x; \sqrt{E_i}, \epsilon) = \frac{1}{E_i}K_0(\sqrt{E_i}Y, x; \epsilon)
$$

$$= \frac{Y^2}{2}\hat{h}(\sqrt{E_i}Y; \epsilon) + \frac{\epsilon^\nu}{E_i}\hat{U}(x; \epsilon),
$$

where $\hat{h}(y; \epsilon) = 1 + O(|k_0|^2\epsilon)$ is given in (108) and, consequently, $K_0$ is a $C^{\nu-2m+2}$ function with respect to $(Y, x)$ with a bounded $C^\nu$ norm up to $\ell = n - 2m + 2$, because $\hat{h}(y; \epsilon)$ is $C^{\nu-2m+2}$ with respect to $y$ with a bounded $C^\nu$ norm up to $\ell = n - 2m + 2$. 

The scaling (129) transforms the domain $D_{\varepsilon}$ in (128) into

$$
\tilde{D} = \{ (y, x, s) \in \mathbb{R} \times \mathbb{R} / 2\pi k_0 \mathbb{Z} \times \mathbb{T} : K_0(y, x; \sqrt{E_i}, \varepsilon) = E / E_i, \ c_a E_i \leq E \leq c_b E_i \} 
$$

$$
= \{ (y, x, s) \in \mathbb{R} \times \mathbb{R} / 2\pi k_0 \mathbb{Z} \times \mathbb{T} : K_0(y, x; \sqrt{E_i}, \varepsilon) = \varepsilon, \ c_a \leq \varepsilon \leq c_b \}. \quad (132)
$$

Next we will define the action-angle variables $(A, \psi)$ associated with the Hamiltonian $K_0(y, x; \sqrt{E_i}, \varepsilon)$ in the domain $\tilde{D}$. Note that the Hamiltonian $K(y, x, s; \sqrt{E_i}, \varepsilon)$ is $2\pi k_0$-periodic in $s$ and $2\varepsilon$-periodic in $\varepsilon$, where $K_0(y, x; \sqrt{E_i}, \varepsilon)$ is $2\pi$-periodic in $x$ and independent of $s$. Therefore, the domain $\tilde{D}$ is nothing else but $k_0$ copies of the domain $D^* \times \mathbb{T}$, where

$$
D^* = \{ (y, x) \in \mathbb{R} \times \mathbb{T} : K_0(y, x; \sqrt{E_i}, \varepsilon) = \varepsilon, \ c_a \leq \varepsilon \leq c_b \}. \quad (133)
$$

Note that, by expression (131) for $K_0$, the equation

$$
K_0(y, x; \sqrt{E_i}, \varepsilon) = \varepsilon
$$

has the same form as equation (116) with $\delta = \varepsilon' / E_i$ and it defines two functions $Y = Y_\pm(x, \varepsilon)$ on $D^*$, given in (119), which are of the form

$$
Y_\pm(x, \varepsilon) = \pm \sqrt{2 \left( e - \frac{\varepsilon'}{E_i} \tilde{U}(x; \varepsilon) \right)} \left( 1 + O_{C_n-2n-2}(\varepsilon^2) \right).
$$

Since, by construction of $\tilde{U}(x; \varepsilon)$, on $x = 0$ there is a global maximum such that $-c \leq \tilde{U}(x; \varepsilon) \leq 0$, in the domain $D^*$ we have

$$
0 \leq c_a \leq \varepsilon \leq e - \frac{\varepsilon'}{E_i} \tilde{U}(x; \varepsilon) \leq e + c \frac{\varepsilon'}{E_i} \leq c_b + \text{const},
$$

where we have used $E_i \geq \varepsilon'$ and therefore $\tilde{c}_a \leq Y_\pm(x, \varepsilon) \leq \tilde{c}_b + \text{const}$ and $Y_\pm$ is $O_{C_n-2n-2}(1)$, for some constants $\tilde{c}_a$ and $\tilde{c}_b$.

We consider in $D^*$ the action-angle variables

$$
A = \frac{1}{2\pi} \int_0^{2\pi} Y_\pm(x, \varepsilon) \, dx,
$$

$$
\psi = \frac{2\pi}{T(e)} \tau(x, \varepsilon), \quad (134)
$$

where $\tau(x, \varepsilon)$ is the time along the orbit of the Hamiltonian $K_0(y, x; \sqrt{E_i}, \varepsilon)$ with energy $\varepsilon$ given by

$$
\tau(x, \varepsilon) = \int_0^x \frac{\partial Y_\pm}{\partial e}(u, \varepsilon) \, du. \quad (135)
$$

We have chosen the origin of time at $x = 0$ and with this choice $T(e) = \tau(2\pi, e)$ is the period of the periodic orbit.

From expression (134) it is obvious that $A$ satisfies $\tilde{c}_a \leq A \leq \tilde{c}_b$ and that $A$ is $O_{C_n-2n-2}(1)$.

The action-angle variables $(A, \psi)$ introduced in (134) have already been studied in proposition 8.35 of [DLS06a] for the case when they become singular, that is when the domain $D^*$ depends on $\varepsilon$. In our case, we can adapt the result in proposition 8.35 of [DLS06a] for the domain $D^*$ not depending on $\varepsilon$. We obtain that we can express the integrable Hamiltonian $\sqrt{E_i} K_0(y, x; \sqrt{E_i}, \varepsilon)$ in (131) into action-angle variables $(A, \psi)$ in the domain $D^*$ and the change of coordinates is away from the singularity in this domain. More precisely, there exists a $C^{\nu-2n+2}$ change of variables in $D^*$

$$
X : D^* \mapsto D^*,
$$

$$
(A, \psi) \mapsto (Y, x) \quad (136)
$$
given in (134) with $D^{**} = \{(A, \psi) : \tilde{c}_a \leq A \leq \tilde{c}_b, \psi \in \mathbb{T}\} = [\tilde{c}_a, \tilde{c}_b] \times \mathbb{T}$ and $\tilde{c}_a, \tilde{c}_b$, suitable constants independent of $\varepsilon$ and $E_l$, such that

i. $K_0(\mathcal{X}(A, \psi); \sqrt{E_l}, \varepsilon) = G(A; \sqrt{E_l}, \varepsilon)$.

ii. $|\mathcal{X}|_{C^0(D^{**})} \leq 1$, $|\mathcal{X}^{-1}|_{C^0(D^{**})} \leq 1$, $0 \leq n_0 \leq n - 2m + 2$.

iii. $|G|_{C^0(D^{**})} \leq 1$ and $|G''|_{C^0(D^{**})} \geq 1$

where the constants in the above inequalities do not depend on $\varepsilon$ and $E_l$.

Now, we consider the Hamiltonian $K$ in (130) expressed in action-angle variables,

$$\tilde{K}(A, \psi, s; \sqrt{E_l}, \varepsilon) = \sqrt{E_l}G(A; \sqrt{E_l}, \varepsilon) + \frac{\varepsilon^{m+1}}{\sqrt{E_l}}\tilde{S}(A, \psi, s; \sqrt{E_l}, \varepsilon),$$

(137)

where $\tilde{K} = K \circ \mathcal{X}$ and $\tilde{S} = S \circ \mathcal{X}$.

The Hamiltonian (137) is of the form (80) with $K_0 = \sqrt{E_l}G(A; \sqrt{E_l}, \varepsilon)$ and $K_1 = \varepsilon^{m+1}E_l^{-1/2}\tilde{S}(A, \psi, s; \sqrt{E_l}, \varepsilon)$ and $2\pi k_0$-periodic in $\psi$.

The functions $G$ and $\tilde{S}$ are $C^{r-2m+2}$ and $C^{r-2m}$ with bounded $C^r$ norms up to $r = n - 2m + 2$ and $r = n - 2m$ in the variables $(A, \psi)$, respectively. Since by hypotheses of theorem 3.30 we have that $r > n \geq 2m + 6$, $G$ and $\tilde{S}$ have a bounded $C^6$ norm in the variables $(A, \psi)$. Therefore, using Faa–di Bruno formula (C.3) and the bound for the $C^6$ norm in the variables $(y, x)$ for $D^{**}$ in expression (111) jointly with the bounds for the change of coordinates $\mathcal{X}$ in item (ii) we have that, for any $s \in \mathbb{T}$,

$$\left|\frac{\varepsilon^{m+1}}{\sqrt{E_l}}\tilde{S}(\cdot; s; \sqrt{E_l}, \varepsilon)\right|_{C^6(D^{**})} \leq |k_0|^{-4}E_l^{-1/2}\varepsilon^{m+1-\alpha(6+2m)},$$

where $D^{**}_{k_0} = [\tilde{c}_a, \tilde{c}_b] \times \mathbb{R}/2\pi k_0\mathbb{Z}$. Moreover, by item (iii) in this proof we have that

$$\sqrt{E_l}\left|G''(\cdot; \sqrt{E_l}, \varepsilon)\right|_{C^6(D^{**})} \geq \sqrt{E_l}.$$

Therefore, we can apply KAM theorem 3.22 to Hamiltonian (137) with $n_0 = 5, \beta = 1 - q$, for any $q > 0$, $\delta = \delta(\varepsilon)$, $|k_0|^{-4}E_l^{-1/2}\varepsilon^{m+1-\alpha(6+2m)}$ and $M = M(\varepsilon) = \text{const} \sqrt{E_l}$ and we obtain the following:

1. There exists a set of values $A_l$, such that the Hamiltonian $K \circ \mathcal{X}$ has the invariant tori given by

$$T_l = \{(A, \psi, s) \in D^{**}_{k_0} \times \mathbb{T} : A = A_l + \mathcal{A}_l(\psi, s; \sqrt{E_l}, \varepsilon)\}$$

where $\mathcal{A}$ are $C^{q-\alpha}$ functions in the variables $(\psi, s)$, for any $q > 0$ and

$$\left|\mathcal{A}_l(\cdot; \sqrt{E_l}, \varepsilon)\right|_{C^q(\mathbb{R}/2\pi k_0\mathbb{Z} \times \mathbb{T})} \leq |k_0|E_l^{-3/4}\varepsilon^{m+1-\alpha(6+2m))/2}.$$

2. The motion of these tori is $C^{2-\alpha}$-conjugate to a rigid translation of frequencies $(\omega(A_l), 1)$, where $\omega(A_l)$ is a Diophantine number of constant type and Markov constant $|k_0|E_l^{-1/4}\varepsilon^{m+1-\alpha(6+2m))/2}$.

3. The union of neighbourhoods of size $|k_0|E_l^{-3/4}\varepsilon^{m+1-\alpha(6+2m))/2}$ of these tori cover all the region $D^{**}_{k_0} \times \mathbb{T}$.

In the variables $(Y, x, s) = (\mathcal{X}(A, \psi), s)$, the torus $T_l$ satisfies $K_0(Y, x; \sqrt{E_l}, \varepsilon) = G(A_l + \mathcal{A}_l(\psi, s; \sqrt{E_l}, \varepsilon); \sqrt{E_l}, \varepsilon)$, so that, introducing $G(A_l; \sqrt{E_l}, \varepsilon) = e_l$ and using the
estimates in items (ii) and (iii) in this proof as well as Faa–Di Bruno formulae, one obtains that the tori are given by
\[ K_0(Y, x; \sqrt{E_i}, \varepsilon) = G(A_i; \sqrt{E_i}, \varepsilon) + \mathcal{O}_\varepsilon \left( |G|_{C^1} |A_i|_{C^1} |X^{-1}|_{C^1}^2 \right) \]
\[ = e_i + \mathcal{O}_\varepsilon \left( |k_0| E_i^{-3/4} \varepsilon^{(m+1-\alpha (6+2m))/2} \right). \] (138)

Going back to the variables \((y, x, s)\) performing the scaling \(y = \sqrt{E_i} Y\) and using the expression for \(K_0\) given in (131) one obtains that the tori are given by
\[ K_0(y, x; \varepsilon) = E_i + \mathcal{O}_\varepsilon \left( |k_0| E_i^{-1/4} \varepsilon^{m+1-\alpha (2m)} \right), \]
where \(E_i = E_i e_i\).

By compactness of \(D_t\), the covering \(\{ \text{int}(D_{E_i}) \}_{i=1}^\infty\) of \(D_t\) admits a finite subcovering \(D_t = \bigcup_{i=1}^N \text{int}(D_{E_i})\), and we get the claimed results in part (1) of theorem 3.30.

The proof of parts (2) and (3) of this theorem follows as in [DLS08]. The only difference is that we introduce a sequence of domains as we did in this proof in the far region and we perform adequate scalings which allow us to get better estimates for the functions describing the searched tori. More precisely, consider the region \(D_0\) (the case for \(D_{\alpha_0}\) is analogous) and introduce the domain
\[ D_{E_i} = \{(y, x, s) \in D_0 : K_0(y, x; \varepsilon) = F, \ c_\alpha F_i \leq F \leq c_\beta F_i\}, \]
where \(c_\alpha, c_\beta\) are analogous to (128) in part (1). Since the energy \(F_i \leq \varepsilon^{\gamma/2}\) in \(D_0\) (see (122)), from the expression for the main term of \(y\) given by \(\ell(x, E)\) in (118), the coordinate \(y\) ranges from \(\sqrt{F_i}\) to \(\varepsilon^{\gamma/2}\). Hence we perform the scaling \(y = \varepsilon^{\gamma/2} Y\) and we proceed as in lemma 8.36 in [DLS06a]. We obtain that the original system is transformed into a Hamiltonian system of \(C^{\gamma-2m}\) Hamiltonian with respect to \((Y, x, s)\) of the form
\[ K(Y, x, s; \varepsilon^{\gamma/2}, \varepsilon) = \varepsilon^{\gamma/2} K_0(Y, x; \varepsilon) + \varepsilon^{m+1-\gamma/2} S(\varepsilon^{\gamma/2} Y, x, s; \varepsilon), \]
with
\[ K_0(Y, x; \varepsilon^{\gamma/2}, \varepsilon) = \frac{Y^2}{2} \hat{h}(\sqrt{E_i} Y; \varepsilon) + \hat{U}(x; \varepsilon), \]
where \(\hat{h}(y; \varepsilon) = 1 + \mathcal{O}(\varepsilon)\) is given in (108). The Hamiltonian is defined now on the domain
\[ \hat{D} = \{(y, x, s) \in \mathbb{R} \times \mathbb{R}/2\pi k_0 \mathbb{Z} \times \mathbb{T} : K_0(Y, x; \varepsilon^{\gamma/2}) = F / F_i, \ c_\alpha^0 F_i \leq F \leq c_\beta^0 F_i\} \]
\[ = \{(y, x, s) \in \mathbb{R} \times \mathbb{R}/2\pi k_0 \mathbb{Z} \times \mathbb{T} : K_0(Y, x; \varepsilon^{\gamma/2}) = e, \ c_\alpha^0 F_i / \varepsilon^{\gamma/2} \leq e \leq c_\beta^0 F_i / \varepsilon^{\gamma/2}\}. \]

Next, we define the action angle variables in the domain \(\hat{D}\) by formulae (134). The only change is that we need to take into account that instead of expression (8.77) in [DLS06a] we have
\[ c_\alpha^0 F_i / \varepsilon^{\gamma/2} \leq e - \hat{U}(x; \varepsilon) \leq c_\beta^0 F_i / \varepsilon^{\gamma/2} + c \leq \text{const}, \]
and by (111) the perturbation \(\varepsilon^{m+1-\gamma/2} S(\varepsilon^{\gamma/2} Y, x, s; \varepsilon^{\gamma/2})\) can be bounded in the \(C^6\) norm in the variables \((Y, x)\) by \(e^{-\gamma/2} \varepsilon^{m+1-\alpha (6+2m)} |k_0|^{-4}\).

