On the Metric Dimension of Infinite Graphs *

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Abstract. A set of vertices $S$ resolves a graph $G$ if every vertex is uniquely determined by its vector of distances to the vertices in $S$. The metric dimension of a graph $G$ is the minimum cardinality of a resolving set. An infinite graph is a graph with an infinite set of vertices. In this work we study the metric dimension of infinite graphs such that all vertices have finite degree. We give necessary conditions for those graphs to have finite metric dimension and characterize infinite trees with finite metric dimension. We also establish some results about the metric dimension of the cartesian product of finite and infinite graphs, and give the metric dimension of the cartesian product of several families of graphs.

Key words: infinite graphs, resolving sets, metric dimension, infinite trees, cartesian product of graphs.

1 Introduction

A graph $G$ is an ordered pair of disjoint sets $(V, E)$ where $V$ is nonempty and $E$ is a subset of unordered pairs of $V$. The vertices and edges of $G$ are the elements of $V = V(G)$ and $E = E(G)$ respectively. Two vertices $u$ and $v$ are adjacent if $\{u, v\} \in E(G)$. We say that a graph $G$ is finite (resp. infinite) if the set $V(G)$ is finite (resp. infinite). The degree of a vertex $u \in V(G)$ is the number of edges containing $u$. An infinite graph is locally finite if every vertex has finite degree. All infinite graphs considered in this work are locally finite. A $u-v$ trail of length $k$ is a sequence $u_0, u_1, \ldots, u_k$ of vertices such that $u_0 = u$, $u_k = v$ and $\{u_i, u_{i+1}\}$ is an edge for all $k \in \{0,1,\ldots,k-1\}$. A trail with no repeated vertices is a path. We define the distance between two

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vertices \( u \) and \( v \) in a graph \( G \) as the minimum length of all \( u - v \) paths, if there is at least a \( u - v \) path, and infinite, if there is no \( u - v \) path. We denote it by \( d_G(u,v) \), or simply \( d(u,v) \) if the graph \( G \) is clear. A graph is connected if the distance between two any vertices is finite.

Traditionally, infinite graphs have not attracted as much attention as finite graphs, although very prominent authors as D. König [9], C. St. J. A. Nash-Williams [12], C. Thomassen [16], R. Diestel and many other authors [6] expressed interest in them.

We say that a vertex \( x \) in a graph \( G \) resolves two vertices \( u, v \) if \( d(u,x) \neq d(v,x) \). A subset of vertices \( S \) is a a resolving set of \( G \) if for any two vertices, there exists a vertex in \( S \) that resolves them. A resolving set with minimum cardinality is a metric basis. If a graph \( G \) has at least a finite resolving set, the metric dimension \( \beta(G) \) of \( G \) is the cardinality of a metric basis, otherwise we say that the metric dimension of \( G \) is infinite. If \( S = \{x_1, \ldots, x_n\} \) is a finite set of vertices of \( G \), we denote by \( r(u|S) \) the vector of distances from \( u \) to the vertices of \( S \), that is, \( r(u|S) = (d(u,x_1), \ldots, d(u,x_n)) \). Then, \( S \) is a resolving set if and only if \( r(u|S) \neq r(v|S) \) for all vertices \( u \neq v \). If \( S \) is a finite metric basis we say that \( r(u|S) \) are the metric components of vertex \( u \) respect to \( S \).

A resolving set does not necessarily contain a metric basis. For example, it is easy to verify that any two distinct vertices of a path form a resolving set, but a metric basis of this graph contains one of its endpoints and no more vertices. This fact makes more difficult to determine the metric dimension of a graph.

Resolving sets in general graphs were defined by Harary and Melter [11] and Slater [15]. Resolving sets have been widely investigated [5,3,7] and arise in many diverse areas including coin weighting [1], network discovery and verification [2], robot navigation [10,14], connected joins in graphs [13], and strategies for the Mastermind game [8].

