On the metric dimension of graphs

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**RESOLVING SETS**

* $G = (V, E)$ is a connected graph.

- A vertex $x$ of $G$ **RESOLVES** a pair of vertices $u, v$ of $G$ if:

  $$d(x, u) \neq d(x, v).$$

- A vertex subset $S \subseteq V$ is a **RESOLVING SET** of $G$ if:

  every two distinct vertices of $G$ are resolved by some vertex of $S$. 
**METRIC DIMENSION**

* $S \subseteq V$ is a resolving set of $G = (V, E)$.

▶ $S$ is a **metric basis** if it is a minimum resolving set.

* $S = \{u_1, u_2, \ldots, u_r\}$ is a metric basis of $G$.

▶ The **metric dimension** of $G$ is: $\beta(G) = r$.

* $x$ is a vertex of $G$.

▶ The **metric coordinates** of $x$ are:

$$ (x_1, \ldots, x_r), \text{ where } x_i = d(u_i, x). $$
$S = \{x, y\}$ is a metric basis

$\beta(G) = 2$

$d(x, a) = 5$

$d(y, a) = 1$
Some Basic Known Results

- The problem of computing the metric dimension of an arbitrary graph is NP-hard.

<table>
<thead>
<tr>
<th>name</th>
<th>path</th>
<th>cycle</th>
<th>complete</th>
<th>bicomplete</th>
<th>wheel</th>
<th>hypercube</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td>$P_n$</td>
<td>$C_n$</td>
<td>$K_n$</td>
<td>$K_{r,s}$</td>
<td>$W_{1,r}$</td>
<td>$Q_r = [K_2]^r$</td>
</tr>
<tr>
<td>$</td>
<td>V(G)</td>
<td>$</td>
<td>$n \geq 1$</td>
<td>$n \geq 3$</td>
<td>$n \geq 2$</td>
<td>$r + s \geq 3$</td>
</tr>
<tr>
<td>$\beta(G)$</td>
<td>1</td>
<td>2</td>
<td>$n - 1$</td>
<td>$n - 2$</td>
<td>3</td>
<td>$r \ (r \leq 4)$</td>
</tr>
</tbody>
</table>

- If $T$ is a tree s.t. $\lambda(T) \geq 1$, then $\beta(T) = |Ext(T)| - \lambda(T)$.

- If $G = T + e$, then $\beta(T) - 2 \leq \beta(T + e) \leq \beta(T) + 1$.

- $r \notin \{3, 6\}$: $\beta(W_{1,r}) = \left\lfloor \frac{2^r + 2}{5} \right\rfloor$. 
ICGT05, HYERES

leaves

exterior major vertices

<p>| | | | | |</p>
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<tr>
<td>x</td>
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<td></td>
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</tr>
<tr>
<td>y</td>
<td>506</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>z</td>
<td>560</td>
<td></td>
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**BOUNDARY VERTICES**

- $u, v$ are two vertices of a graph $G$.

  - $v$ is a boundary vertex of $u$ if:
    \[
    \forall w \in N(v), \ d(u, w) \leq d(u, v).
    \]

  - The set of all boundary vertices of $u$ is its boundary $\partial(u)$.

  - The boundary of $G$ is the set $\partial(G) = \bigcup_{u \in V(G)} \partial(u)$.

■ **THEOREM:** The boundary $\partial(G)$ is a resolving set of $G$.

■ **COROLLARY:** $\beta(G) \leq |\partial(G)|$. 
\[ \partial(G) = \{a, b, c, d, e\} \]
**BOUNDS AND CHARACTERIZATIONS**

- $|G| - d^{\beta(G)} \leq \beta(G) \leq |G| - d$ ($d = \text{diam}(G)$).

- $\beta(G) = 1 \iff G = P_n$.

- $\beta(G) = |G| - 1 \iff G = K_n$.

- $t \geq 2$: $\beta(G) = |G| - 2 \iff G \in \{K_{s,t}, K_s + \overline{K}_t, K_s + (K_1 \cup K_t)\}$.

