Locating domination in bipartite graphs and their complements

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Abstract. A set $S$ of vertices of a graph $G$ is distinguishing if the sets of neighbors in $S$ for every pair of vertices not in $S$ are distinct. A locating-dominating set of $G$, LD-set for short, is a dominating distinguishing set. The location-domination number of $G$, $\lambda(G)$, is the minimum cardinality of an LD-set. In this work we study relationships between $\lambda(G)$ and $\lambda(G)$ for bipartite graphs. The main result is the characterization of connected bipartite graphs $G$ satisfying $\lambda(\overline{G}) = \lambda(G) + 1$. To this aim, we define an edge-labeled graph $G^S$ associated with a distinguishing set $S$ that turns out to be very helpful.

Keywords. Locating-domination, bipartite graphs, complement, distinguishing sets

1 Introduction

Let $G = (V, E)$ be a simple, finite graph. The neighborhood of a vertex $u \in V$ is $N(u) = \{v : uv \in E\}$. We say that two vertices $u$ and $v$ are twins if $N(u) = N(v)$ or $N(u) \cup \{u\} = N(v) \cup \{v\}$. A set $S \subseteq V$ is distinguishing if $N(u) \cap S \neq N(v) \cap S$ for every pair of different vertices $u, v \notin S$. A locating-dominating set, LD-set for short, is a distinguishing dominating set. The location-domination number of $G$, denoted by $\lambda(G)$, is the minimum cardinality of a locating-dominating set. A locating-dominating set of cardinality $\lambda(G)$ is called an LD-code [3]. LD-codes and the location-domination parameter have been intensively studied during the last decade (see [2] and its references).

The complement of $G$, denoted by $\overline{G}$, is the graph on the same vertices such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. This work is devoted to approach the relationship between $\lambda(G)$ and $\lambda(\overline{G})$ for connected bipartite graphs. It follows immediately from the definitions that a set $S \subseteq V$ is distinguishing in $G$ if and only if it is distinguishing in $\overline{G}$. A straightforward consequence of this fact are the following results.

© Partially supported by projects MTM2014-60127-P, MTM2015-63791-R, Gen. Cat. DGR 2014SGR46. This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 734922.
Proposition 1 ([2]). If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$, then $S$ is an LD-set of $\overline{G}$ if and only if $S$ is a dominating set of $\overline{G}$;

Proposition 2 ([1]). For every graph $G$, $|\lambda(G) - \lambda(\overline{G})| \leq 1$.

According to the preceding inequality, $\lambda(\overline{G}) \in \{\lambda(G) - 1, \lambda(G), \lambda(G) + 1\}$ for every graph $G$, all cases being feasible for some connected graph $G$. We intend to determine graphs such that $\lambda(\overline{G}) > \lambda(G)$, that is, we want to solve the equation $\lambda(\overline{G}) = \lambda(G) + 1$. This problem was completely solved in [2] for the family of block-cactus. In this work, we carry out a similar study for bipartite graphs. For this purpose, we first introduce in Section 2 the graph associated with a distinguishing set. This graph turns out to be very helpful to derive some properties related to LD-sets and the location-domination number of $G$, and will be used to state the main results in Section 3.

2 The graph associated with a distinguishing set

Let $G = (V, E)$ be a graph of order $n$ and let $S \subseteq V$ be a distinguishing set of $G$.

Definition 1. The graph associated with $S$, denoted by $G^S$, is the edge-labeled graph defined as follows:

i) $V(G^S) = V \setminus S$;

ii) If $x, y \in V(G^S)$, then $xy \in E(G^S)$ if and only if the sets of neighbors of $x$ and $y$ in $S$ differ in exactly one vertex $u(x, y) \in S$;

iii) The label $\ell(xy)$ of edge $xy \in E(G^S)$ is the only vertex $u(x, y) \in S$ described in the preceding item.