Therefore we can apply proposition 8.38 in [DLS06a] and proceed as in the proof of parts (2) and (3) of theorem 8.30 in [DLS06a] replacing the estimates in terms of \(\varepsilon\) in (2.1), equation (8.50) and (2.5), \(\varepsilon / \varepsilon^{\gamma/2}\) by \(\varepsilon^{m+1-\alpha (6+2m)} |k_0|^{-4}\) and \(\varepsilon^{\gamma/2} / \varepsilon^{\gamma/2}\) by \(F_i \varepsilon^{\gamma/2}\), and multiplying by the constant \(C_{\alpha_0} = \text{const} |k_0|^3\) of KAM theorem 3.22, to obtain the estimates in 2(a), equation (125) and 2(e). Finally, by compactness of \(D_0\), we get the claimed results.
Proof of corollary 3.31. It is totally analogous to the proof of corollary 8.31 in [DLS06a] and it follows from theorem 3.30 just applying the implicit function theorem.

We apply theorem 3.30, with \( m \geq 10 \) and \( \beta = \gamma + 1 + v/2, \) where \( 1 \leq \gamma < 2 + v \) and \( v \leq 1/16. \) From these conditions it follows that \( \beta > \gamma \) and \( m \geq 14(\beta - \gamma) + 3\gamma/2 \) and therefore, we obtain that the invariant tori in the domains \( D_\ell, D_o \) and \( D_m \) are given by the implicit equations (124), (125) and (126), which are of the form

\[
K_0(y, x; \varepsilon) = E + \delta g(y, x, E; \varepsilon)
\]

with \( |g|_{C^1} \leq \text{const} \) and

\[
E = E_i, \quad \delta = \varepsilon^{(a_1+2(b_2+2))}, \quad E_i^{1/4}|k_0|,
\]

\[
E = F_i, \quad \delta = \varepsilon^{(a_1+2(b_2+2v)/2+64)}, \quad F_i^{-1}|k_0|,
\]

\[
E = G_i, \quad \delta = \varepsilon^{(a_1+2(b_2+2v)/2+64)}|G_i|^{-1}|k_0|,
\]

respectively.

Equation (139) is equivalent to the equation

\[
M(y, x, s, t; \delta, \varepsilon) \equiv y - \mathcal{Y}_\pm(x, t) = 0,
\]

where \( t = E + \delta g(y, x, s, E; \varepsilon) \) and \( \mathcal{Y}_\pm(x, t) \) is given in equation (119). The above equation has been studied in full detail in lemma 8.39 of [DLS06a]. It is not difficult to check that one has

\[
\left| \frac{\partial M}{\partial y} - 1 \right| \leq \text{const} \delta \varepsilon^{-\gamma/2},
\]

which is a bound analogous to (8.95) in lemma 8.39 in [DLS06a], where the factor \( \varepsilon^{-\gamma/2} \) comes directly from expression (113) of \( K_0. \) So, as long as \( \delta \varepsilon^{-\gamma/2} \leq \delta_0 \ll 1, \) for some constant \( \delta_0 \) independent of \( \varepsilon, \) we can apply the implicit function theorem in order to get the invariant tori of items 1, 2 and 3 written as graphs of the action \( y \) over the angles \( (x, s) \) as

\[
y = f_\pm^\circ(x, s; \varepsilon)
\]

where \( v = E_i, F_i, G_i, \) respectively and

\[
f_\pm^\circ(x, s; \varepsilon) = \mathcal{Y}_\pm(x, v) + O(\delta \varepsilon^{-\gamma/2}).
\]

Let us check first that condition \( \delta \varepsilon^{-\gamma/2} \ll 1 \) is fulfilled. Note, first, that by the choice \( m \geq 10 \) and \( \beta = \gamma + 1 + v/2, \) where \( v \leq 1/16, E_i \leq \varepsilon^{2v} \) and \( F_i, G_i \geq \varepsilon^\beta, \) one obtains in the three cases of (140), that \( |\delta| \leq |k_0|\varepsilon^\beta, \) which clearly implies \( \delta \varepsilon^{-\gamma/2} \leq |k_0|\varepsilon^{1+\gamma/2} \leq \delta_0, \) for some constant \( \delta_0 \ll 1 \) since, by expression (112), \( |k_0| \leq \varepsilon^{-1+\gamma/2} \leq \varepsilon^{-1}. \) Thus, we obtain results in items (1), (2), (3) and

\[
|f_\circ - \mathcal{Y}(x, v)|_{C^1} \leq \delta \varepsilon^{-\gamma/2} \leq |k_0|\varepsilon^{1+\gamma/2} = |k_0|\varepsilon^{1+\gamma/2},
\]

as claimed in (4a). In an analogous way one gets (4b).

Finally, from results (1e), (2e) and (3e) in theorem 3.30 and definitions of \( D_i, D_o \) and \( D_m \) given in (121), (122) and (123) we have

\[
|E_i - E_{i+1}| \leq \varepsilon^{(a_1+2(b_2+2))} (E_i^{1/4} + E_{i+1}^{1/4})|k_0|
\]

\[
|F_i - F_{i+1}| \leq \varepsilon^{(a_1+2(b_2+2v)/2+64)} (F_i^{-5} + F_{i+1}^{-5})|k_0|
\]

\[
|G_i - G_{i+1}| \leq \varepsilon^{(a_1+2(b_2+2v)/2+64)} (|G_i|^{-5} + |G_{i+1}|^{-5})|k_0|
\]
and taking into account that $E_1 \sim F_1 \sim \varepsilon^\gamma$ and $F_1 \sim G_{\text{lg}} \sim \varepsilon^\beta$ we get

$$|E_1 - F_1| \leq \varepsilon^{m+1} + \varepsilon^{(m+2)(\beta - \gamma)/2} |k_0|,$$

$$|F_1 - G_{\text{lg}}| \leq (\varepsilon^\beta + \varepsilon^{m+1} + \varepsilon^{(m+2)(\beta - \gamma)/2} - \varepsilon^{10(\beta - \gamma)/2}) |k_0|.$$ 

Since $\beta = \gamma/2 + 1 + \nu/2$, $m \geq 10$, all these exponents are bigger than $\beta$ as claimed in item (4c).

The last estimate in item (4c) follows from the inequalities above and the following bounds:

$$\begin{align*}
|\frac{\partial f_E}{\partial E}| &\leq \varepsilon - \frac{\gamma}{2}, \\
|\frac{\partial D f_E}{\partial E}| &\leq \varepsilon - \frac{\gamma}{2}, \\
\text{analogous to (8.91) given by lemma 8.39 in [DLS06a].}
\end{align*}$$

\square

Remark 3.32. In the case considered in [DLS06a], where the perturbation $h$ in (1) is assumed to be a trigonometric polynomial in the angular variables $(\phi, t)$, there exists a finite number of resonances so $L$ can be chosen independent of $\varepsilon$, that is $\alpha = 0$. Moreover $\gamma$ is simply replaced by the values $j = 1, 2$ in [DLS06a] corresponding to resonances of order 1 and 2, respectively. In this case, corollary 3.31 only requires $m \geq 9$ and $r = n \geq 24$ since there is no need of a truncation process, so that Hamiltonian in (1) only needs to be $C^{26}$. This improves substantially the regularity required in [DLS06a], since Hamiltonian (1) was assumed to be $C^{60}$ because $m$ was chosen $= 26$.

**Invariant tori in the original variables.** Theorem 3.30 gives the KAM tori, both primary and secondary, in the variables $(y, x, s)$. From equations (124), (125) and (126) in theorem 3.30, we know that these tori are given approximately by the level sets of the Hamiltonian $K_0(y, x; \varepsilon)$ in (113).

We can write them in the original variables $(I, \phi, s)$ using the change of coordinates given by theorem 3.11 and changes (24), (102) and (105). More precisely, we have that the relation with the original variables is given by

$$y = k_0 I + l_0 + O_{C^2}(|k_0| \varepsilon^{1-4\alpha}), \\
x = k_0 \phi + l_0 s + O_{C^2}(|k_0| \varepsilon^{1-4\alpha}),$$

(141)

whose inverse in terms of the $I$ variable can be written in the form

$$I = -\frac{l_0}{k_0} + \frac{1}{k_0} y + \zeta(y, x, s; \varepsilon),$$

where $\zeta$ is $O_{C^2}(\varepsilon^{1-4\alpha})$.

Using expressions (107) and (108) these invariant objects are given by the level sets of a $C^{4-\varrho}$ function $F$, for any $\varrho > 0$, which has the form

$$F(I, \phi, s; \varepsilon) = \frac{\gamma^2}{2} (1 + \varepsilon k_0^2 \hat{h}(y; \varepsilon)) + \varepsilon^\gamma \hat{U}(x; \varepsilon) + O_{C^2}(|k_0|^3 \varepsilon^{\gamma/2 + 1 + 6/2}),$$

where $y$ and $x$ are given in (141) in terms of $(I, \phi, s)$. By the definition of $\gamma$ in (112) jointly with $\hat{U}$ and $U$ in (114) and (109), respectively, we get expression (96) given in theorem 3.28.

Moreover, from items (1), (2) and (3), together with the estimates in item (4a) in corollary 3.31 we have that KAM tori can be written as graphs in the variables $(y, x, s)$ of functions of the form

$$y = f_E^*(x, s; \varepsilon) = \mathcal{Y}_{\hat{u}}(x, E) + O_{C^2}(|k_0| \varepsilon^{1+\gamma}).$$
Using the mentioned changes, we obtain that the tori inside the region $D_{BG}$, are given in the original variables $(I, \varphi, s)$ by

$$I = \frac{\lambda}{k_0} - \frac{l_0}{k_0} + 2\varphi (\theta, E) + O_C(1)$$

with $\theta = k_0 \varphi + l_0 s$, where $\varphi$ is given (119).

For the purposes of this paper, it will be convenient to keep the expression of the variable $y$ in terms of the variables $\varphi$ and $s$, that is

$$y = f^\pm_E(\varphi, s) = 2(\theta, E) + O_C(|k_0|^1\nu)$$

where $\theta = k_0 \varphi + l_0 s$.

Finally, from corollary 3.31 we know that there exist invariant tori $T_E, T'_E$ of energies $E, E'$ such that

$$|E - E'| = O(|k_0|^1\nu) = O(|k_0|^{3/2}\nu/|k_0, l_0|^{r/2})$$

and there exist also points $(y_1, x, s) \in T_E$ and $(y_2, x, s) \in T'_E$ with

$$|y_1 - y_2| = O_C(|k_0|^1\nu),$$

so in terms of their $I$ variables it follows that

$$|I_1 - I_2| \leq \frac{1}{|k_0|}|y_1 - y_2| + \frac{1}{|k_0|} |\frac{\partial \xi}{\partial y}| |y_1 - y_2|$$

and by the definition of $\gamma$ given in (112), we obtain the claimed results in item (iii) of theorem 3.28.

### 3.4. Proof of theorem 3.1

The proof of theorem 3.1 follows directly from the results obtained in propositions 3.24, 3.26 and theorem 3.28.

Choosing $n = 2m + 6$ and assuming $m \geq 10$ and $r > 2(m + 1)^2$, the hypotheses on $r, n$ and $m$ in the mentioned propositions and theorem are satisfied. Moreover, the choice $\eta = \min((m - 1 - \alpha m)/2, v/2 - 3(1 + \nu)/r)$ with $1/(r/6 - 1) < \nu < 1/16$, fits clearly with the assumptions on $\eta$ in propositions 3.24 and 3.26, and also with the one in theorem 3.28.

By propositions 3.24 and 3.26, the tori obtained in the non-resonant region and in the resonant region with small gaps are primary and they are given by the level sets of the same function $F = I + O_C(\nu)$, so they are flat up to $O_C(1)$. Both regions form the flat tori region. The explicit approximate expressions for the invariant tori are given implicitly by the function (89) and as a graph of the action $I$ over the variables $(\varphi, s)$ by (90), both functions in proposition 3.26.

By hypothesis $H3$, theorem 3.28 provides a sequence of invariant KAM tori (both primary and secondary) for the big gaps region. In a connected component of this region of the form (84), these tori are given by the level sets of a function $F$ in (96) and as a graph of the action $I$ over the angle variables $(\varphi, s)$, in (97). Moreover, the distance between the consecutive tori is $O(1)$ in terms of the action variable and $O(\nu)$ in terms of the energy.
4. Construction of a transition chain

In the previous section, we have proved that in the NHIM $\tilde{\Lambda}_c$ there exists a discrete foliation of invariant tori $\mathcal{T}_i$ (primary and secondary) with graphs at a distance $O_c(\varepsilon^{1+\eta})$, for some $\eta > 0$. We have also shown that these tori are close to being the level sets of the averaged Hamiltonian, and we have given its first order perturbative calculation for the flat tori region $\mathcal{D}_F$ and the big gaps region $\mathcal{D}_{BG}$.

