Extremal Graph Theory for Metric Dimension and Diameter

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Abstract

Let $\mathcal{G}_{\beta,D}$ be the set of graphs with metric dimension $\beta$ and diameter $D$. The first contribution is to characterize the graphs in $\mathcal{G}_{\beta,D}$ with order $\beta + D$ for all values of $\beta$ and $D$. The second contribution is to determine the maximum order of a graph in $\mathcal{G}_{\beta,D}$ for all values of $D$ and $\beta$. Only a weak upper bound was previously known.

Keywords: Graph, resolving set, metric dimension, metric basis, diameter, order.

1 Introduction

Let $G$ be a connected graph. A vertex $x \in V(G)$ resolves a pair of vertices $v, w \in V(G)$ if $\text{dist}(v, x) \neq \text{dist}(w, x)$. A set of vertices $S \subseteq V(G)$ resolves $G$, and $S$ is a resolving set of $G$, if every pair of distinct vertices of $G$ are resolved

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by some vertex in $S$. Informally, $S$ resolves $G$ if every vertex of $G$ is uniquely determined by its vector of distances to the vertices in $S$. A resolving set $S$ of $G$ with the minimum cardinality is a metric basis of $G$, and $\beta(G) := |S|$ is the metric dimension of $G$. Resolving sets in general graphs were first defined by Slater [7] and Harary and Melter [4]. Resolving sets have since been widely investigated [2,5,9], and arise in diverse areas including coin weighing problems [8], network discovery and verification [1], robot navigation [5], connected joins in graphs [6], and strategies for the Mastermind game [3]. For non-negative integers $\beta$ and $D$, let $G_{\beta,D}$ be the class of connected graphs with metric dimension $\beta$ and diameter $D$. Consider the following two extremal questions: (1) What is the minimum order of a graph in $G_{\beta,D}$? (2) What is the maximum order of a graph in $G_{\beta,D}$?

The first question was independently answered by Yushmanov [9], Khuller et al. [5], and Chartrand et al. [2], who proved that the minimum order of a graph in $G_{\beta,D}$ is $\beta + D$. Thus it is natural to consider the following problem: Characterize the graphs in $G_{\beta,D}$ with order $\beta + D$. Such a characterization is simple for $\beta = 1$. In particular, Khuller et al. [5] and Chartrand et al. [2] independently proved that paths $P_n$ (with $n \geq 2$ vertices) are the only graphs with metric dimension 1. Thus $G_{1,D} = \{P_{D+1}\}$. The characterization is again simple at the other extreme with $D = 1$. In particular, Chartrand et al. [2] proved that the complete graph $K_n$ (with $n \geq 1$ vertices) is the only graph with metric dimension $n-1$ (see Proposition 2.1). Thus $G_{\beta,1} = \{K_{\beta+1}\}$. Chartrand et al. [2] studied the case $D = 2$, and obtained a non-trivial characterization of graphs in $G_{\beta,2}$ with order $\beta + 2$ (see Proposition 2.2). The first contribution of this paper is to characterize the graphs in $G_{\beta,D}$ with order $\beta + D$ for all values of $\beta \geq 1$ and $D \geq 3$, thus completing the characterization for all values of $D$. This result is stated and proved in Section 2. We then study the second question above: What is the maximum order of a graph in $G_{\beta,D}$? Previously, only a weak upper bound was known. In particular, Khuller et al. [5] and Chartrand et al. [2] independently proved that every graph in $G_{\beta,D}$ has at most $D^\beta + \beta$ vertices. This bound is tight only for $D \leq 3$ or $\beta = 1$. Our second contribution is to determine the (exact) maximum order of a graph in $G_{\beta,D}$ for all values of $D$ and $\beta$. This result is stated and proved in Section 3.

2 Graphs with minimum order

Twin vertices. Let $u$ be a vertex of a graph $G$. The open neighborhood of $u$ is $N(u) := \{v \in V(G) : uv \in E(G)\}$, and the closed neighborhood of $u$ is $N[u] := N(u) \cup \{u\}$. Two distinct vertices $u, v$ are adjacent twins if $N[u] = N[v]$,
and non-adjacent twins if \( N(u) = N(v) \). Observe that if \( u, v \) are adjacent twins then \( uv \in E(G) \), and if \( u, v \) are non-adjacent twins then \( uv \notin E(G) \); thus the names are justified. If \( u, v \) are adjacent or non-adjacent twins, then \( u, v \) are twins. A consequence of the definitions is that if \( u, v \) are twins in a connected graph \( G \), then \( \text{dist}(u, x) = \text{dist}(v, x) \) for every vertex \( x \in V(G) \setminus \{u, v\} \). This implies that if \( u, v \) are twins in a connected graph \( G \) and \( S \) resolves \( G \), then \( u \) or \( v \) is in \( S \). Moreover, if \( u \in S \) and \( v \notin S \), then \( (S \setminus \{u\}) \cup \{v\} \) resolves \( G \).