In this paper we study resolving sets in infinite graphs. The first question that arises is to determine infinite graphs with finite resolving sets. In Section 2 we give necessary conditions for this and in Section 3 we characterize acyclic connected infinite graphs, i.e. infinite trees, with finite metric dimension. On the other hand, interesting examples of graphs can be obtained as cartesian products of graphs. The metric dimension of cartesian products of (finite) graphs is studied in [3]. In Section 4 we give bounds for the metric dimension of cartesian products of finite and infinite graphs and determine the metric dimension for diverse families of graphs. We omit proofs because of limited space.

### 2 Metric Dimension of Infinite Graphs

Infinite graphs may have finite or infinite metric dimension. In fact, there exist connected infinite graphs with metric dimension \( n \) for all \( n \geq 0 \). For
example, we define the infinite path, \( P_\infty \), as an infinite graph with set of vertices \( V = \{ u_i : i \geq 0 \} \) and two vertices \( u_i, u_j \) are adjacent if and only if \( |i - j| = 1 \). We say that \( u_0 \) is the endpoint of \( P_\infty \). In a similar way, we define for \( k \geq 2 \) the \( k \)-infinite path, \( P_{k\infty} \), as the graph formed by \( k \) pairwise disjoint infinite paths and a new vertex adjacent to their \( k \) endpoints. Obviously, \( \beta(P_\infty) = 1 \), and it is easy to verify that \( \beta(P_{2\infty}) = 2 \) and \( \beta(P_{k\infty}) = k - 1 \), if \( k \geq 3 \). Metric basis for the graphs \( P_\infty \), \( P_{2\infty} \) and \( P_{k\infty} \) are shown as black vertices in Figure 1.

Fig. 1. Black vertices are metric basis of the graphs \( P_\infty \), \( P_{2\infty} \) and \( P_{k\infty} \).

On the other hand, if we hang vertices of degree 1 to each vertex of an infinite path, we obtain the infinite comb with infinite metric dimension (see Figure 2).

Fig. 2. The infinite comb has infinite metric dimension.

**Proposition 1.** The infinite comb has infinite metric dimension.

An infinite graph is uniformly locally finite if there exits a positive integer \( M \) such that the degree of every vertex is at most \( M \). For example, the \( k \)-infinite path and the infinite comb are uniformly locally finite, and the graph obtained by hanging to the infinite path \( i \) vertices of degree 1 to the vertex at distance \( i \geq 0 \) from the endpoint is non uniformly locally finite (see Figure 3).

**Theorem 1.** If \( G \) is an infinite graph with finite metric dimension \( \beta(G) = n \), then the degree of all vertices of \( G \) is at most \( 3^n - 1 \).

**Corollary 1.** If \( G \) is an infinite graph with finite metric dimension, then it is uniformly locally finite.
Fig. 3. An example of non uniformly locally finite graph.

The reciprocal is not true, and the infinite comb is a counterexample.

In going on from finite to infinite graphs, a common technique is to study the desired property or parameter for the finite subgraphs of an infinite graph. However, this seems to go nowhere for our purposes. In fact, an infinite graph \( G \) with finite metric dimension may have an induced subgraph \( H \) with infinite metric dimension. For example, the graph of Figure 4 has dimension two. However, it has the infinite comb as induced subgraph, which has infinite metric dimension.

Fig. 4. A graph with \( \beta(G) = 2 \): squared vertices form a metric basis. However, black vertices induce a subgraph with infinite metric dimension.

A metric ray with endpoint \( u_0 \) in a graph \( G \) is an infinite set of vertices \( V = \{ u_0, u_1, u_2, u_3, \ldots \} \) such that for all \( k \geq 0 \) vertex \( u_k \) is adjacent to \( u_{k+1} \), and \( d(u_0, u_k) = k \).

Probably, the oldest and best known result about infinite graphs is the König’s Lemma.

**Lemma 1 ([9]).** For every vertex \( v \) of an infinite connected graph, there exists a metric ray with \( v \) as endpoint.

From this lemma the next result follows.