The goal of this section is to prove proposition 4.1, which states that, assuming that the non-degeneracy conditions $H_2''$, $H_3''$ and $H_3''''$ in theorem 2.1 hold, there exists transversality between the foliation of invariant tori in $\tilde{\Lambda}_c$ provided by theorem 3.1 and its image under the scattering map $S_\varepsilon$ as submanifolds of $\tilde{\Lambda}_c$:

$$S_\varepsilon(\mathcal{T}_i) \pitchfork_{\tilde{\Lambda}_c} \mathcal{T}_{i+1} \Rightarrow W^u_T \pitchfork W^s_T.$$

Hence, proposition 4.1 provides a transition chain through applications of the scattering map.

**Proposition 4.1.** Consider Hamiltonian (1) satisfying the hypotheses of theorem 2.1. Pick two KAM tori $T_{\pm}$ such that $|I(x_{\pm}) - I_{\pm}| \leq \varepsilon^{1+\eta}$ for some $x_{\pm} \in \mathcal{T}_{\pm}$ and $\eta > 0$ (these tori exist thanks to theorem 3.1). Then, there exists a transition chain $\{\mathcal{T}_i\}_{N(\varepsilon)}^0$, where $N(\varepsilon) = C/\varepsilon$, in such a way that

1. The transition chain is obtained through applications of the scattering map. That is, $S_\varepsilon(\mathcal{T}_i) \pitchfork_{\tilde{\Lambda}_c} \mathcal{T}_{i+1}$.
2. $T_0 = T_-, T_{N(\varepsilon)} = T_+$.

**Proof.** The proof of proposition 4.1 is postponed to section 4.2 and is based on the results in the following section 4.1.

4.1. The scattering map and the transversality of heteroclinic intersections

The main result of this section is lemma 4.2, stated below, which considers a foliation $\mathcal{F}_F$ whose leaves are the level sets of a certain function $F$ and provides an expression for the action of the scattering map $S_\varepsilon$ on this foliation in terms of the Hamiltonian function $S_0$, generating its deformation. Moreover, it gives criteria to establish transversality between the foliation $\mathcal{F}_F$ and its image under the scattering map $S_\varepsilon$.

**Lemma 4.2.** Consider the foliation $\mathcal{F}_F$ whose leaves $L^F_E$ are the level sets of a certain function $F$:

$$L^F_E = \{(I, \varphi, s) \in (I_-, I_+), \mathbb{T}^2; F(I, \varphi, s; \varepsilon) = E\}, \quad E \in (E_1, E_2).$$

Let $S_\varepsilon$ be the scattering map introduced in (19), and $S_\varepsilon = S_0 + \varepsilon S_1 + O(\varepsilon^2)$ its Hamiltonian function given in (21) with $S_0 = -L^*$, where $L^*$ is the reduced Poincaré function introduced in (11). Then, $S_\varepsilon(L^F_E)$, the image sets of the leaves $L^F_E$ of $\mathcal{F}_F$ under the scattering map $S_\varepsilon$, satisfy $S_\varepsilon(L^F_E) = L^{F \circ S^{-1}}_E$ and therefore the equation $F \circ S^{-1}_\varepsilon = E$, where the expression $F \circ S^{-1}_\varepsilon$ is given by

$$F \circ S^{-1}_\varepsilon = F - \varepsilon\{F, S_0\} + \frac{\varepsilon^2}{2}\{\{F, S_0\}, S_0\} - \{F, S_1\} + O(\varepsilon^3),$$

(142)
where \( \{ F, S_i \} = \partial_i F \partial \phi S_i - \partial_i \partial \phi S_i \) is the Poisson bracket of the functions \( F \) and \( S_i \). Moreover, the image of a leaf \( L_{E}^F \) under the scattering map \( S_{\epsilon} \) intersects another leaf \( L_{E'}^F \), for some \( E' \), if and only if there exists \( x \in L_{E}^F \) such that \( F \circ S_{\epsilon}(x) = E' \), where the expression \( F \circ S_{\epsilon} \) is given by

\[
F \circ S_{\epsilon} = F + \epsilon \{ F, S_0 \} + \frac{\epsilon^2}{2}(\{ F, S_0 \}, S_0) + \{ F, S_1 \}) + \mathcal{O}(\epsilon^3).
\]

Assuming that

\[
\frac{||\{ F, F \circ S_{\epsilon}^{-1} \}||}{||\nabla F||^2} \geq C \epsilon,
\]

where \( C \) is a constant independent of \( \epsilon \) and \( E \), the angle between the surfaces \( L_{E}^F \) and \( S_{\epsilon}(L_{E}^F) \) at the intersection points is bounded from below by \( C \epsilon \). Therefore, foliations \( \mathcal{F}_F \) and \( \mathcal{F}_{F \circ S_{\epsilon}^{-1}} \) intersect transversally.

**Remark 4.3.** For the case of a function \( F \) which is \( O_{\mathcal{C}^1}(1) \), the scattering map increases (decreases) the energy \( E \) by order \( \epsilon \), provided that the first order term \( \{ F, \mathcal{L}^* \} \) in (143) satisfies \( \{ F, \mathcal{L}^* \} \neq 0 \).

**Remark 4.4.** Using expression (142) and \( S_0 = -\mathcal{L}^* \), the condition for the transversality of the foliations (144) reads

\[
\frac{||\{ F, \{ F, \mathcal{L}^* \} \} + \frac{1}{2} \epsilon (-\{ F, \{ F, \mathcal{L}^* \}, \mathcal{L}^* \} + \{ F, \{ F, S_1 \} \}) + \mathcal{O}(\epsilon^2)||}{||\nabla F||^2} \geq C.
\]

Note that if \( F \) is \( O_{\mathcal{C}^2}(1) \) the term \( \epsilon \) can be neglected and the condition reduces to

\[
\frac{||\{ F, \{ F, \mathcal{L}^* \} \}||}{||\nabla F||^2} \geq C.
\]

Also note that an equivalent condition to (144) is

\[
\frac{||\{ F, F \circ S_{\epsilon} \}||}{||\nabla F||^2} \geq C \epsilon.
\]

**Proof.** In section 2.3.2 we have shown that there exists a Hamiltonian function \( S_{\epsilon} \) generating the deformation of the scattering map \( S_{\epsilon} \) and we have given its first order perturbative computation in equation (21). Hence, taking into account that \( S_{\epsilon} = S_0 + \epsilon S_1 + \mathcal{O}(\epsilon^2) \), it is clear that (see [CH82] for instance) \( F \circ S_{\epsilon} \) is given by

\[
F \circ S_{\epsilon} = F + \epsilon \{ F, S_0 \} + \frac{\epsilon^2}{2}(\{ F, S_0 \}, S_0) + \{ F, S_1 \}) + \mathcal{O}(\epsilon^3),
\]

with \( S_0 = -\mathcal{L}^* \). The expression for \( F \circ S_{\epsilon}^{-1} \) follows identically.

In order to show the transversality between the foliations \( \mathcal{F}_F \) and \( \mathcal{F}_{F \circ S_{\epsilon}^{-1}} \), we need to obtain lower bounds for the angle of intersection. More precisely, the angle \( \alpha \) between the normal vectors to the tangent planes to the surfaces \( S_{\epsilon}(L_{E}^F) \) and \( L_{S_{\epsilon}^{-1}}^F \) is given by

\[
\sin(\alpha) = \frac{|\nabla (F \circ S_{\epsilon}^{-1}) \times \nabla F|}{|\nabla (F \circ S_{\epsilon}^{-1})| |\nabla F|} = \frac{|\{ F, F \circ S_{\epsilon}^{-1} \}|}{|\nabla (F \circ S_{\epsilon}^{-1})| |\nabla F|},
\]

where \( F \circ S_{\epsilon}^{-1} \) is given in expression (142). From this expression one can see that \( \sin(\alpha) \) is \( \mathcal{O}(\epsilon) \) and condition (144) gives the required transversality.

As we have argued in the previous section the tori in \( \hat{\Lambda}_1 \) have different behaviour depending whether they are close to or far from the separatrix. Thus, the tori in the flat tori region and
in the big gaps region far from the resonance are rather flat, whereas they are bent in the big
gaps region close to a resonance. The fact that the tori are not flat has the consequence that
the dominant effect of comparing a torus with the image under the scattering map of another
torus, will include some extra terms. For this reason, we will divide the study into three cases:
on the one hand, the flat tori region and on the other hand the resonant region with big gaps,
where we will distinguish between far from and close to the resonance.

4.1.1. The flat tori region. In lemma 4.5, we apply lemma 4.2 to the flat tori region \( \mathcal{D}_F \).
By theorem 3.1, in one connected component of this region the invariant tori are given by the
leaves \( L^F_E \) of a foliation \( \mathcal{F}_F \), where \( F \) is of the form (89). Moreover they can be written as
a graph of the action \( I \) over the angle variables \((\varphi, s)\): \( I = \lambda_E(\varphi, s; \varepsilon) \), where \( \lambda_E \) is given in
(90).

**Lemma 4.5.** Let us consider a foliation \( \mathcal{F}_F \) contained in a connected component of the flat
tori region \( \mathcal{D}_F \), where the function \( F \) is of the form (89), so that the equation \( F(I, \varphi, s; \varepsilon) = E \)
defines a smooth surface given as a graph \( \lambda_E(\varphi, s; \varepsilon) \), with \( \lambda_E \) as in (90).

Assume that hypothesis \( H2^* \) is fulfilled. More precisely, the reduced Poincaré function \( \mathcal{L}^* \)
defined in (11) verifies, for any value of \((I, \varphi, s) \in \mathcal{D}_F\) such that \((I, \varphi - Is) \in H^*_E\) (respectively,
such that \((I, \varphi - Is) \in H^*_F\)), that the function
\[
\bar{\vartheta} \mapsto \frac{\partial \mathcal{L}^*}{\partial \bar{\vartheta}}(I, \bar{\vartheta})
\]
for \( \bar{\vartheta} = \varphi - Is \) is positive (respectively negative) and non-constant for \( \bar{\vartheta} \) on some set \( \mathcal{J}^*_E \) (see
(12)). Then the foliations \( \mathcal{F}_F \) and \( \mathcal{F}_F, S^{-1} \) intersect transversally.

More precisely, any surface \( S_\varepsilon(L^F_E) \) intersects at some point the surface \( L^F_E \) for any \( E' > 0 \)
(respectively \( E' < 0 \)), \(|E' - E| = O(\varepsilon)\). The angle between the surfaces \( S_\varepsilon(L^F_E) \) and \( L^F_E \) at
the intersection can be bounded from below by \( C\varepsilon \), where \( C \) is a constant independent of \( \varepsilon \)
and \( E \).

**Proof.** We will apply lemma 4.2 with \( F(I, \varphi, s; \varepsilon) = I + O(\varepsilon^{1-4\alpha}) \) and \( I = \lambda_E(I, \varphi, s; \varepsilon) = E + O(\varepsilon^{1-4\alpha}) \) for \( \alpha < 1/26 \). We will see that provided hypothesis \( H2^* \) is fulfilled, condition
(144) of lemma 4.2 is satisfied.

We first apply the scattering map to the implicit surface
\[
L^F_E = \{(I, \varphi, s) \in \mathcal{D}_F, F(I, \varphi, s) = E\},
\]
and recall that \( S_\varepsilon(L^F_E) \) intersects a leaf \( L^F_E \) at a point \((I, \varphi, s) \in L^F_E \) if \( F \circ S_\varepsilon(I, \varphi, s; \varepsilon) = E' \),
where, using expression (143), \( F \circ S_\varepsilon \) is given by
\[
(F \circ S_\varepsilon)(I, \varphi, s; \varepsilon) = E - \varepsilon \{F, \mathcal{L}^*\} + O(\varepsilon^2), \tag{148}
\]
with,
\[
\{F, \mathcal{L}^*\} = - \frac{\partial \mathcal{L}^*}{\partial \varphi} \frac{\partial F}{\partial I} + \frac{\partial F}{\partial \varphi} \frac{\partial \mathcal{L}^*}{\partial I} = - (1 + O(\varepsilon^{1-4\alpha})) \frac{\partial \mathcal{L}^*}{\partial \bar{\vartheta}} + O(\varepsilon^{1-4\alpha})
\]
\[
= - \frac{\partial \mathcal{L}^*}{\partial \bar{\vartheta}} + O(\varepsilon^{1-4\alpha})
\]
with \( \bar{\vartheta} = \varphi - Is \). Evaluating on \( I = E + O(\varepsilon^{1-4\alpha}) \), equation (148) reads
\[
(F \circ S_\varepsilon)(I, \varphi, s; \varepsilon) = E + \varepsilon \frac{\partial \mathcal{L}^*}{\partial \bar{\vartheta}}(E, \varphi - Es) + O(\varepsilon^{1-4\alpha}).
\]
By hypothesis $H2''$ in theorem 2.1 the scattering map increases (respectively decreases) the energy by order $\varepsilon$, for $(I, \varphi, s) \in D_F$ such that $(I, \varphi - Is) \in H^*_E$ (respectively for $(I, \varphi) \in D_F$ such that $(I, \varphi - Is) \in H^*_E$). In particular, the surface $S_e(L^E_F)$ intersects all surfaces $L^E_F$ such that $|E' - E| = O(\varepsilon)$.

Moreover, in order to see that they intersect transversally we need to check that condition (144) is satisfied. Note that in this case, by remark 4.4, condition (146) implies (144). Thus, we first compute

$$\{F, [F, L^*]\} = \left( \frac{\partial F}{\partial I} \right)^2 \frac{\partial^2 L^*}{\partial \varphi^2} + O_C(\varepsilon^{1-4\gamma})$$

$$= (1 + O_C(\varepsilon^{1-4\gamma})) \frac{\partial^2 L^*}{\partial \varphi^2} + O_C(\varepsilon^{1-4\gamma})$$

$$= \frac{\partial^2 L^*}{\partial \varphi^2} + O_C(\varepsilon^{1-4\gamma}).$$

Since, by assumption, the function $\frac{\partial L^*}{\partial \varphi}(E, \tilde{\theta})$ is non-constant for $\tilde{\theta}$ in $J_E^*$, there exists an interval $J_E \subset J_E^*$ where

$$\left| \frac{\partial^2 L^*}{\partial \varphi^2} \right| \geq C > 0,$$

and using

$$|\nabla F| = 1 + O_C(\varepsilon^{1-4\gamma}),$$

we have that condition (144) is satisfied and the angle between the surfaces $S_e(L^E_F)$ and $L^E_F$ at the intersection can be bounded from below by $C\varepsilon$, where $C$ is a constant independent of $\varepsilon$ and $E$. □

**Remark 4.6.** By theorem 3.1, two consecutive tori are, at most, at a distance of $O(\varepsilon^{1+\eta})$, for some $\eta > 0$, in terms of the $I$ variable. Moreover, these tori are $O_C(\varepsilon^{1-4\gamma})$-close to the level sets of the action $I$.

