Watching Systems in Complete Bipartite Graphs

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  Detection devices and graphs
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  Watching systems
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Complete bipartite graphs
  Bounds of the watching number
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Detection devices

- Detection devices located at some vertices of a graph
- Detect and locate an object placed at any vertex of a graph

- Dominating/total dominating sets
- Locating sets
Detection devices

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**Detection devices and graphs**

**Dominating set**

**Locating set**

**Locating-dominating set**

**Identifying set**
Definitions

\[ G = (V, E) \text{ graph,} \]

- \( N(u) = \{ v : uv \in E \} \)
- \( N[u] = \{ u \} \cup N(u) \)
- twin vertices: \( N[u] = N[v] \)
- twin-free graph: it has no pair of twin vertices
- dominating set: \( S \subseteq V \text{ s.t. for all } v \in V \setminus S, S \cap N(v) \neq \emptyset \)
- dominating number, \( \gamma(G) \): minimum size of a dominating set of \( G \)
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Identifying codes

[Karpovsky, Chakrabarty, Levitin, 1998]

**Identifying code** in a graph $G = (V, E)$:

$S \subseteq V$ s.t. the sets $N[v] \cap C$, $v \in V(G)$, are all nonempty and distinct.

- *label* of vertex $v$: $L_C(v) = N[v] \cap C$
- *identifying number*, $i(G)$: minimum size of an identifying code of $G$
- Identifying codes exist only in twin-free graphs.
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Detection devices and graphs
Complete bipartite graphs
Identifying codes
Watching systems

[Auger, Charon, Hudry, Lobstein, 2010]

*Watching system* in a graph $G = (V, E)$ graph:

$W = \{w_1, w_2, \ldots, w_k\}$ where $w_i = (l(w_i), A(w_i))$, with

$l(w_i) = v_i \in V(G)$ and $A(w_i) \subseteq N[v_i]$, for all $i \in \{1, 2, \ldots, k\}$, s.t.
the sets $L_W(v) = \{w \in W : v \in A(w_i)\}$ are all nonempty and distinct.

- $w_i$ is a *watcher* located at vertex $l(w_i)$ that checks its *watching zone*, $A(w_i)$

- $L_W(v)$ is the label of vertex $v$

Several watchers at the same vertex, each watcher checks its watching zone
Watching systems

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Watching number

- **watching number**, $w(G)$: minimum size of a watching system of $G$
- **minimum watching system**: watching system of cardinality $w(G)$

- Watching systems exist for all graphs
- $w(G) \leq i(G)$ if there exists at least an identifying code in $G$
- A watching system remains so if we add edges
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Example

$G = K_{1,6}: i(G) = 6, w(G) = 3$

$W = \{w_1, w_2, w_3\}, l(w_i) = 7$

$A(w_1) = \{1, 4, 5, 7\}, A(w_2) = \{2, 4, 6, 7\}, A(w_3) = \{3, 5, 6, 7\}$
General bounds of the watching number

- $w(G) \geq \lceil \log_2(n + 1) \rceil$
- Complete graphs, stars, graphs s.t. $\Delta = n - 1$ attain this bound
- $w(G) \geq \gamma(G)$
- $w(G) \leq \gamma(G) \lceil \log_2(\Delta + 2) \rceil$
- $w(G) \leq i(G)$, if $G$ is twin-free
- $w(G) \leq w(H)$ for any spanning subgraph $H$ of $G$
- $w(G) \leq \frac{2n}{3}$, if $G$ is a connected graph of order 3 or $\geq 5$ [Auger, Charon, Hudry, Lobstein, to appear]
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Watching number and identifying number of some families

\[ w(P_n) = \left\lceil \frac{n+1}{2} \right\rceil \quad i(P_n) = \left\lceil \frac{n+1}{2} \right\rceil \]

\[ w(C_n) = \begin{cases} 
3, & \text{if } n = 4; \\
\left\lceil \frac{n}{2} \right\rceil, & \text{otherwise.}
\end{cases} \quad i(C_n) = \begin{cases} 
3, & \text{if } n = 4, 5; \\
\frac{n}{2}, & \text{if } n \geq 6 \text{ even}; \\
\frac{n+3}{2}, & \text{if } n \geq 7 \text{ odd.}
\end{cases} \]
Complete bipartite graphs

$K_{r,s}$, $2 \leq r \leq s$

- $\gamma(K_{r,s}) = 2$
- $i(K_{r,s}) = r + s - 2$

$W = \{w_i : i \in [m]\}$ watching system in $K_{r,s}$

- $V(K_{r,s}) = V_1 \cup V_2$, $|V_1| = r$, $|V_2| = s$
- $\mathcal{L}(W) = \{l(w_i) : i \in [m]\} \subseteq V$
- $\mathcal{L}_1(W) = \mathcal{L}(W) \cap V_1$, $\mathcal{L}_2(W) = \mathcal{L}(W) \cap V_2$, 
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- \( \mathcal{L}_1(W) = \mathcal{L}(W) \cap V_1, \quad \mathcal{L}_2(W) = \mathcal{L}(W) \cap V_2, \)
Bounds

\[ w_0(r, s) = \lceil \log_2(r + s + 1) \rceil \]