We can represent the graph $G^S$ with the vertices lying on $|S| + 1$ levels, from bottom (level 0) to top (level $|S|$), in such a way that vertices with exactly $k$ neighbors in $S$ are at level $k$. There are at most $\binom{|S|}{k}$ vertices at level $k$. An edge of $G^S$ has its endpoints at consecutive levels. Moreover, if $e = xy \in E(G^S)$, with $\ell(e) = u \in S$, and $x$ is at exactly one level higher than $y$, then the neighborhood of $x$ is $S$ obtained by adding vertex $u$ to the neighborhood of $y$ is $S$. Hence, $x$ and $y$ have the same neighborhood in $S \setminus \{u\}$. Therefore, the existence of an edge in $G^S$ with label $u \in S$ means that $S \setminus \{u\}$ is not a distinguishing set. The converse is not necessarily true: it is possible that $S \setminus \{u\}$ were not distinguishing because it does not distinguish $u$ from another vertex not in $S$. See Figure 1.1 for an example of an LD-set-associated graph. Next proposition states some properties of this graph.
Proposition 3. Let $S \subseteq V$ be a distinguishing set of size $r$ of a graph $G$ of order $n$. Then, the graph $G^S$ satisfies the following conditions.

i) $G^S$ is a bipartite graph of order $n - r$ and incident edges have different labels.

ii) If $xy \in E(G^S)$ and $\ell(xy) = u \in S$, then $x$ and $y$ are not distinguished by $S \setminus \{u\}$.

iii) All the vertices belonging to a connected component of the subgraph of $G^S$ induced by the edges with label in $S'$ have the same neighborhood in $S \setminus S'$.

item[iv)] The set $S$ is a distinguishing set of $\overline{G}$ and $G^S = \overline{G}^S$. Moreover, the representation by levels of the graph $\overline{G}^S$ is obtained by turning upside down the representation of the graph $G^S$.

3 The bipartite case

This section is devoted to solve the equation $\lambda(\overline{G}) = \lambda(G) + 1$ when we restrict ourselves to bipartite graphs. As we have seen in Section 1, $\lambda(\overline{G}) \in \{\lambda(G) - 1, \lambda(G), \lambda(G) + 1\}$, and for every pair $r, s$, $3 \leq r \leq s$, it is easy to give bipartite graphs such that $\lambda(\overline{G}) = \lambda(G) - 1$ (the bi-star $K_{2(r,s)}$) and bipartite graphs such that $\lambda(\overline{G}) = \lambda(G)$ (the biclique $K_{r,s}$). We want to analyze now the case...
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$\lambda(G) = \lambda(G) + 1$. In the sequel, $G = (V, E)$ stands for a bipartite connected graph of order $n \geq 3$, being $U$ and $W$ their stable sets and $1 \leq |U| = r \leq s = |W|$.

**Proposition 4.** $G$ satisfies $\lambda(G) \leq \lambda(G)$ if any of the following conditions holds:

i) $1 \leq r \leq 2$ or $3 \leq r = s$;

ii) $G$ has an LD-code with vertices at both stable sets;

iii) $r < s$ and $W$ is an LD-code of $G$.

**Corollary 1.** If $\lambda(G) = \lambda(G) + 1$, then $U$ is the only LD-code of $G$ and $3 \leq r < s \leq 2^r - 1$.

**Theorem 1.** Let $3 \leq r < s$. Then, $\lambda(G) = \lambda(G) + 1$ if and only if the following conditions hold:

i) $W$ has no twins;

ii) There exists a vertex $w \in W$ such that $N(w) = U$;

iii) For every vertex $u \in U$, the graph $G^U$ associated with $U$ has at least two edges with label $u$.

By Proposition 3, and taking into account that $G$ is a bipartite graph, the third condition of the preceding theorem is equivalent to the existence of at least two pairs of twins in $W$ in the graph $G - u$, for every vertex $u \in U$.

**Corollary 2.** If $r \geq 3$ and $\lambda(G) = \lambda(G) + 1$, then $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$.

**Theorem 2.** For every pair $(r, s)$, $r, s \in \mathbb{N}$, such that $3 \leq r$ and $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$, there exists a bipartite graph $G(r, s)$ such that $\lambda(G) = \lambda(G) + 1$.

**References**