Hence, we conclude that the image under the scattering map of a torus $T_i$ in the flat tori region, given by $I = I_i + O(\varepsilon^{1+\eta})$, intersects transversally another torus of this region given by $I = I_{i+1} + O(\varepsilon^{1-4\gamma})$ with $|I_{i+1} - I_i| = O(\varepsilon)$:

$$S_e(T_i) \pitchfork T_{i+1}.$$

### 4.1.2. Big gaps region.

In lemma 4.7 we are going to apply lemma 4.2 in one connected component of the big gaps region $D_{BG}$. By theorem 3.1, the invariant tori are given by the leaves $L^E_F$ of a foliation $F_F$ for a certain function $F$ of the form (96). Moreover, they can be written as a graph of the action $I$ over the angle variables $(\varphi, s)$: $I = \lambda^E_F(\varphi, s)$, with $\lambda^E_F$ as in (97). Recall that in this foliation, the leaves with $E > 0$ are primary KAM tori whereas the leaves with $E < 0$ are secondary.

The dominant terms in $F$ and in the expressions $\lambda^E_F$ of these tori depend on the resonance $-l_0/k_0$ and the distance to the separatrix, which is measured in terms of $E$. Thus, on the one hand tori are bent when they approach the separatrix, that is, when $E \to 0$, and on the other hand tori are flatter when the size $\varepsilon |(k_0, l_0)|^{-1/\gamma}$ of the gap decreases, which is controlled by $k_0$ and therefore by $\gamma$ (see (112) for a definition of $\gamma$).

In the following lemma 4.7 we consider the different cases and we prove that conditions $H2''$, $H3''$ and $H3'''$ ensure the existence of a transversal intersection between the foliation $F_F$ and its image under the scattering map $F_{F,S^{-1}}$. 


Lemma 4.7. Let us consider a connected component of the big gaps region $\mathcal{D}_{BG}$ defined in (84). Recall from formula (96) together with expressions (112) and (114) that, in this component, the function $F$ defining the foliation is of the form

$$F(I, \varphi, s; \varepsilon) = \frac{y^2}{2} (1 + \varepsilon k_0^2 \hat{H}(y; \varepsilon)) + \varepsilon^2 \hat{U}(x; \varepsilon) + \mathcal{O}_C(|k_0|^3 e^{y^2/2 + \varepsilon y^2/2}),$$

where $y$ and $s$ are given in (141) in terms of $(I, \varphi, s)$, and for some $0 \leq \rho < \pi$ and some range of energies $-\varepsilon^2 \leq E \leq \varepsilon^2$, the equation $F(I, \varphi, s; \varepsilon) = E$ defines two smooth surfaces $L^\pm_E$ given as graphs $I = \lambda^\pm_E(\varphi, s; \varepsilon)$, with $\lambda^\pm_E$ given in (97), such that

$$y = \hat{j}^\pm_E(\varphi, s; \varepsilon) = \mathcal{Y}_d(\theta, E) + \mathcal{O}_C(|k_0| e^{y^2}),$$

where

$$\mathcal{Y}_d(\theta, E) = \pm (1 + \varepsilon) \ell(\theta, E) + \varepsilon \mathcal{Y}_d(\ell(\theta, E)),$$

for $\rho \leq \theta = k_0 \varphi + \ell_0 s \leq 2\pi - \rho$ and $\ell(\theta, E) = \sqrt{2(E - \varepsilon^2 \hat{U}(\theta; 0))}$ with $\hat{U}(\theta; \varepsilon)$ defined in (114) and $\mathcal{Y}_d$ satisfying (120).

Assume that hypothesis H2* is fulfilled, more precisely, that the reduced Poincaré function $\mathcal{L}^*$ verifies, for any value of $(I, \varphi, s) \in \mathcal{D}_{BG}$ such that $(I, \varphi - 1s) \in \mathcal{H}_+^*$ (respectively, such that $(I, \varphi - 1s) \in \mathcal{H}_-^*$), that the function

$$\hat{\vartheta} \mapsto \frac{\partial \mathcal{L}^*}{\partial \vartheta}(I, \hat{\vartheta})$$

is non-constant for $\hat{\vartheta} = \varphi - 1s$ positive (respectively negative) and non-constant for $\hat{\vartheta} \in \mathcal{J}^*_+$.

For $|(k_0, l_0)| < C_1 \varepsilon^{-1/\rho}$, where

$$C_1 = C_1(\mathcal{L}^*, \hat{U}) = \left(\frac{2M_\vartheta}{M_{\mathcal{L}^*}}\right)^{1/r},$$

and $M_{\vartheta} = \max_{\vartheta \in \mathcal{J}^*_0/k_0} \left| \frac{\partial \mathcal{L}^*}{\partial \vartheta}(\theta, 0) \right|$ and $M_{\mathcal{L}^*} = \max_{\theta \in \mathcal{J}^*_0/k_0} |\hat{U}(\theta; 0)|$, assume hypothesis H3* on $(k_0, l_0)$ in theorem 2.1, which is that the function

$$\theta \mapsto \frac{k_0 \hat{U}^{k_0/k_0}(\theta; 0) \bar{c}_{\mathcal{L}^*}(\theta; k_0 \vartheta)}{2 \hat{U}(\theta; 0) \partial^2 \mathcal{L}^*/\partial \vartheta^2 - \theta}$$

is non-constant for $\theta / k_0 \in \mathcal{J}^*_0/k_0$.

For $C_1 \varepsilon^{-1/\rho} \leq |(k_0, l_0)| \leq C_2 \varepsilon^{-1/\rho}$ with $C_1$ given in (153) and

$$C_2 = C_2(\mathcal{L}^*, \hat{U}) = \left(\frac{32M_\vartheta}{M_{\mathcal{L}^*}}\right)^{1/r},$$

where $M_{\vartheta}$ and $M_{\mathcal{L}^*}$ are as in the previous case, we assume the following hypothesis, which is condition H3* on $(k_0, l_0)$ in theorem 2.1:

There exists a constant $C$, independent of $E$ and $\varepsilon$, and an interval $\mathcal{J} \subset \mathcal{J}^*_0/k_0$ such that given any $E, \varepsilon$ in this region and $\theta / k_0 \in \mathcal{J}$,

$$\left| \frac{1}{2(E - \varepsilon^2 \hat{U}(\theta; 0))} \left( 2E \partial^2 \mathcal{L}^*/\partial \vartheta^2 \left( \frac{-l_0}{k_0} \frac{\theta}{k_0} \right) + \hat{U}(\theta; 0) \partial^2 \mathcal{L}^*/\partial \vartheta^2 \left( \frac{-l_0}{k_0} \frac{\theta}{k_0} \right) \right) \right| \geq C.$$
More precisely, any surface $S_{c}(L_{E}^{F,-})$ intersects at some point the surface $L_{E}^{F,-}$ for any $E' < E$ (respectively $E' > E$) such that $|E' - E| \leq C|k_{0}|\varepsilon\max(|E|^{1/2},\varepsilon^{1/2})$. Analogously, any surface $S_{c}(L_{E}^{F,+})$ intersects at some point the surface $L_{E}^{F,+}$ for any $E' > E$ (respectively $E' < E$) such that $|E' - E| \leq C|k_{0}|\varepsilon\max(|E|^{1/2},\varepsilon^{1/2})$. (In some cases, it is also possible that a certain surface $S_{c}(L_{E}^{F,-})$ intersects the surface $L_{E}^{F,+}$ with $E'$ such that $|E' - E| \leq C|k_{0}|\varepsilon\max(|E|^{1/2},\varepsilon^{1/2})$.)

The angle between the surfaces $S_{c}(L_{E}^{F,-})$ and $L_{E}^{F,+}$ at the intersection is bounded from below by $C\varepsilon$, where $C$ is a constant independent of $\varepsilon$ and $E$.

**Remark 4.8.** Lemma 10.16 in [DLS06a] gives a computable sufficient condition that guarantees that hypothesis $H3_{\omega}$ on $(k_{0}, l_{0})$ is verified independently of $\varepsilon$ and $E$. Indeed, let

\[
\begin{align*}
a(\theta) &= \frac{a^{2}}{\partial \theta^{2}} \left( \frac{l_{0}}{k_{0}} \theta \right), \\
b(\theta) &= -\frac{1}{2} \left( k_{0}\tilde{U}(k_{0}, \theta; 0) \frac{\partial L_{c}}{\partial \theta} \left( \frac{l_{0}}{k_{0}}, \theta \right) + 2\tilde{U}(\theta; 0) \frac{\partial^{2} L_{c}}{\partial \theta^{2}} \left( \frac{l_{0}}{k_{0}}, \theta \right) \right), \\
c(\theta) &= \pm \frac{\sqrt{2}}{2} \frac{\partial L_{c}}{\partial \theta} \left( \frac{l_{0}}{k_{0}}, \theta \right) \frac{\partial^{2} L_{c}}{\partial \theta^{2}} \left( \frac{l_{0}}{k_{0}}, \theta \right),
\end{align*}
\]

if there exist $\theta_{1}, \theta_{2}$ and $\theta_{3}$ in some interval $\mathcal{J}$ verifying

\[
\begin{align*}
\left| \tilde{a}(\theta_{1}) \right| &\neq 0, \\
\left| \tilde{b}(\theta_{1}) \right| &\neq 0, \\
\left| \tilde{c}(\theta_{1}) \right| &\neq 0,
\end{align*}
\]

where

\[
\begin{align*}
\tilde{a}(\theta) &= a(\theta)^{2}, \\
\tilde{b}(\theta) &= 2a(\theta)b(\theta) - c(\theta)^{2}, \\
\tilde{c}(\theta) &= b(\theta)^{2} - c(\theta)^{2}\tilde{U}(\theta; 0),
\end{align*}
\]

then there exist a constant $C$ and three intervals $\theta_{i} \in \mathcal{J}_{i} \subset \mathcal{J}$, $i = 1, 2, 3$ such that for any $\theta \in \mathcal{J}_{i}$

\[
\begin{align*}
\frac{a(\theta)E + b(\theta)e^{\varepsilon^{\gamma}} + c(\theta)e^{k_{0}\sqrt{E - \varepsilon^{\gamma}\tilde{U}(\theta; 0)}}}{E - \varepsilon^{\gamma}\tilde{U}(\theta; 0)} &\geq C,
\end{align*}
\]

which is hypothesis $H3_{\omega}$ on $(k_{0}, l_{0})$.

**Proof.** We will apply lemma 4.2 to the foliation $\mathcal{F}_{E}$ given by the function $F$ in (149).

We first apply the scattering map to the implicit surface $L_{E}^{F} = \{(I, \varphi, s) \in \mathcal{D}_{BG}, F(I, \varphi, s; \varepsilon) = E \}$, and recall that $S_{c}(L_{E}^{F})$ intersects a leaf $L_{E}^{F}$ at a point $(I, \varphi, s) \in L_{E}^{F}$ if $F \circ S_{c}(I, \varphi, s; \varepsilon) = E'$, where, using expression (143) with $S_{0} = -L_{c}$, $F \circ S_{c}$ on $L_{E}^{F}$ is given by

\[
(F \circ S_{c})(I, \varphi, s; \varepsilon) = E - \varepsilon[F, L_{c}'] + \frac{\varepsilon^{2}}{2}([F, L_{c}'], L_{c}') + [F, S_{1}] + O(\varepsilon^{3}).
\]

Note that the terms in expression (159) involve the derivatives of $F$ on $L_{E}^{F}$. Using the expression for $F$ in (149) and the expression of $y$ and $x$ in terms of $(I, \varphi, s)$ in (144), we have that

\[
\begin{align*}
\frac{\partial F}{\partial I}(I, \varphi, s; \varepsilon) &= [y(1 + \varepsilon k_{0}^{2}\tilde{h}(y; \varepsilon)) + \frac{y^{2}}{2} + \varepsilon k_{0}^{2}\tilde{h}(y; \varepsilon) \\
&+ O_{c}(k_{0}^{2}\varepsilon^{2/4 + 1/4})]k_{0} + O_{c}(|k_{0}|\varepsilon^{1-\delta_{0}}),
\end{align*}
\]
and using the expression of the leaf $L_{E}^{F,\pm}$ as a graph of $y$ over the angle variables $(\varphi, s)$ given in (150), on $L_{E}^{F,\pm}$ we have

$$
\frac{\partial F}{\partial I}(I, \varphi, s; \varepsilon) = \pm k_{0}\ell(\theta, E) + O(|k_{0}|^{3}\varepsilon|\ell| + k_{0}^{2}\varepsilon^{1+\eta} + |\ell||k_{0}|\varepsilon^{1-4\alpha})
$$  \hspace{1cm} (160)

and analogously on $L_{E}^{F,\pm}$ we have,

$$
\frac{\partial F}{\partial \varphi}(I, \varphi, s; \varepsilon) = \varepsilon^{\gamma}k_{0}U^{k_{0}-l_{0}}(\theta, \varepsilon) + O_{C}(|k_{0}|^{3}\gamma^{2+1+\eta}).
$$  \hspace{1cm} (161)