- $d \geq 3$, $\beta(G) = |G| - d \Rightarrow \exists a \in \text{Per}(G)$ without twins.

- $\beta(G) = |G| - 3 \iff G \in \{\ldots, \ldots, \ldots\}$.\[\begin{array}{cccc}
  & K_1 & K_r & K_s & K_1 \\
\end{array}\]
**CARTESIAN PRODUCT**

$\star G_1 = (V_1, E_1)$ $G_2 = (V_2, E_2)$ are two connected graphs.

$\blacktriangleright$ The cartesian product $G_1 \square G_2$ is the graph with:

$\diamond V(G_1 \square G_2) = V_1 \times V_2$.

$\diamond (u_1, u_2)$ is adjacent to $(v_1, v_2)$ iff

\[
\begin{cases}
    u_1 = v_1 \text{ and } u_2v_2 \in E_2 \\
    \text{ or } \\
    u_1v_1 \in E_1 \text{ and } u_2 = v_2
\end{cases}
\]

$\blacksquare d((u_1, u_2), (u_1, u_2)) = d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2)$.

$\blacksquare$ **THEOREM:** \( \beta(G_1 \square G_2) \geq \max[\beta(G_1), \beta(G_2)] \).
**SOME KNOWN RESULTS ON THE C.P.**

- $\beta(G) \leq \beta(G \square K_2) \leq \beta(G) + 1$.

- $\beta(P_m \square P_n) = 2$.

- $\lim_{n \to \infty} \beta(Q_n) \cdot \frac{\log_2(n)}{n} = 2$

- $\beta(P_{m_1} \square P_{m_2} \square \ldots \square P_{m_d}) = d$ (*wrong result: DAM, 70 (1996)*).
**Cartesian Products [Bounds]**

- \( \max[\beta(G), \beta(H)] \leq \beta(G \square H) \).

- \(|G| \geq 3, |H| \geq 3: \beta(G \square H) \leq \min\{\beta(G) + |H|, \beta(H) + |G|\} - 2 \)

- \( \beta(G \square K_n) \leq \max\{n - 1, 2 \cdot \beta(G)\} \).
- \( \beta(G \square P_n) \leq \beta(G) + 1 \).

- \( \beta(G \square C_n) \leq \beta(G) + 2 \).
- \( \beta(G \square T) \leq \beta(G) + |\text{Ext}(T)| - 1 \).

- For all \( k \geq 1 \) and \( n \geq 2 \) there is a \( k \)-connected graph \( G_{n,k} \) for which \( \beta(G_{n,k}) \leq 2k \) and \( \beta(G_{n,k} \square G_{n,k}) \geq n \).

- For all \( k \geq 1 \) there in no function \( f \) such that: \( \beta(G \square H) \leq f(\beta(G), \beta(H)) \), for all \( k \)-connected graphs \( G \) and \( H \).
CARTESIAN PRODUCTS [EXACT VALUES]

- \( m \leq n \Rightarrow \dim(K_m \square K_n) = \begin{cases} 
  \frac{n - 1}{3} & \text{if } 2m - 2 < n, \\
  \frac{2m + 2n - 2}{3} & \text{if } 2m - 2 \geq n.
\end{cases} \)

- \( m \geq 4: \beta(K_m \square C_n) = \begin{cases} 
  m, & \text{if } m = 4 \text{ and } n \text{ odd,} \\
  m - 1, & \text{otherwise.}
\end{cases} \)

- \( \beta(C_m \square C_n) = \begin{cases} 
  3, & \text{if } m \text{ or } n \text{ is odd} \\
  4, & \text{otherwise.}
\end{cases} \)

- \( \beta(C_m \square P_n) = \begin{cases} 
  2, & \text{if } m \text{ odd} \\
  3, & \text{if } m \text{ even (and } n \neq 1) \end{cases} \)