Bounds:

- \[ w_0(r, s) \leq w(K_{r,s}) \leq \lceil \log_2 r \rceil + \lceil \log_2 s \rceil \]

Both bounds are tight:

- \[ w(K_{3,16}) = w_0(3, 16) = 5 \]
- \[ w(K_{8,11}) = \lceil \log_2 8 \rceil + \lceil \log_2 11 \rceil = 7 \]

Particular case:

- \[ w(K_{2,s}) = w_0(2, s) = \lceil \log_2(s + 3) \rceil \]
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Watching Systems in Complete Bipartite Graphs

Consider $K_{r,s}$, $2 \leq r \leq s$:

- If a watching system has 2 watchers at a same vertex, we obtain another watching system by placing one of them at another vertex of the same stable set.
- A watching system with all watchers located in the same stable set has size at least $\max\{r, \lceil \log_2(r + s + 1) \rceil \}$.
- A watching system with at least a watcher in each stable set has size $> w_0(r, s)$. 

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Attaining the lower bound

If $2 \leq r \leq s,$

- If $K_{r,s} \neq K_{5,5},$ $w(K_{r,s}) = w_0(r,s)$ if and only if $r \leq w_0(r,s).$
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Not attaining the lower bound

If $r > w_0(r, s)$,

- There is a minimum watching system $W$ satisfying $|L_1(W)| \geq |L_2(W)|$
- $w(K_{r,s}) = \min \{m : m = h+k, r \leq k+2^h-1, s \leq h+2^k-1\}$
- If $6 \leq r = s$, then $w(K_{r,r}) \neq w_0(r, r)$
- For each $r \geq 3$, there is a minimum watching system of $K_{r,r}$ such that $0 \leq |L_1(W)| - |L_2(W)| \leq 1$
- For each $r \geq 3$, if $n_h = h+2^h$,

$$w(K_{r,r}) = \begin{cases} 
2h, & \text{if } n_{h-1} < r < n_h \text{ for some } h \geq 2; \\
2h+1, & \text{if } r = n_h \text{ for some } h \geq 2.
\end{cases}$$
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  2h + 1, & \text{if } r = n_h \text{ for some } h \geq 2.
  \end{cases}$$
Feasible values

\[ w(K_{r,s}) = w_0(r, s), \text{ if } r \leq w_0(r, s); \]
\[ w_0(r, s) \leq w(K_{r,s}) \leq r, \text{ if } r > w_0(r, s). \]

Given \( a, b, c \) with \( 2 \leq a \leq b \leq c \), find \( r, s \), such that \( 2 \leq r \leq s \) and \( w_0(K_{r,s}) = a, w(K_{r,s}) = b, \max\{r, w_0(r, s)\} = c \).
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Feasible values

Existence of $r, s$ such that $w_0(K_{r,s}) = a$, $w(K_{r,s}) = b$, and $\max\{r, w_0(r, s)\} = c$:

- If $2 \leq a = b = c$, a solution is $r = a$ and $s = 2^a - a - 1$
- If $2 \leq a = b < c$, there is no solution
- If $2 \leq a < b = c$, there is solution if and only if $a \geq \log_2(2^{c-3} + c + 3)$.
- If $2 \leq a < b < c$, if there is a solution, then $a + \lceil \log_2(c - a + 3) \rceil - 2 \leq b \leq a + \lceil \log_2(c - a + 1) \rceil$
Watching number of $K_{r,s}$

$w(K_{5,5}) = 4$, and for $s \geq r \geq 3$, not both equal to 5:

\[
\begin{align*}
  w(K_{r,s}) &= w_0, & \text{if } r \leq w_0; \\
  w(K_{r,s}) &= w_0 + 1, & \text{if } r = w_0 + 1; \\
  w(K_{r,s}) &\in \{w_0 + 1, w_0 + 2\}, & \text{if } r = w_0 + 2; \\
  w(K_{r,s}) &\in \{w_0 + \lceil \log_2(r - w_0 + 1) \rceil, \\
  &\quad w_0 + \lceil \log_2(r - w_0 + 2) \rceil - 1, \\
  &\quad w_0 + \lceil \log_2(r - w_0 + 3) \rceil - 2\} & \text{if } r \geq w_0 + 3.
\end{align*}
\]

The identifying number of the complete bipartite graph $K_{r,s}$ is $r + s - 2$!
Summary

- Watching systems as an extension of identifying codes
  - Watching systems exist in all graphs
  - $w(G) \leq i(G)$ if $G$ has at least an identifying code

- Watching systems and watching number of complete bipartite graphs

- Open problems
  - Watching number in bipartite graphs and other families
  - Graphs with minimum watching number
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