Hence, on $L_{E}^{F,\pm}$ we can disregard $\frac{\partial F}{\partial \varphi}$ in front of $\frac{\partial F}{\partial I}$ and we have,

$$
\{F, \mathcal{L}^{*}\} = -\frac{\partial F}{\partial I} \frac{\partial \mathcal{L}^{*}}{\partial \varphi} + \frac{\partial F}{\partial \varphi} \frac{\partial \mathcal{L}^{*}}{\partial I} = \mp k_{0}\ell(\theta, E) \frac{\partial \mathcal{L}^{*}}{\partial \theta} \left(-\frac{l_{0}}{k_{0}} + \frac{1}{k_{0}}\ell(\theta, E), \varphi - \left(-\frac{l_{0}}{k_{0}} + \frac{1}{k_{0}}\ell(\theta, E)\right)s\right) + O(\xi),
$$  \hspace{1cm} (162)

where

$$
\xi = \xi(\varepsilon, |k_{0}|, |\ell|) = |k_{0}|^{3}\varepsilon|\ell| + k_{0}^{2}\varepsilon^{1+\eta} + |\ell||k_{0}|\varepsilon^{1-4\alpha} + |k_{0}|\varepsilon^{\gamma}.
$$  \hspace{1cm} (163)

Regarding the term of order $\varepsilon^{2}$ in expression (159), we see that among all the terms in \(\frac{1}{2}\varepsilon^{2}(\{(F, \mathcal{L}^{*}), \mathcal{L}^{*}\} + \{F, \mathcal{S}_{1}\})\) there is a dominant one. To that end we note first that all the terms that appear in the derivatives up to second order for $F$ with respect to $(I, \varphi, s)$ on $L_{E}^{F,\pm}$ are $O(|k_{0}|\ell|, k_{0}^{2}\varepsilon^{\alpha})$, except

$$
\frac{\partial^{2}F}{\partial I^{2}} = k_{0}^{2} + O(k_{0}^{3}\varepsilon^{1-4\alpha}).
$$  \hspace{1cm} (164)

Hence, in the expression $\{(F, \mathcal{L}^{*}), \mathcal{L}^{*}\} + \{F, \mathcal{S}_{1}\}$ on $L_{E}^{F,\pm}$, all the terms are of order $k_{0}^{2}\varepsilon^{\alpha}$, for some $\alpha > 0$, except

$$
\frac{\partial^{2}F}{\partial I^{2}} \left(\frac{\partial \mathcal{L}^{*}}{\partial \theta}(I, \theta)\right)^{2}.
$$

Therefore, using this feature and (162), expression (159) for $F \circ S_{\xi}$ on $L_{E}^{F,\pm}$, is given by

$$
F \circ S_{\xi}(I, \varphi, s; \varepsilon) = E \pm \varepsilon k_{0}\ell(\theta, E) \frac{\partial \mathcal{L}^{*}}{\partial \theta} \left(-\frac{l_{0}}{k_{0}} + \frac{\theta}{k_{0}}\right) + \frac{\varepsilon^{2}}{2}k_{0}^{2} \left(\frac{\partial \mathcal{L}^{*}}{\partial \theta}\right)^{2} + O(k_{0}^{2}\varepsilon^{2\varphi}, |k_{0}|\varepsilon^{\gamma+1}, |\ell|^{2})
$$  \hspace{1cm} (165)

\begin{align*}
&= E + \varepsilon M_{\pm}(\theta; \varepsilon) + O(k_{0}^{2}\varepsilon^{2\varphi}, |\xi|),
\end{align*}

where $\xi$ is given in (163) and

$$
M_{\pm}(\theta; \varepsilon) = k_{0} \frac{\partial \mathcal{L}^{*}}{\partial \theta} \left(-\frac{l_{0}}{k_{0}} + \frac{\theta}{k_{0}}\right) \left(\pm \sqrt{2(E - \varepsilon^{\gamma} U(\theta; 0)) + \varepsilon k_{0}^{2} \frac{1}{2} \frac{\partial \mathcal{L}^{*}}{\partial \theta} \left(-\frac{l_{0}}{k_{0}} + \frac{\theta}{k_{0}}\right)}\right). \hspace{1cm} (166)
$$

Therefore, the size of the heteroclinic jumps provided by the scattering map is determined by the size of the term $M_{\pm}$ in (166).

In order to check the transversality of the heteroclinic intersections we use condition (144), which involves, in any case, the computation of the Poisson bracket $\{F, \{F, \mathcal{L}^{*}\}\}$ and the gradient of $F$ (see formula (145)).

From expressions (160) and (161) it follows that on $L_{E}^{F,\pm}$,

$$
|\nabla F(I, \varphi, s; \varepsilon)| = |k_{0}|\ell(\theta, E)| + O(\xi),
$$  \hspace{1cm} (167)

where $\xi$ is given in (163).
On the other hand, the computation of \( \{ F, \{ F, \mathcal{L}^* \} \} \) involves several terms. However, using the expression for \( \{ F, \mathcal{L}^* \} \) obtained in (162) and the expression and estimates for the derivatives up to second order for \( F \) with respect to \( (I, \varphi, s) \) given in (160)–(161)–(164), one can see that the dominant terms in \( \{ F, \{ F, \mathcal{L}^* \} \} \) involve
\[
\left( \frac{\partial F}{\partial I} \right)^2 \frac{\partial^2 \mathcal{L}^*}{\partial \varphi^2} = \frac{\partial F}{\partial \varphi} \frac{\partial^2 F}{\partial I^2} \frac{\partial \mathcal{L}^*}{\partial \varphi}
\]
and therefore we have that on \( L_{E, \pm}^E \),
\[
\{ F, \{ F, \mathcal{L}^* \} \} = k_0^2 \ell(\theta, E) \frac{\partial^2 \mathcal{L}^*}{\partial \varphi^2}(\bar{\theta}, E) - k_0 \varepsilon \gamma \bar{U}_{k_0, i_0}(\theta, \varepsilon) k_0 k_0^2 \frac{\partial \mathcal{L}^*}{\partial \varphi}(\bar{\theta}, E) + O(|k_0||\varepsilon|). \tag{168}
\]
where \( \varepsilon \) is given in (163).

In expression (166) there appear two quantities that can be comparable or not depending on \( k_0 \) and \( E \). Note first that \( |\varepsilon(\cdot, E)| = \max(E^{1/2}, \varepsilon^{\nu/2}) \), with \( 1 \leq \nu < 2 + v \), for some \( v > 0 \). In consequence, when the size of the energy is big \( (|E| > \varepsilon^\nu) \), we have \( \varepsilon(\theta; E) = O(\varepsilon^{\nu/2}) \) and therefore the term involving \( \varepsilon(\theta, E) \) in expression (166) dominates. On the other hand, if the energy is small, that is \( |E| \) is smaller than or comparable to \( \varepsilon^{\nu/2} \), then \( \varepsilon(\theta; E) = O(\varepsilon^{\nu/2}) \), which by expression (112) is also \( O(|k_0|\varepsilon^{1+\nu/2}) \), for some \( v > 0 \). In this case, the dominant term in expression (166) will depend on the size of \( k_0 \).

Hence, we choose \( \mu \) such that \( 0 < \mu < \nu \) and we distinguish two cases: the case when tori are close to the resonance, which corresponds to small values of the energy \( (\varepsilon^\nu \leq E \leq \varepsilon^\mu) \) and the case when they are reasonably far from a resonance, which corresponds to greater values of the energy \( (\varepsilon^\mu \leq E \leq \varepsilon^\nu) \).

**Far from the resonance:** \( \varepsilon^\mu \leq E \leq \varepsilon^\nu \). The case far from a resonance is analogous to the flat tori region, studied in the previous section, because in this case
\[
\varepsilon(\theta, E) = \sqrt{2(E - \varepsilon^\gamma \bar{U}(\theta; \varepsilon))} = \sqrt{2E} \sqrt{1 - \frac{\varepsilon^\gamma}{E} \bar{U}(\theta; \varepsilon)}
\]
Consequently, since \( \sqrt{2E} \geq \sqrt{2\varepsilon^{\nu/2}} \) and \( \varepsilon^{\nu/2} > \varepsilon^{\nu/2} \geq |k_0|\varepsilon^{1/2} \), expression (165) can be written as
\[
F \circ S_{e}(I, \varphi, s; \varepsilon) = E \pm \varepsilon k_0 \sqrt{2E} \frac{\partial \mathcal{L}^*}{\partial \bar{\theta}} \left( \frac{l_0}{k_0}, \frac{\theta}{k_0} \right) + O(k_0|E|^{1/2}\varepsilon^{1+\nu/2}, \varepsilon|E|).
\]

Therefore, by the hypothesis \( \textbf{H2}'' \) on \( \frac{\partial \mathcal{L}^*}{\partial \bar{\theta}}(I, \bar{\theta}) \), we have that the image of \( L_{E, \pm}^E \) under the scattering map, for \( E \) large, intersects all surfaces \( L_{E, \pm}^E \) such that \( |E - E| = O(\varepsilon|k_0||E|^{1/2}) \).

In order to prove the transversality of intersections, we need to check condition (146). Using that the term involving \( \varepsilon(\theta, E) \) is the dominant one in expression (168) for \( \{ F, \{ F, \mathcal{L}^* \} \} \) and expression (167) for \( \nabla F \), condition (146) for the transversality of the intersections is
\[
\left| \frac{\partial^2 \mathcal{L}^*}{\partial \bar{\theta}^2} \right| \geq C > 0,
\]
which is clearly satisfied by the hypothesis on \( \textbf{H2}'' \) on \( \frac{\partial \mathcal{L}^*}{\partial \bar{\theta}}(I, \bar{\theta}) \).

**Close to the resonance:** \( -\varepsilon^{\nu} \leq E \leq \varepsilon^\mu \). The case close to a resonance is more technical because the size of the energy is now comparable to the term \( \varepsilon^{\nu} \bar{U} \) and therefore \( \varepsilon(\cdot, E) = O(\varepsilon^{\nu/2}) \). Hence, in expression (166) there appear two quantities that can be comparable or not
depending on \( k_0 \). On the one hand, there is \( \sqrt{2(E - \varepsilon^y \tilde{U}(\theta; 0))} \), which is related to the size of the gap and on the other hand there is \( \varepsilon k_0 \frac{1}{2} \frac{\partial C^*}{\partial \theta} (-\frac{l_0}{k_0}, \frac{\theta}{k_0}) \), which is related to the size of the heteroclinic jumps provided by the scattering map \( S_e \). Hence we distinguish three situations depending on \( k_0 \):

i. The case when the first term \( \sqrt{2(E - \varepsilon^y \tilde{U}(\theta; 0))} \) is smaller than the second term

\[ \varepsilon k_0 \frac{1}{2} \frac{\partial C^*}{\partial \theta} (-\frac{l_0}{k_0}, \frac{\theta}{k_0}), \]

say smaller than \( 1/2 \) the maximum of the second one:

\[ \sqrt{2M_\varepsilon^y} < \varepsilon |k_0| M_{\varepsilon^y}, \]

(169)

where \( M_{\varepsilon^y} = \max \left\{ \left. \frac{|\alpha^\varepsilon|}{\theta/k_0}, \theta/k_0 \right| : \theta/k_0 \in J^*_{l_0/k_0} \right\} \) and \( M_{\varepsilon^y} = \max \{|\tilde{U}(\theta; 0)| : \theta/k_0 \in J^*_{l_0/k_0}\} \).

As \( \varepsilon^y = \varepsilon |k_0|^2 |(k_0, l_0)|^{-\tau} \) by (112), (169) is equivalent to

\[ M_{\varepsilon^y} |k_0|^2 |(k_0, l_0)|^{-\tau} < \frac{1}{\varepsilon} k_0^2 M_{\varepsilon^y}, \]

or \( |(k_0, l_0)|^{-\tau} < \frac{M_{\varepsilon^y}}{2M_{\varepsilon^y}} \), that is \( |(k_0, l_0)|^{-\tau} > \frac{2M_{\varepsilon^y}}{M_{\varepsilon^y}} |k_0|^2 \), or finally

\[ |(k_0, l_0)| > C_2 \varepsilon^{-1/\tau}, \]

where \( C_2 \) is the constant given in (155). In this case,

\[ \pm \sqrt{2(E - \varepsilon^y \tilde{U}(\theta; 0))} + \varepsilon k_0 \frac{1}{2} \frac{\partial C^*}{\partial \theta} \left(-\frac{l_0}{k_0}, \frac{\theta}{k_0}\right) \geq \frac{1}{4} \varepsilon |k_0| \left| \frac{\partial C^*}{\partial \theta} \left(-\frac{l_0}{k_0}, \frac{\theta}{k_0}\right) \right|. \]

and from expression (165), we have that tori are essentially flat and this is equivalent to

the flat tori case. Hence, condition \( H_2^e \) assures that the foliations intersect transversally.

ii. The case when the second term \( \varepsilon k_0 \frac{1}{2} \frac{\partial C^*}{\partial \theta} (-\frac{l_0}{k_0}, \frac{\theta}{k_0}) \) is smaller than the first one

\( \sqrt{2(E - \varepsilon^y \tilde{U}(\theta; 0))} \), say smaller than \( 1/2 \) the maximum of the first one:

\[ \frac{1}{2} \varepsilon |k_0| M_{\varepsilon^y} < \frac{1}{2} \sqrt{2M_{\varepsilon^y} \varepsilon^y}, \]

(170)

where \( M_{\varepsilon^y} \) and \( M_{\varepsilon^y} \) are as in the previous case. As before, using that \( \varepsilon^y = \varepsilon |k_0|^2 |(k_0, l_0)|^{-\tau} \)

by (112), (170) is equivalent to

\[ \frac{1}{2} \varepsilon^y k_0^2 M_{\varepsilon^y} < \frac{1}{2} M_{\varepsilon^y} \varepsilon |k_0|^2 |(k_0, l_0)|^{-\tau}, \]

or \( |(k_0, l_0)|^{-\tau} > \frac{2M_{\varepsilon^y}}{M_{\varepsilon^y}} \), that is \( |(k_0, l_0)|^{-\tau} < \frac{2M_{\varepsilon^y}}{M_{\varepsilon^y}} |k_0|^2 \), or finally

\[ |(k_0, l_0)| < C_1 \varepsilon^{-1/\tau}, \]

where \( C_1 \) is the constant given in (155). In this case, \( |(k_0, l_0)| < C_1 \varepsilon^{-1/\tau} \).