<table>
<thead>
<tr>
<th>G \backslash H</th>
<th>\begin{array}{c} K_n \ C_n \ P_n \end{array}</th>
</tr>
</thead>
<tbody>
<tr>
<td>\begin{array}{c} K_m \ C_m \ P_m \end{array}</td>
<td>\begin{array}{c} n - 1, \left\lfloor \frac{2m + 2n - 2}{3} \right\rfloor \ m - 1, m \ 3, 4 \end{array}</td>
</tr>
<tr>
<td>\begin{array}{c} K_m \ C_m \ P_m \end{array}</td>
<td>\begin{array}{c} m - 1 \end{array}</td>
</tr>
<tr>
<td>\begin{array}{c} K_m \ C_m \ P_m \end{array}</td>
<td>\begin{array}{c} m - 1 \end{array}</td>
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<tr>
<td>\begin{array}{c} K_m \ C_m \ P_m \end{array}</td>
<td>\begin{array}{c} m - 1 \end{array}</td>
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</table>
Cartesian Product of a Cycle by a Path

- \( m \geq 3, n \geq 2: \beta(C_m) = 2, \beta(P_n) = 1 \)
- \( \max[\beta(G), \beta(H)] \leq \beta(G \square H) \).
- \( \beta(G \square P_n) \leq \beta(G) + 1 \).

\[ \Rightarrow 2 = \beta(C_m) \leq \beta(C_m \square P_n) \leq \beta(C_m) + 1 = 3. \]

\( \Rightarrow \beta(C_m \square G) = 2 \iff G \) is a path and \( m \) is odd.

**Theorem:** \( \beta(C_m \square P_n) = \begin{cases} 
2, & \text{if } m \text{ odd} \\
3, & \text{if } m \text{ even (and } n \neq 1) 
\end{cases} \)
**DOUBLY RESOLVING SETS**

- Two vertices $v, w$ of $G \neq K_1$ are **DOUBLY RESOLVED** by a pair of vertices $x, y$ of $G$ if:
  \[ d(v, x) - d(w, x) \neq d(v, y) - d(w, y). \]

- A set $S \subseteq V$ is a **DOUBLY RESOLVING SET** of $G$ if every pair of distinct vertices of $G$ are doubly resolved by two vertices in $S$. $\psi(G)$ is the minimum cardinality of a doubly resolving set.

\[ 1 \leq \beta(G) \leq \psi(G) \leq |G| - 1 \quad \text{and} \quad 2 \cdot \beta(G \square G) \geq \psi(G) \]

\[ \beta(G \square H) \leq \beta(G) + \psi(H) - 1 \quad \text{and} \quad \beta(G \square H) \leq \beta(G) + 2\beta(H \square H) - 1 \]

<table>
<thead>
<tr>
<th>name</th>
<th>path</th>
<th>odd cycle</th>
<th>even cycle</th>
<th>complete</th>
<th>tree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td>$P_n$</td>
<td>$C_n$</td>
<td>$C_n$</td>
<td>$K_n$</td>
<td>$T_n$</td>
</tr>
<tr>
<td>$</td>
<td>V(G)</td>
<td>$</td>
<td>$n \geq 2$</td>
<td>$n \geq 3$</td>
<td>$n \geq 4$</td>
</tr>
<tr>
<td>$\beta(G)$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>$n - 1$</td>
<td>$</td>
</tr>
<tr>
<td>$\psi(G)$</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>$n - 1$</td>
<td>$</td>
</tr>
</tbody>
</table>
C. P. OF AN ODD CYCLE BY A PATH

\[ \psi(C_m) = \begin{cases} 2, & \text{if } m \text{ odd} \\ 3, & \text{if } m \text{ even} \end{cases} \]

\[ \psi(P_n) = 2 \]

\[ \beta(G) \leq \beta(G \square P_n) \leq \beta(G) + \psi(P_n) - 1 = \beta(G) + 1 \]

\[ \beta(G) \leq \beta(C_m \square G) \leq \beta(G) + \psi(C_m) - 1 \leq \begin{cases} \beta(G) + 1, & \text{if } m \text{ odd} \\ \beta(G) + 2, & \text{if } m \text{ even} \end{cases} \]

\[ \Rightarrow m \text{ odd } \Rightarrow \beta(C_m \square P_n)) = 2. \]

http://es.arXiv.org/find/math/1/PELAYO/0/1/0/past/3/0