**Lemma 2.** Let \( S \) be a finite subset of vertices of an infinite connected graph \( G \), and let \( P = \{ u_0, u_1, u_2, \ldots \} \) be a metric ray such that \( S \cap P = \emptyset \). Then there always exists a vertex \( u_i \in P \) such that for any integer \( k \geq 0 \), \( r(u_{i+k}|S) = r(u_i|S) + (k, \ldots, k) \).

For every metric ray \( P \) in \( G \), there is at least one vertex satisfying the conditions of Lemma 2. Let denote by \( u_P \) one such vertex.
Here we give a necessary condition for infinite graphs with finite metric dimension using the notion of doubly resolving set introduced in [3].

Two vertices $x$ and $y$ in a connected graph $G$ (finite or infinite) \textit{doubly resolve} a pair of vertices $u$ and $v$ if $d(u, x) - d(v, x) \neq d(u, y) - d(v, y)$. If $S$ and $U$ are two subsets of vertices of $G$, we say that $S$ \textit{doubly resolves} $U$ if every pair of distinct vertices in $U$ are doubly resolved by two vertices in $S$. $S$ is a \textit{doubly resolving set} of $G$ if $S$ doubly resolves $V(G)$. If $G \neq K_1$ has at least a finite doubly resolving set, we define $\psi(G)$ as the minimum cardinality of a doubly resolving set. Otherwise, we say that $\psi(G) = \infty$.

**Lemma 3.** Let $G$ be an infinite connected graph and let $U$ be an infinite subset of vertices in $G$. Then there does not exist a finite set that doubly resolves $U$.

**Corollary 2.** If $G$ is an infinite connected graph, then $\psi(G) = \infty$.

**Lemma 4.** Let $G$ be an infinite connected graph and let $S$ be a resolving set for $G$. Suppose that $P$ is a set of pairwise disjoint metric rays such that $P \cap S = \emptyset$ for all $P \in \mathcal{P}$. Then $S$ doubly resolves the set of vertices $\{u_P | P \in \mathcal{P}\}$.

The above assertions allow us to give a necessary condition for those infinite graphs with finite metric dimension.

**Theorem 2.** If $G$ is an infinite connected graph with finite metric dimension, then it does not contain an infinite number of pairwise disjoint metric rays.

The reciprocal of Theorem 2 is not true: it is easy to verify that the graph of Figure 5 has infinite metric dimension, however it contains at most two disjoint metric rays.

![Fig. 5. A graph with infinite metric dimension not containing infinite pairwise disjoint metric rays.](image)

Note that the conditions of Corollary 1 and Theorem 2 are independent. There are graphs non uniformly locally finite and not containing infinite pairwise disjoint metric rays, as the graph of Figure 5. On the other hand, we define the \textit{infinite grid} as the infinite graph with set of vertices $V(G) = \{(x, y) : x, y \in \mathbb{Z}\}$, and a vertex $(x, y)$ is adjacent to the four vertices $(x, y - 1), (x, y + 1), (x - 1, y)$ and $(x + 1, y)$ (see Figure 6). This graph is uniformly locally finite, but has infinite pairwise disjoint metric rays. Observe also that both conditions together are not sufficient to assure finite metric dimension, since the infinite comb satisfies both of them, but has infinite metric dimension.
3 Metric Dimension of Infinite Trees

An infinite tree is a connected acyclic infinite graph. Next result characterizes infinite trees with finite metric dimension.

**Theorem 3.** An infinite tree has finite metric dimension if and only if the set of vertices of degree at least three is finite.

Metric dimension and metric basis of trees are well known (see [4] and [10]). In a similar way it is possible to determine the metric dimension and all metric basis of infinite trees that have finite metric dimension. Let $v$ be a vertex of an infinite tree $T$ and let $L_v$ be the number of connected components of $T \setminus v$ that are finite or infinite paths.

**Proposition 2.** If $T \neq P_\infty \Box P_\infty$ is an infinite tree with finite metric dimension, then

$$\beta(T) = \sum_{v \in T} \max\{L_v - 1, 0\},$$

and all metric basis can be obtained in the following way: for each vertex $v$ such that $L_v = k \geq 2$ select $k - 1$ vertices of distinct connected components of $T \setminus v$ that are finite or infinite paths.