This is the case when the size of the gaps in the foliation of primary tori is bigger than the size of the heteroclinic jumps provided by the scattering map. Hence, if we consider the surface \( L_{E'}^- \), by hypothesis \( H_2^e \) we have that

\[ -\varepsilon k_0 \sqrt{2(E - \varepsilon^y \tilde{U}(\theta; 0))} \frac{\partial C^*}{\partial \theta} \left(-\frac{l_0}{k_0}, \frac{\theta}{k_0}\right) \]

is a negative function, and therefore by equation (165) \( S(L_{E'}^-) \) intersects surfaces \( L_{E'}^- \)

with \( E' < E \) (respectively \( E' > E \)) such that \( |E' - E| \leq |k_0|^4 \varepsilon^{4/2} \). An analogous result is obtained for \( L_{E'}^+ \) with \( E' > E \) (respectively \( E' < E \)).
iii. Finally, it remains to consider the case when

\[ C_1 \varepsilon^{-1/r} \leq |(k_0, l_0)| \leq C_2 \varepsilon^{-1/r}, \]

where \( C_1 \) and \( C_2 \) are the constants depending on \( L^* \) and \( \tilde{U} \) given explicitly in (153) and (155), respectively. In this case we have that the terms \( \sqrt{2} (E - \varepsilon \gamma \tilde{U} (\theta; 0)) + \frac{1}{2} \varepsilon k_0 \frac{\partial L^*}{\partial \theta} \left( \frac{l_0}{k_0}, \frac{\theta}{k_0} \right) \) in expression (166) are comparable. This case is the hardest to study because the size of the gap is of the same order as the heteroclinic jumps. This causes different geometries for \( S_{\varepsilon} (L_{E, e}^F) \) that could happen depending on the numerical values of the leading coefficients.

We focus on the case of \( S_{\varepsilon} (L_{E, e}^F) \) and the function (152) positive. The case for \( S_{\varepsilon} (L_{E, e}^F) \) and the function (152) negative is analogous. Hence, by hypothesis \( \text{H2'''e} \), the main term \( M_e \) in \( F \) given in (166) can have different signs depending on the size of \( \epsilon (\theta; \varepsilon) \). According to that, we distinguish the following cases:

(a) The first case is when

\[ -\sqrt{2} (E - \varepsilon \gamma \tilde{U} (\theta; 0)) + \varepsilon k_0 \frac{1}{2} \frac{\partial L^*}{\partial \theta} \left( \frac{l_0}{k_0}, \frac{\theta}{k_0} \right) \bigg|_{C_1} \leq \varepsilon^{1+\gamma}. \]

This case corresponds to points in \( L_{E, e}^F \) that are \( O(C_1 (\varepsilon^{2+\gamma}))-close to homoclinic jumps \( S_{\varepsilon} (L_{E, e}^F) \cap L_{E, e}^F \). They are not good for diffusion.

(b) The second case is when

\[ -\sqrt{2} (E - \varepsilon \gamma \tilde{U} (\theta; 0)) + \varepsilon k_0 \frac{1}{2} \frac{\partial L^*}{\partial \theta} \left( \frac{l_0}{k_0}, \frac{\theta}{k_0} \right) \bigg|_{C_1} > \varepsilon^{1+\gamma}. \]

This case corresponds to points in heteroclinic jumps \( S_{\varepsilon} (L_{E, e}^F) \cap L_{E, e}^F \) and we can distinguish two situations that can take place.

On the one hand, if

\[ -\sqrt{2} (E - \varepsilon \gamma \tilde{U} (\theta; 0)) > \varepsilon k_0 \frac{1}{2} \frac{\partial L^*}{\partial \theta}, \]

which is the case when the heteroclinic jumps are smaller than the gap, \( S_{\varepsilon} (L_{E, e}^F) \) intersects surfaces \( L_{E, e}^F \) with \( E' < E \) and \( |E' - E| \leq |k_0| \varepsilon^{1+\gamma/2} \). Thus, for small values of energy \( E > 0 \), the scattering map will connect a surface with energy \( E > 0 \) with a surface \( E' < 0 \), which corresponds to a heteroclinic connection of a primary tori with a secondary one.

On the other hand, when

\[ -\sqrt{2} (E - \varepsilon \gamma \tilde{U} (\theta; 0)) < \varepsilon k_0 \frac{1}{2} \frac{\partial L^*}{\partial \theta}, \]

which is the case when the heteroclinic jumps are bigger than the gaps created between the primary tori, we obtain that \( S_{\varepsilon} (L_{E, e}^F) \) will intersect the surfaces \( L_{E, e}^F \) with \( E' > E \) (respectively \( E' < E \)) and \( |E' - E| \leq |k_0| \varepsilon^{1+\gamma/2} \). In this case the scattering map will connect two tori with positive energy, that is, two primary tori, and cross the gap with just one application of the scattering map.

Once we have a heteroclinic connection that crosses the separatrix loop, we can consider \( S_{\varepsilon} (L_{E, e}^F) \), which corresponds to the upper branch of the level set \( F (I, \varphi, s; \varepsilon) = E, E > 0 \).

In this case, by hypothesis \( \text{H2''e} \), in expression (165) the main term \( M_e \) in \( F \) given in (166) is always positive, so \( S_{\varepsilon} (L_{E, e}^F) \) will intersect surfaces \( L_{E, e}^F \) with \( E' > E \) (respectively \( E' < E \)) and \( |E' - E| \leq |k_0| \varepsilon^{1+\gamma/2} \).

Now, we want to check that the intersections for the cases (ii) and (iii) take place transversally by means of condition (144). For the case described in item (ii) in this proof,
condition (146) implies condition (144). So, using expression (168) for \( \{F, \{ F, \mathcal{L}^* \} \} \) and expression (167) for \( \nabla F \) on \( L_i^F \), we have that the condition (144) is satisfied provided that

\[
\left| \frac{\pm 1}{2(E - \varepsilon \gamma \bar{U}(\theta; 0))} \left( 2E \frac{\partial^2 \mathcal{L}^*}{\partial \theta^2} \left( -\frac{l_0}{k_0}, \frac{\theta}{k_0} \right) - \varepsilon \gamma \right) \left[ k_0 \bar{U}^k \bar{b} (\theta; 0) \frac{\partial \mathcal{L}^*}{\partial \theta} \left( -\frac{l_0}{k_0}, \frac{\theta}{k_0} \right) + 2 \bar{U}(\theta; 0) \frac{\partial^2 \mathcal{L}^*}{\partial \theta^2} \left( -\frac{l_0}{k_0}, \frac{\theta}{k_0} \right) \right] \right| \geq C.
\]

By lemma 10.10 in [DLS06a], hypothesis H3' on \((k_0, l_0)\) implies the previous condition and therefore the angle between the surfaces \( S_i(L_i^F) \) and \( L_i^F \), at the intersection can be bounded from below by \( C \varepsilon \), for some suitable constant independent of \( \varepsilon \).

For the particular case \([(k_0, l_0)] \sim \varepsilon^{-1/\gamma} \) described in item (iii), we need to check condition (145). Using expression (166) for the dominant term \( M_{k, l} \) in \( F \), it is not difficult to see that the dominant term in the numerator of (144) involves the terms

\[
\left( \frac{\partial F}{\partial I} \right)^2 \frac{\partial^2 \mathcal{L}^*}{\partial q^2} - \frac{\partial F}{\partial I} \frac{\partial^2 \mathcal{L}^*}{\partial q^2} \frac{\partial^2 \mathcal{L}^*}{\partial \phi^2} + \frac{\partial F}{\partial I} \frac{\partial^2 \mathcal{L}^*}{\partial q^2} \frac{\partial^2 \mathcal{L}^*}{\partial \phi^2}
\]

\[
= (k_0 I + l_0)^2 \frac{\partial^2 \mathcal{L}^*}{\partial \theta^2} (\theta, E) - \varepsilon \gamma \bar{U}^k \bar{b} (\theta, E) k_0 \frac{\partial \mathcal{L}^*}{\partial \theta} (\theta, E)
\]

\[
+ \varepsilon (k_0 I + l_0)^2 \frac{\partial \mathcal{L}^*}{\partial \phi} (\theta, E) \frac{\partial \mathcal{L}^*}{\partial \phi} (\theta, E)
\]

Using the expression for \( \nabla F \) in (167), we have that the condition (144) is satisfied provided that

\[
\left| \frac{\pm 1}{2(E - \varepsilon \gamma \bar{U}(\theta; 0))} \left( 2E \frac{\partial^2 \mathcal{L}^*}{\partial \theta^2} \left( -\frac{l_0}{k_0}, \frac{\theta}{k_0} \right) - \varepsilon \gamma \right) \left[ k_0 \bar{U}^k \bar{b} (\theta; 0) \frac{\partial \mathcal{L}^*}{\partial \theta} \left( -\frac{l_0}{k_0}, \frac{\theta}{k_0} \right) + 2 \bar{U}(\theta; 0) \frac{\partial^2 \mathcal{L}^*}{\partial \theta^2} \left( -\frac{l_0}{k_0}, \frac{\theta}{k_0} \right) \right] \right| \geq C,
\]

for some constant \( C \). By hypothesis H3'' on \((k_0, l_0)\) in theorem 2.1 we know that the previous condition is satisfied for \( \theta/k_0 \in \mathcal{J} \subset \mathcal{J}_{k_0/l_0} \). Consequently, the angle of intersection can be bounded again from below by \( C \varepsilon \), for some suitable constant independent of \( \varepsilon \).

**Remark 4.9.** By theorem 3.1 we know that the tori in a connected component of the big gaps region are given by the expression \( I = \lambda_E^\pm (\varphi, s; \varepsilon) \), for \( E = E_i \), and \( -\varepsilon \gamma \ll E_i \ll L^2 \), with \( \lambda_E \) given in (97). Moreover, they satisfy

\[
|E_i - E_{i+1}| \leq |k_0| \varepsilon^{\gamma/2 + 0 + \gamma} \leq |k_0| \max(|E_i|^{1/2}, \varepsilon^{\gamma/2})
\]

and they are \( O(\varepsilon^{\gamma+\varepsilon}) \)-closely spaced, in terms of the \( I \) variable.

Hence, we conclude that the image under the scattering map of a torus \( T_i \), \( S_i(T_i) \) in the big gaps region, given by \( I = \lambda_E^\pm (\varphi, s; \varepsilon) \), intersects transversally another torus \( T_{i+1} \) of this region given by \( \tilde{I} = \lambda_{E_{i+1}}^\pm (I, \varphi, s; \varepsilon) \), with \( |E_{i+1} - E_i| = O(\varepsilon^{\gamma/2 + 0 + \gamma}) \) (equivalently \( |I - \tilde{I}| \ll \varepsilon^{\gamma+\varepsilon} \)):

\[
S_i(T_i) \cap T_{i+1}.
\]
4.2. Proof of proposition 4.1

The proof is just a combination of the results obtained in section 4.1. We start with a torus $T_0$, which is $O(\varepsilon^{1+\eta})$-close to the submanifold $I = I_\omega$. Assume that this torus belongs to the flat tori region with averaged energy $E_0$. The case when $T_0$ belongs to a big gaps region is analogous. Then, we apply lemma 4.5 and remark 4.6 and we get that $S_\varepsilon(T_0)$ intersects transversally all primary tori with averaged energy in the mentioned interval $(E_0 - C\varepsilon, E_0 + C\varepsilon)$. We pick a primary KAM torus $T_1$ provided by theorem 3.1 with energy $E_1$ in the interval and we repeat the argument until we reach a big gaps region. Assuming that we have applied it $K$ times, we have that the torus $T_0$ has heteroclinic connections with all the tori whose energy lies in the interval $(E_0 - KC\varepsilon, E_0 + KC\varepsilon)$, or equivalently, in the interval $(I_\omega - K^*C\varepsilon, I_\omega + K^*C\varepsilon)$ in terms of action variables.

When the domain $(I_\omega - K^*C\varepsilon, I_\omega + K^*C\varepsilon) \times \mathbb{T}^2$ for which the torus $T_0$ has a heteroclinic connection overlaps with a big gaps region $[\frac{-l_0}{k_0}, \frac{-l_0}{k_0} + Lk_0] \times \mathbb{T}^2$ we use lemma 4.7 and remark 4.9 to show that we can cross the gap created by the resonance $\frac{-l_0}{k_0}$ just connecting either a primary KAM torus with a secondary one and again with a primary one or two primary KAM tori. Hence, we can construct a piece of chain that starts in $T_0$ and reaches all the way to $T_r$, where $T_r$ is a primary KAM torus whose equation is $I = \frac{-l_0}{k_0} + Lk_0 + O(\varepsilon)$ and is contained again in the flat tori region.

Therefore, we can keep constructing a transition chain just repeating the procedure stated before for the primary KAM torus $T_r$ until we reach $T_{N(\varepsilon)}$. □

5. Example

Consider the Hamiltonian

$$H_\varepsilon(p, q, I, \varphi, t) = \pm \left(\frac{p^2}{2} + \cos q - 1\right) + \frac{I^2}{2} + \varepsilon \cos q \ g(\varphi, t),$$

(171)

which is a generalization of the famous example introduced by Arnol’d in [Arn64]. This is the same Hamiltonian in the example discussed in [DLS06a], except that the function $g$ is chosen as a periodic function with an infinite number of harmonics in the angles $(\varphi, t)$,

$$g(\varphi, t) = \sum_{(k,l) \in \mathbb{N}^2} a_{k,l} \cos(k\varphi + lt),$$

(172)

where, for simplicity, we have chosen $g$ to be an even function and with an explicit formula for its Fourier coefficients, say $a_{k,l} = \rho^k r^l$ and $0 < \rho, r < 1$ real numbers to be chosen small enough. Note that

$$g(\varphi, t) = \frac{1 + \rho r \cos(\varphi + t) - \rho \cos \varphi - r \cos t}{(1 - 2\rho \cos \varphi + \rho^2)(1 - 2r \cos t + r^2)}.$$

The Hamiltonian of one degree of freedom $P_\varphi(p, q) = \pm \left(p^2/2 + \cos q - 1\right)$ is the standard pendulum when we choose the + sign, and its separatrix for positive $p$ is given by $q_0(t) = 4 \arctan \frac{\varepsilon}{p_0(t)}$, $p_0(t) = 2/cosh t$.