For a graph \( G \), a set \( T \subseteq V(G) \) is a twin-set of \( G \) if \( v, w \) are twins in \( G \) for every pair of distinct vertices \( v, w \in T \). It is easy to prove that if \( T \) is a twin-set of a graph \( G \), then either every pair of vertices in \( T \) are adjacent twins, or every pair of vertices in \( T \) are non-adjacent twins. If \( T \) is a twin-set of a connected graph \( G \) with \( |T| \geq 3 \), it can be proved that \( \beta(G) = \beta(G \setminus S) + |S| \) for every subset \( S \subseteq T \) with \( |S| \leq |T| - 2 \).

The Twin Graph. Let \( G \) be a graph. Define a relation \( \equiv \) on \( V(G) \) by \( u \equiv v \) if and only if \( u = v \) or \( u, v \) are twins. \( \equiv \) is an equivalence relation. For each vertex \( v \in V(G) \), let \( v^* \) be the set of vertices of \( G \) that are equivalent to \( v \) under \( \equiv \). Let \( \{v^*_1, \ldots, v^*_k\} \) be the partition of \( V(G) \) induced by \( \equiv \), where each \( v_i \) is a representative of the set \( v^*_i \). The twin graph of \( G \), denoted by \( G^* \), is the graph with vertex set \( V(G^*) := \{v^*_1, \ldots, v^*_k\} \), where \( v^*_i v^*_j \in E(G^*) \) if and only if \( v_i v_j \in E(G) \). Two vertices \( v^* \) and \( w^* \) of \( G^* \) are adjacent if and only if every vertex in \( v^* \) is adjacent to every vertex in \( w^* \) in \( G \). Each vertex \( v^* \) of \( G^* \) is a maximal twin-set of \( G \). \( G[v^*] \) is a complete graph if the vertices of \( v^* \) are adjacent twins, or \( G[v^*] \) is a null graph if the vertices of \( v^* \) are non-adjacent twins. So it makes sense to consider the following types of vertices in \( G^* \). We say that \( v^* \in V(G^*) \) is of type: (i) \( (1) \) if \( |v^*| = 1 \); (ii) \( (K) \) if \( G[v^*] \cong K_r \) and \( r \geq 2 \); (iii) \( (N) \) if \( G[v^*] \cong N_r \) and \( r \geq 2 \); where \( N_r \) is the null graph with \( r \) vertices and no edges. A vertex of \( G^* \) is of type \( (1K) \) if it is of type \( (1) \) or \( (K) \). A vertex of \( G^* \) is of type \( (1N) \) if it is of type \( (1) \) or \( (N) \). Observe that the graph \( G \) is uniquely determined by \( G^* \), and the type and cardinality of each vertex of \( G^* \). In particular, if \( v^* \) is adjacent to \( w^* \) in \( G^* \), then every vertex in \( v^* \) is adjacent to every vertex in \( w^* \) in \( G \). If \( G \) is a graph with \( \text{diam}(G) \geq 3 \) then \( \text{diam}(G) = \text{diam}(G^*) \). Theorem 2.3 below characterizes the graphs in \( G_{\beta,D} \) for \( D \geq 3 \) in terms of the twin graph. Chartrand et al. [2] characterized the graphs in \( G_{\beta,D} \) for \( D \leq 2 \). For consistency with Theorem 2.3, we describe the characterization by Chartrand et al. [2] in terms of the twin graph.

Proposition 2.1 ([2]) The following are equivalent for \( G \) with \( n \) vertices: i) \( G \) has metric dimension \( \beta(G) = n - 1 \); ii) \( G \cong K_n \); iii) \( \text{diam}(G) = 1 \); (iv) the twin graph \( G^* \) has one vertex, which is of type \( (1K) \).
Proposition 2.2 ([2]) The following are equivalent for $G$ with $n \geq 3$ vertices: i) $G$ has metric dimension $\beta(G) = n - 2$; ii) $G$ has metric dimension $\beta(G) = n - 2$ and diameter $\text{diam}(G) = 2$; iii) the twin graph $G^*$ of $G$ satisfies: a) $G^* \cong P_2$ with at least one vertex of type $(N)$, or b) $G^* \cong P_3$ with one leaf of type (1), the other leaf of type $(1K)$, and the degree-2 vertex of type $(1K)$.