4 Metric Dimension of Cartesian Products

The cartesian product of graphs $G$ and $H$, denoted by $G \Box H$, is the graph with vertex set $V(G) \times V(H) = \{(a, v) : a \in V(G), v \in V(H)\}$, where $(a, v)$ is adjacent to $(b, w)$ whenever $a = b$ and $v, w \in E(H)$, or $v = w$ and $a, b \in E(G)$. Observe that if $G$ and $H$ are connected, then $G \Box H$ is connected. In particular, $d_{G \Box H}((a, v), (b, w)) = d_G(a, b) + d_H(v, w)$. A number of interesting examples can be viewed as the cartesian product of two graphs. For example, the two dimensional infinite grid is the graph $P_\infty \Box P_\infty$. Since the cartesian product
of two infinite graphs has always infinite pairwise disjoint metric rays, it follows from Theorem 2 that:

**Corollary 3.** Given two infinite graphs $G$ and $H$, then $\beta(G \square H) = \infty$.

Given a subset $S$ of vertices in $G \square H$, its projection onto $G$ is the set of vertices $x \in V(G)$ for which there exists a vertex $u \in V(H)$ such that $(x, u) \in S$. Similarly is defined the projection of $S$ onto $H$. Some results obtained for finite graphs in [3] can be extended to infinite graphs.

**Proposition 3 (see [3]).** Let $G$, $H$ be finite or infinite connected graphs. If $S$ is a resolving set of $G \square H$, then the projection of $S$ onto $G$ (resp. onto $H$) is a resolving set of $G$ (resp. of $H$) and, consequently, $\beta(G \square H) \geq \max\{\beta(G), \beta(H)\}$.

**Corollary 4.** If $G$ is an infinite connected graph with infinite metric dimension, then for any graph $H$ we have $\beta(G \square H) = \infty$.

Another result that can be extended to infinite graphs is the following:

**Theorem 4 (see [3]).** If $G$ is an infinite connected graph with finite metric dimension and $H$ is a finite connected graph with at least two vertices, then the metric dimension of $G \square H$ is finite and $\beta(G \square H) \leq \beta(G) + \psi(H) - 1$.

We summarize the preceding results in Table 1.

| $|V(G)|$ | $|V(H)|$ | $\beta(G \square H)$ |
|--------|--------|----------------------|
| $< \infty$ | $< \infty$ | $< \infty$ |
| $= \infty$ | $= \infty$ | $= \infty$ |
| $= \infty$ | $< \infty$ | $\leq \beta(G)$ if $\beta(G) < \infty$, $= \infty$ if $\beta(G) = \infty$. |

**Table 1.** Possibilities for the metric dimension of the cartesian product of graphs.

Next, we determine the metric dimension of the cartesian product of the infinite graphs $P_\infty$ and $P_{2\infty}$ by finite paths, cycles and complete graphs. Theorem 4 gives us an upper bound in terms of the parameter $\psi$ of paths, cycles and complete graphs, determined in [3]. We have obtained the following results.

**Lemma 5.** If $H$ is a connected graph, then $\beta(P_{2\infty} \square H) = 2$ if and only if $H$ is the trivial graph, $K_1$.

**Proposition 4.** For all $n \geq 2$, $\beta(P_\infty \square P_n) = 2$ and $\beta(P_{2\infty} \square P_n) = 3$.

**Proposition 5.** For all $n \geq 3$,

$$\beta(P_\infty \square C_n) = \begin{cases} 2 & \text{if } n \text{ is odd}, \\ 3 & \text{if } n \text{ is even} \end{cases}$$

and $\beta(P_{2\infty} \square C_n) = \begin{cases} 3 & \text{if } n \text{ is odd}, \\ 4 & \text{if } n \text{ is even}. \end{cases}$

**Proposition 6.** For all $n \geq 4$, $\beta(P_\infty \square K_n) = \beta(P_{2\infty} \square K_n) = n - 1$. 
References