An important feature of the Hamiltonian (171) is that the three-dimensional NHIM

$$\tilde{\Lambda} = \{(0, 0, I, \varphi, s) : (I, \varphi, s) \in \mathbb{R} \times \mathbb{T}^2\}$$

is preserved without any deformation for any $\varepsilon$: $p = q = 0 \Rightarrow \hat{p} = \hat{q} = 0$. However, in contrast with the example in [Arn64], the perturbation does not vanish on $\tilde{\Lambda}$. Indeed, the
Hamiltonian (171) restricted to $\tilde{\Lambda}$ takes the form $I^2/2 + \varepsilon g(\varphi, t)$. Hence, the two-dimensional whiskered tori

$$T^0 = \{(0, 0, I, \varphi, s) : (\varphi, s) \in \mathbb{T}^2\}$$

are not preserved for $\varepsilon \neq 0$, and resonances (49) take place at $I = -l/k$ for each $(k, l) \in \mathbb{N}^2$, $\gcd(k, l) = 1$. Therefore, we have a dense set of gaps of size $O(\varepsilon^{1/2} \sqrt{ak,l})$ centred at $I = -l/k$ and among them the ones such that $\sqrt{ak,l} \gtrsim \varepsilon^{1/2}$ give rise to resonances with big gaps, so the example (171) presents the large gap problem for $I < 0$.

Hence, for any finite range of $I$, $[I_-, I_+] \subset \mathbb{R}$ we will prove the existence of diffusing orbits.

The Melnikov potential (9) of the Hamiltonian (171) is given by

$$\mathcal{L}(I, \varphi, s) = \sum_{(k,l) \in \mathbb{N}^2} A_{k,l}(I) \cos(k\varphi + ls),$$

with

$$A_{k,l}(I) = 2\pi \frac{(kI + l)}{\sinh \frac{\pi}{2}(kI + l)} a_{k,l}. \quad (173)$$

Next, we will see that for $0 < \rho, r \ll 1$ we can find open sets of $(I, \varphi, s) \in [I_-, I_+] \times \mathbb{T}^2$, such that the function $\tau \mapsto \mathcal{L}(I, \varphi - I\tau, s - \tau)$ has non-degenerate critical points at $\tau = \tau^*(I, \varphi, s)$ which verify the hypothesis $H_2'$.

Recall that hypothesis $H_2'$ deals with the existence of transverse intersections of the stable and unstable manifolds of $\tilde{\Lambda}$. Hence, the non-degenerate critical points of the function $\tau \mapsto \mathcal{L}(I, \varphi - I\tau, s - \tau)$ give rise to transverse intersections.

In order to check hypothesis $H_2'$, we will use the results in the example given in section 13 of [DLS06a] by means of the following argument. Assuming that $\rho, r$ are small enough, the function $g(\varphi, s)$ is well approximated by its truncated first order trigonometric polynomial $g^{[\leq 1]}(\varphi, s) = 1 + \rho \cos \varphi + r \cos s$. More precisely,

$$g(\varphi, s) = 1 + \rho \cos \varphi + r \cos s + O(\rho, r) \quad := g^{[\leq 1]}(\varphi, s) + g^{[> 1]}(\varphi, s).$$

Hence, as long as $0 < \rho, r \ll 1$, if hypothesis $H_2'$ is verified for the trigonometric polynomial $g^{[\leq 1]}(\varphi, s)$, it will also be verified for the perturbation $g(\varphi, s)$.

Note that the Fourier coefficients $A_{k,l}(I)$ are nothing else but the Fourier coefficients $a_{k,l}$ multiplied by a certain function depending on $I$ that decreases exponentially as $|I|$ goes to infinity. Hence, arguing as we did for the perturbation $g$, we approximate the function $\mathcal{L}(I, \varphi, s)$ by its first order trigonometric polynomial $\mathcal{L}^{[\leq 1]}(I, \varphi, s) = A_{0,0} + A_{1,0}(I) \cos \varphi + A_{0,1} \cos s$, that is

$$\mathcal{L}(I, \varphi, s) = A_{0,0} + A_{1,0}(I) \cos \varphi + A_{0,1} \cos s + O(\rho, r) \quad := \mathcal{L}^{[\leq 1]}(I, \varphi, s) + \mathcal{L}^{[> 1]}(I, \varphi, s). \quad (174)$$

Recall that we are looking for non-degenerate critical points of

$$\mathcal{L}(\tau) := \mathcal{L}(I, \varphi - I\tau, s - \tau) = \sum_{(k,l) \in \mathbb{N}^2} A_{k,l}(I) \cos(k\varphi + ls - \tau(Ik + l)), \quad (175)$$

with $A_{k,l}(I)$ as in (173).
Using that the Melnikov function $\mathcal{L}$ is well approximated by $\mathcal{L}^\leq 1$, fixed $(I, \varphi, s)$, we only need to study the evolution of $\mathcal{L}^\leq 1(I, \varphi, s)$ along the straight lines $R : \tau \in \mathbb{R} \mapsto (\varphi - I\tau, s - \tau) \in \mathbb{T}^2$ on the torus. This study has already been performed in the example in section 13 in [DLS06a], where the reader can find more details. For simplicity of the exposition we will take $0 < \rho < r$, with $\alpha = \pi/(2 \sinh(\pi/2))$, so that, for any fixed $I$, we have $A_{0,1} > A_{1,0}(I) > 0$, but one can see that it is enough to assume that $0 < \rho < r$ for the existence of diffusion. Therefore the function $(\varphi, s) \mapsto \mathcal{L}^k(I, \varphi, s)$ possesses exactly four non-degenerate critical points: a maximum at $(0, 0)$, a minimum at $(\pi, \pi)$ and two saddles at $(0, \pi)$ and $(\pi, 0)$ (see figure 2).

Around the two extremum points, its level curves are closed (and indeed convex) curves which fill out a basin ending at the level curve of one of the saddle points. Therefore, any straight line (176) that enters into some extremum basin is tangent to one of the convex closed level curves, giving rise to a non-degenerate extremum of $\tau \in \mathbb{R} \mapsto \mathcal{L}^\leq 1(I, \varphi - Is, s - \tau)$. So, degenerate extrema of $\tau \in \mathbb{R} \mapsto \mathcal{L}^\leq 1(I, \varphi - Is, s - \tau)$ can only exist for straight lines that never enter inside such extremum basins. It is clear that this never happens for irrational values of $I$ because it implies a dense straight line (and infinite non-degenerate extrema for $\tau \in \mathbb{R} \mapsto \mathcal{L}^\leq 1(I, \varphi - Is, s - \tau)$). On the other hand, the straight lines with rational slopes enter inside both extremum basins at least twice, except for the slopes $I = 0, \pm 1$. In these cases, one can check directly, using that $A_{0,1} > A_{1,0}(I) > 0$, that the function $\tau \in \mathbb{R} \mapsto \mathcal{L}^k(I, \varphi - Is, s - \tau)$ has one non-degenerate maximum and one non-degenerate minimum in any interval of length $2\pi$.

When we take into account $\mathcal{L}^\geq 1$ in the Melnikov potential $\mathcal{L}$ in (174), it is clear that in the compact subset $[I_-, I_+] \times \mathbb{T}^2$, as long as $0 < \rho, r \ll 1$, the function $\tau \in \mathbb{R} \mapsto \mathcal{L}(I, \varphi - I\tau, s - \tau)$ has non-degenerate extrema, and for every $I$ we can find a smooth function $\tau = \tau^*(I, \varphi, s)$ defined in an open set of $(\varphi, s) \in \mathbb{T}^2$.

Moreover, since $\mathcal{L}$ is periodic with respect to $(\varphi, s)$ and non-constant with non-degenerate extrema along any straight line, $\partial_\varphi \mathcal{L}^*$, where $\mathcal{L}^*$ is given in (11), is also periodic and non-constant and indeed changes sign. Therefore, for every $I$, there exists a non-empty set $\mathcal{J}_I$ where $\partial_\varphi \mathcal{L}^* > 0$ (and a non-empty set $\mathcal{J}_I^-$ where $\partial_\varphi \mathcal{L}^* < 0$), so hypothesis $H2^*$ is fulfilled. Indeed the set of points where $\partial_\varphi \mathcal{L}^*$ vanishes is a discrete set.

Figure 2. Graph and level curves of the Melnikov potential $\mathcal{L}^\leq 1(I, \varphi, s)$ with $\rho = 1/16$, $r = 1/(8\alpha)$ with $\alpha = \pi/(2 \sinh(\pi/2))$ and $I = 0$. In this case, $A_{0,0} = 4$, $A_{1,0} = 1/4$ and $A_{0,1} = 1/2$. 

\[ \begin{align*} &0 \leq \pi/2 \leq \pi, \quad 3.2 \leq 3.4 \leq 3.6 \leq 3.8 \leq 4 \leq 4.2 \leq 4.4 \leq 4.6 \leq 4.8 \leq \phi \leq s. \end{align*} \]
Conditions $H^3$, $H^3'$ and $H^3''$ can also be checked in the example (172) for any resonance $I = -l_0/k_0$ and any $M \geq 1$, independently of $\epsilon$.

Thus, if we consider $I = -l_0/k_0$ for any $(k_0, l_0) \in \mathbb{N}^2$, $k_0 \neq 0$ and $\gcd(k_0, l_0) = 1$, the function $U^{k_0, l_0}$ in hypothesis $H^3$ on $(k_0, l_0)$ has the following expression:

$$U^{k_0, l_0}(\theta) = \sum_{i=1}^{M} a_{k_0, l_0} \cos(i\theta) = a_{k_0, l_0} \cos(\theta) + O_2(\rho, \rho),$$

(177)

where $\theta = k_0 \varphi + l_0 \psi$ and $M \geq 1$ can be chosen arbitrarily.

Therefore, $\theta_1 = 0$ and $\theta_2 = \pi$ are the unique critical points for the function $U^{k_0, l_0}(\theta)$. Hence hypothesis $H^3$ on $(k_0, l_0)$ is clearly verified.

Next, for $I = -l_0/k_0$ we want to check hypothesis $H^3'$ on $(k_0, l_0)$. This condition requires to show that the function $f$ in (15) is not constant. To that end, we will consider two values of $\theta$ and we will show that their images for this function are different. For instance, note that the function $f$ in (15) takes the same values as $U^{k_0, l_0}$ evaluated on its critical points $\theta_1$ and $\theta_2$ as long as $\frac{\partial^2 L}{\partial \theta^2}(I, \theta_i/k_0) \neq 0$, for $i = 1, 2$. Hence, hypothesis $H^3'$ on $(k_0, l_0)$ is clearly satisfied if the function $U^{k_0, l_0}$ has two extrema $\theta_i$ taking different values which satisfy $\frac{\partial^2 L}{\partial \theta^2}(I, \theta_i/k_0) \neq 0$, which is the case as can be checked just looking at non-degenerate extrema of the function $L$.

They give rise to non-degenerate extrema of the function $L^i$, which coincide with the ones of the function $U^{k_0, l_0}$.

Similarly, we can check hypothesis $H^3''$ on $(k_0, l_0)$. In this case we need to show that the determinant (157) given in remark 4.8 does not vanish. It is clearly non-zero if we choose, for the two first columns, the two critical points $\theta_1$ and $\theta_2$ discussed above, and for the third column $\theta_3 \neq 0, \pi$, such that $\frac{\partial^2 L}{\partial \theta^2}(-l_0/k_0, \theta_3/k_0) = 0$, but otherwise $U^{k_0, l_0}(\theta_3) \neq 0$ and $\frac{\partial^2 L}{\partial \theta^2}(-l_0/k_0, \theta_3/k_0) \neq 0$. The existence of this point $\theta_3$ is guaranteed by the fact that if one considers the first order trigonometric polynomial of the reduced Poincaré function $L^i|_{\{\leq 1\}}$, one can see that its critical points are always non-degenerate.

Hence, we apply theorem 2.1 and we conclude that

Proposition 5.1. Given the Hamiltonian (171) with $g$ as in (172), $0 < \rho < r \ll 1$ and $[I_-, I_+] \subset \mathbb{R}^+$, for $|\epsilon| \leq \epsilon^*(\rho, r)$ there exist orbits following the mechanism described in this paper and such that $I(0) \leq I_-, I(T) \geq I_+$, for some $T > 0$.

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Appendix A. Double Fourier series

Proposition A.1. Let $f$ be a $C^r$ function with respect to $(J, \varphi, s, \epsilon), r \geq 1$ and $2\pi$-periodic with respect to $(\varphi, s)$. Then its Fourier coefficients $f_{k, l}(J, \epsilon), (k, l) \in \mathbb{Z}^2$, satisfy, for $\epsilon = 0, \ldots, r$

$$|f_{k, l}|_{C^r} \leq C \frac{|f|_{C^r}}{|(k, l)|^{r-1}}.

(A.1)$$

where $C$ is a constant that depends only on $r$ and $\epsilon$ and $|(k, l)| = \max(|k|, |l|)$. 

Proof. From the expression for the Fourier coefficients of a function $f$

$$f_{k,l}(J; \varepsilon) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}} f(J, \varphi, s; \varepsilon) e^{i(k\varphi+l\varepsilon)} d\varphi ds,$$

taking into account that $f$ is $C^r$ in the variables $(\varphi, s)$, we can integrate $r = n+m$ times by parts ($n$ times with respect to $\varphi$ and $m$ times with respect to $s$) and express the Fourier coefficient $f_{k,l}(J; \varepsilon)$, with $(k, l) \neq (0, 0)$ in the form

$$f_{k,l}(J; \varepsilon) = (-1)^r \frac{1}{(2\pi)^2} \frac{1}{(k\varphi+l\varepsilon)^m} \int_{\mathbb{T}} \frac{\partial^r f(J, \varphi, s; \varepsilon)}{\partial \varphi^n \partial s^m} e^{i(k\varphi+l\varepsilon)} d\varphi ds,$$

so that,

$$|f_{k,l}|_{C^r} \leq \frac{1}{|k|^n|l|^m} \left| \frac{\partial^r f}{\partial \varphi^n \partial s^m} \right|_{C^0} \leq n!m! |f|_{0,r},$$

for any $0 \leq n, m \leq r$ such that $n + m = r$, where $| \cdot |_{C^r}$ is the standard $C^r$ norm defined in (2) and $| \cdot |_{1,t_2}$ is the seminorm defined in (3). Therefore,

$$|f_{k,l}|_{C^r} \leq \frac{r! |f|_{0,r}}{|(k, l)|^r} \leq \frac{r! |f|_{C^r}}{|(k, l)|^r},$$

where $|(k, l)| = \max(|k|, |l|).