To describe our characterization we introduce the following notation. Let $P_{D+1} = (u_0, u_1, \ldots, u_D)$ be a path of length $D$. For $k \in [3, D - 1]$ let $P_{D+1,k}$ be the graph obtained from $P_{D+1}$ by adding one vertex adjacent to $u_{k-1}$. For $k \in [2, D - 1]$ let $P'_{D+1,k}$ be the graph obtained from $P_{D+1}$ by adding one vertex adjacent to $u_{k-1}$ and $u_k$.

Theorem 2.3 Let $G$ be a connected graph of order $n$ and diameter $D \geq 3$. Let $G^*$ be the twin graph of $G$. Let $\alpha(G^*)$ be the number of vertices of $G^*$ of type $(K)$ or $(N)$. Then $\beta(G) = n - D$ if and only if $G^*$ is one of the following graphs:

(i) $G^* \cong P_{D+1}$ and one of the following cases hold:
   (a) $\alpha(G^*) \leq 1$;
   (b) $\alpha(G^*) = 2$, the two vertices of $G^*$ not of type (1) are adjacent, and if one is a leaf of type (K) then the other is also of type (K);
   (c) $\alpha(G^*) = 2$, the two vertices of $G^*$ not of type (1) are at distance 2 and both are of type (N); or
   (d) $\alpha(G^*) = 3$ and there is a vertex of type (N) or (K) adjacent to two vertices of type (N).

(ii) $G^* \cong P_{D+1,k}$ for some $k \in [3, D - 1]$, the degree-3 vertex $u^*_k$ of $G^*$ is any type, each neighbour of $u^*_k$ is type $(1N)$, and every other vertex is type (1).

(iii) $G^* \cong P'_{D+1,k}$ for some $k \in [2, D - 1]$, the three vertices in the cycle are of type $(1K)$, and every other vertex is of type (1).

3 Graphs with maximum order

Theorem 3.1 For all integers $D \geq 2$ and $\beta \geq 0$, the maximum order of a connected graph with diameter $D$ and metric dimension $\beta$ is

$$m(D, \beta) = \left(\left\lfloor \frac{2D}{3} \right\rfloor + 1\right)^\beta + \beta \sum_{i=1}^{\left\lfloor D/3 \right\rfloor} (2i - 1)^{\beta - 1}.$$  

(1)

Lemma 3.2 For every graph $G \in \mathcal{G}_{\beta,D}$, $|V(G)| \leq m(D, \beta)$.

To prove the lower bound in Theorem 3.1 we construct a graph $G \in \mathcal{G}_{\beta,D}$
with as many vertices as in Equation (1). Let \( A = \lceil D/3 \rceil, B = \lceil D/3 \rceil + \lfloor D/3 \rfloor, \) and \( Q = \{(x_1, \ldots, x_\beta) : A \leq x_i \leq D, i \in [1, \beta] \}. \) For each \( i \in [1, \beta] \) and \( r \in [0, A-1], \) let \( P_{i,r} = \{(x_1, \ldots, x_{i-1}, r, x_{i+1}, \ldots, x_\beta) : x_j \in [B-r, B+r], j \neq i \}. \) Let \( P_i = \bigcup \{P_{i,r} : r \in [0, A-1] \} \) and \( P = \bigcup \{P_i : i \in [1, \beta] \}. \) Let \( G \) be the graph with \( V(G) = Q \cup P, \) where \((x_1, \ldots, x_\beta) \) and \((y_1, \ldots, y_\beta) \) in \( V(G) \) are adjacent if and only if \(|y_i - x_i| \leq 1 \) for each \( i \in [1, \beta]. \) Let \( S = \{v_1, \ldots, v_\beta\}, \) where \( v_i = (t, \ldots, t, 0, t, \ldots, t) \in P_i. \) For all \( D, \beta > 0, |V(G)| = m(D, \beta). \)

**Lemma 3.3** For all vertices \( x = (x_1, \ldots, x_\beta) \) and \( y = (y_1, \ldots, y_\beta) \) of \( G, \) \( \text{dist}(x, y) = \max\{|y_i - x_i| : i \in [1, \beta]\} \leq D. \)

**Lemma 3.4** For every \( x = (x_1, \ldots, x_\beta) \) and for each \( v_i \in S \) \( \text{dist}(x, v_i) = x_i. \)

By Lemma 3.3 \( \text{diam}(G) = D. \) By Lemma 3.4 \( S \) resolves \( G. \) If the metric dimension of \( G \) is \(< |S| = \beta \) then, by Lemma 3.2, we get a contradiction.

**References**