Now, taking into account that $D^r f_{k,l}(J; \varepsilon)$ is the Fourier coefficient of the function

$$\frac{\partial^r f(J, \varphi, s, \varepsilon)}{\partial J^r},$$

which is a $C^{r-\ell}$ function, and using the same argument as before we have that

$$|D^r f_{k,l}|_{C^r} \leq \frac{\ell! (r-\ell)! |f|_{C^r}}{|(k, l)|^{r-\ell}} \leq \frac{\ell! (r-\ell)! |f|_{C^r}}{|(k, l)|^{r-\ell}}.$$

From the definition of $| \cdot |_{C^r}$ norm in (2) we have the estimate

$$|f_{k,l}|_{C^r} = \sum_{i=0}^\ell \frac{|D^r f_{k,l}|_{C^r}}{i!} \leq \sum_{i=0}^\ell \frac{(r-i)! |f|_{C^r}}{|(k, l)|^{r-i}} \leq C \frac{|f|_{C^r}}{|(k, l)|^{r-\ell}},$$

where $C$ is a constant that depends only on $\ell$ and $r$, $C = r! + (r-1)! + \ldots + (r-\ell)!$, as we wanted to see. \hfill \Box

We consider the truncation of its Fourier series at order $M$ in the following way:

$$f(J, \varphi, s; \varepsilon) = f^{[\leq M]}(J, \varphi, s; \varepsilon) + f^{[> M]}(J, \varphi, s; \varepsilon),$$

where

$$f^{[\leq M]}(J, \varphi, s; \varepsilon) = \sum_{|k| + |l| \leq 2^M} f_{k,l}(J; \varepsilon) e^{i(k\varphi+l\varepsilon)},$$

and

$$f^{[> M]}(J, \varphi, s; \varepsilon) = \sum_{|k| + |l| > 2^M} f_{k,l}(J; \varepsilon) e^{i(k\varphi+l\varepsilon)}.$$

Proposition A.2. Let $f$ be of class $C^r$ with respect to $(J, \varphi, s, \varepsilon)$, $r \geq 1$ and $2\pi$-periodic with respect to $(\varphi, s)$. The $M$th order remainder $f^{[> M]}$ of the Fourier series of $f$ is bounded in the standard $C^r$ norm, for $\ell = 0, \ldots, r-3$, by

$$|f^{[> M]}|_{C^r} \leq C \frac{|f|_{C^r}}{M^{r-(\ell+2)}},$$

where $C$ is a constant that depends only on $r$ and $\ell$. 


Proof. The proof is very simple and follows from the estimate (A.1) for the Fourier coefficients of a $C^r$ function obtained in the previous proposition. More precisely,

$$
\|f^{(\leq M)}\|_{C^r} \leq \sum_{(k, l) \in \mathbb{Z}^2, |k| + |l| > M} |f_{k,l}|_{C^r} \\
\leq C \sum_{(k, l) \in \mathbb{Z}^2, |k| + |l| > M} \frac{|f|_{C^r}}{|(k,l)|^{r-\ell}} \\
\leq C \sum_{t=M+1}^{\infty} 4t \frac{|f|_{C^r}}{t^{r-\ell}} \\
\leq 4C |f|_{C^r} \int_M^{\infty} t^{\ell-1} dt \\
= 4 \frac{C}{r - \ell} \frac{|f|_{C^r}}{M^{\ell-\ell+2}},
$$

where $C$ is a constant that depends only on $r$ and $\ell$. □

Appendix B. Weighted norms

We consider functions $u \in \tau_M(\mathcal{I} \times \mathbb{T}^2)$, where $\mathcal{I} \subset \mathbb{R}$, introduced in (30), and we can consider the different types of norms introduced in this paper: the standard $C^r$ norm introduced in (2), the Fourier norm introduced in (31) and the Fourier norm with a weight introduced in (32).

The equivalence relations between all these norms are given in the following lemmas:

Lemma B.1. The norms $\|\cdot\|_{C^n}$ and $\|\cdot\|_{C^r}$ defined in (2) and (31), respectively, are equivalent and satisfy the following equivalence relation for $u \in \tau_M(\mathcal{I} \times \mathbb{T}^2)$ and $0 < L \leq 1$:

$$
L^\ell \|u\|_{C^r} \leq \|u\|_{C^{r,L}} \leq CM^2 \|u\|_{C^r}
$$

where $C$ is a constant depending on $\ell$.

Proof. The first inequality is obvious using that $L \leq 1$. For the second one, using again that $L \leq 1$ we have

$$
\|u_{k,l}\|_{C^{r,L}} = \sum_{i=0}^{\ell} L^i \frac{|D^i u_{k,l}|_{C^0}}{i!} \leq \sum_{i=0}^{\ell} \frac{|D^i u_{k,l}|_{C^0}}{i!} = |u_{k,l}|_{C^r},
$$

for $0 \leq n \leq \ell$. Therefore, the result follows directly from the estimate (A.1) for the $C^\ell$ norm of the Fourier coefficients of a $C^r$ function $u$, for $\ell = 0, \ldots, r$. More precisely,

$$
\|u\|_{C^{r,L}} = \sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^\ell \sum_{(k, l) \in \mathbb{Z}^2, |k| + |l| \leq M} |u_{k,l}|_{C^{r,L}} |(k,l)|^{m-n} \\
\leq \sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^\ell \sum_{(k, l) \in \mathbb{Z}^2, |k| + |l| \leq M} |u_{k,l}|_{C^r} |(k,l)|^{m-n} \\
\leq \sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^\ell \sum_{(k, l) \in \mathbb{Z}^2, |k| + |l| \leq M} \tilde{C} \frac{|u|_{C^r}}{|(k,l)|^{r-n}} |(k,l)|^{m-n},
$$
\[ \leq \sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^\ell \sum_{(k, l) \in \mathbb{Z}^2, |k| + |l| \leq M} \tilde{C}|u|_{C^\ell} \]
\[ \leq CM^2|u|_{C^\ell} \]
as we wanted to prove. \qed

**Lemma B.2.** For the seminorm $\| \cdot \|_{j, \ell - j}$ defined in (2), one has that for all $0 \leq j \leq \ell$,
\[ L^j|u|_{j, \ell - j} \leq \|u\|_{C^\ell, L}. \] (B.1)

**Proof.** Again, it follows directly from the fact that $L < 1$ and therefore,
\[ L^j|u_k,l|_{C^*} \leq \sum_{i=0}^{n} L^j \frac{|D^i u_k,l|_{C^0}}{i!} = |u_k,l|_{C^*, L}. \]
for $0 \leq n \leq j$. \qed

**Lemma B.3.** For $0 < L \leq 1$ and $0 \leq \ell \leq r$ we have that for any $u \in \tau_M(I \times \mathbb{T}^2)$ and $v \in \tau_N(I \times \mathbb{T}^2)$
\[ \|uv\|_{C^\ell, L} \leq \|u\|_{C^\ell, L} \|v\|_{C^\ell, L}. \] (B.2)

**Proof.** Let us define
\[ \|u\|_{n, m} = \sum_{(k, l) \in \mathbb{Z}^2, |k| + |l| \leq M} |u_k,l|_{C^\ell} |(k, l)|^{n-m}, \]
then,
\[ \|u\|_{C^\ell, L} = \sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^\ell \|u\|_{n, m}. \] (B.3)

The $\alpha$th Fourier coefficient of $uv$, where $\alpha \in \mathbb{Z}^2$, is
\[ (uv)_\alpha = \sum_{\beta \in \mathbb{Z}^2, |\alpha - \beta| \leq N} u_{\alpha - \beta} v_\beta. \]

Using the Leibniz rule for derivatives we have
\[ |(uv)_\alpha|_{C^\ell, L} \leq \sum_{i=0}^{n} \frac{1}{i!} L^i |D^i (uv)_\alpha|_{C^0} \]
\[ \leq \sum_{i=0}^{n} \frac{1}{i!} \sum_{\beta \in \mathbb{Z}^2, |\alpha - \beta| \leq N} L^i |D^i u_{\alpha - \beta} v_\beta|_{C^0} \]
\[ \leq \sum_{i=0}^{n} \frac{1}{i!} \sum_{\beta \in \mathbb{Z}^2, |\alpha - \beta| \leq N} \left( \sum_{j=0}^{i} \binom{i}{j} L^j |D^{j-i} u_{\alpha - \beta}|_{C^0} L^j |D^j v_\beta|_{C^0} \right) \]
\[ = \sum_{i=0}^{n} \sum_{\beta \in \mathbb{Z}^2, |\alpha - \beta| \leq N} \sum_{j=0}^{i} \frac{L^{i-j} |D^{i-j} u_{\alpha - \beta}|_{C^0} L^j |D^j v_\beta|_{C^0}}{(i-j)! j!} \]
On the other hand, we have
\[
\alpha_{uv} = \max (|\alpha|_{n,m} - |\beta|_{n,m})
\]
Hence, using these two inequalities, we have that
\[
\|uv\|_{n,m} = \sum_{\alpha \in \mathbb{Z}^2, \alpha \leq M + N} |(uv)_{n,m}|_{\alpha} |\alpha|_{m-n}
\]
\[
\leq \sum_{\alpha \in \mathbb{Z}^2, \alpha \leq M + N} \sum_{\beta \in \mathbb{Z}^2, \beta \leq N} |u_{\alpha-\beta}|_{n,m} |v_{\beta}|_{n,m} |\alpha|_{m-n}
\]
\[
+ \sum_{\beta \in \mathbb{Z}^2, \beta \leq N} |v_{\beta}|_{n,m} 2^{m-n} |\alpha|_{m-n} |\beta|_{m-n}
\]
\[
\leq 2^{m-n} \sum_{\alpha \in \mathbb{Z}^2, \alpha \leq M} |u_{\alpha}|_{n,m} |\alpha|_{m-n} \sum_{\beta \in \mathbb{Z}^2, \beta \leq M} |v_{\beta}|_{n,m} |\beta|_{m-n}
\]
\[
= 2^{m-n} \|u\|_{n,m} \|v\|_{n,m}.
\]
Going back to the definition of \(\|uv\|_{c^1,L}\) in (B.3), we have
\[
\|uv\|_{c^1,L} = \sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^\ell \|uv\|_{n,m}
\]
\[
\leq \sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^\ell 2^{m-n} \|u\|_{n,m} \|v\|_{n,m}
\]
\[
\leq \sum_{m=0}^{\ell} \sum_{n=0}^{m} 2^\ell 2^\ell \|u\|_{n,m} \|v\|_{n,m}
\]
\[
\leq \|u\|_{c^1,L} \|v\|_{c^1,L},
\]
as claimed. □
Appendix C. Faa–di Bruno formula

Let \( g \) be a \( C^r(U, V) \) function, with \( U \subset \mathbb{R} \) and \( g(U) \subset W \subset \mathbb{R} \) and \( f \) be a \( C^s(W, \mathbb{R}) \) function with \( r, s > 0 \). Then \( f \circ g \) is a \( C^s(U, \mathbb{R}) \) function, where \( t = \min(r, s) \). By a repeated application of the chain rule, one gets

\[
D^\ell (f \circ g)(x) = \sum_{k=1}^\ell \sum_{j_1+\cdots+j_k=\ell} c_{k,j_1,\ldots,j_k} D^{j_1} f(g(x)) D^{j_2} g(x) \cdots D^{j_k} g(x), \tag{C.1}
\]

for \( \ell = 1, \ldots, t \), where \( c_{k,j_1,\ldots,j_k} \) are combinatorial coefficients. The formula (C.1) is called the Faa–di Bruno formula (see [LO99]).

From equation (C.1), it is easy to see that there exists a constant \( C_\ell \) depending on \( t \) such that

\[
|f \circ g|_{C^\ell} \leq C_\ell |f|_{C^\ell} |g|_{C^{\ell}}, \tag{C.2}
\]

Since we are interested in multi-valued functions, we introduce now a generalized bound. Thus, let us consider a function \( g \) in \( C^r(U, V) \), with \( U \subset \mathbb{R}^n \) and \( g(U) \subset W \subset \mathbb{R}^m \) and a function \( f \) in \( C^s(W, \mathbb{R}) \) with \( r, s > 0 \). As before, \( f \circ g \) is a \( C^s(U, \mathbb{R}) \) function, where \( t = \min(r, s) \). Similarly, we can get an expression for the derivatives of \( f \circ g \), such that for \( \ell = 1, \ldots, t \),

\[
|f \circ g|_{C^\ell} \leq C_\ell \sum_{k=1}^\ell \sum_{j_1+\cdots+j_k=\ell} |f|_{C^\ell} |g|_{C^{\ell}} \cdots |g|_{C^{\ell}}, \tag{C.3}
\]

for \( \ell = 0, \ldots, t \), where \( C_\ell \) is a constant depending on \( \ell \). As before, we can consider the following less precise but more compact bound:

\[
|f \circ g|_{C^\ell} \leq C_\ell |f|_{C^\ell} |g|_{C^{\ell}}, \tag{C.4}
\]

for \( \ell = 1, \ldots, t \), where \( C_\ell \) is a constant depending on \( \ell \).

For some other results related to this, we refer the reader to [LO99].

In some cases, it will be more convenient to use another estimate for the \( |\cdot|_{C^\ell} \) norm instead of the one obtained in (C.4). In formula (C.3) we can separate the term corresponding to \( k = 1 \) in the following way:

\[
|f \circ g|_{C^\ell} \leq C_\ell \left( |f|_{C^\ell} |g|_{C^\ell} + \sum_{k=2}^\ell \sum_{j_1+\cdots+j_k=\ell} |f|_{C^\ell} |g|_{C^{\ell}} \cdots |g|_{C^{\ell}} \right),
\]

for \( \ell = 1, \ldots, t \) and we can bound it in the \( |\cdot|_{C^\ell} \) norm

\[
|f \circ g|_{C^\ell} \leq C_\ell (|f|_{C^\ell} |g|_{C^\ell} + |f|_{C^\ell} |g|_{C^{\ell}}), \tag{C.5}
\]

for \( \ell = 1, \ldots, t \), where \( C_\ell \) is a constant depending on \( \ell \).

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