Quasiperfect Domination in Trees

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Joint work with José Cáceres, Carmen Hernando, Mercè Mora and Maria Luz Puertas.
1. QP-dominating codes and the QP-chain
2. Short QP-chains
3. Trees
1. QP-dominating codes and the QP-chain

2. Short QP-chains

3. Trees
A set \( D \subset V(G) \) of a graph \( G \) is a dominating set if every vertex \( u \) not in \( D \) has at least a neighbour in \( D \), i.e., \( N(u) \cap D \neq \emptyset \).

The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality of a dominating set of \( G \).

A dominating set of cardinality \( \gamma(G) \) is called a \( \gamma \)-code.

\( \gamma(P) = 3 \), since red vertices form a \( \gamma \)-code.
A set $D \subset V(G)$ of a graph $G$ is a *dominating set* if every vertex $u$ not in $D$ has at least a neighbour in $D$, i.e., $N(u) \cap D \neq \emptyset$. The *domination number* of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A dominating set of cardinality $\gamma(G)$ is called a $\gamma$-code.
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A set $D \subset V(G)$ of a graph $G$ is a **dominating set** if every vertex $u$ not in $D$ has **at least** a neighbour in $D$, i.e., $N(u) \cap D \neq \emptyset$.

The **domination number** of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$.

A dominating set of cardinality $\gamma(G)$ is called a **$\gamma$-code**.

$\gamma(P) = 3$, since red vertices form a $\gamma$-code.
A dominating set \( D \) is a perfect dominating set if every vertex \( u \) not in \( D \) has exactly one neighbour in \( D \), i.e., \( |N(u) \cap D| = 1 \).

The perfect domination number, denoted by \( \gamma_1 \), is the minimum cardinality of a perfect dominating set of \( G \).

A perfect dominating set of cardinality \( \gamma_1 \) is called a \( \gamma_1 \)-code.

\( \gamma_1(P) = 4 \), since red vertices form a \( \gamma_1 \)-code.
A dominating set $D$ is a **perfect dominating set** if every vertex $u$ not in $D$ has **exactly** one neighbour in $D$, i.e., $|N(u) \cap D| = 1$. 
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The **perfect domination number**, denoted by $\gamma_{11}(G)$, is the minimum cardinality of a perfect dominating set of $G$. 

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$\gamma_{11}(P) = 4$, since red vertices form a $\gamma_{11}$-code.
A dominating set $D \subset V(G)$ of a graph $G$ is a $k$-quasiperfect dominating set if every vertex of $V \setminus D$ has at most $k$ neighbours in $D$, i.e., for each $u \in V \setminus D$, $1 \leq |N(u) \cap D| \leq k$.

The $k$-quasiperfect perfect domination number, denoted $\gamma_1^k(G)$, is the minimum cardinality of a $k$-quasiperfect dominating set of $G$.

A $\gamma_1^k$-code is a $k$-QP dominating set of cardinality $\gamma_1^k(G)$.

Note that $n = 10$, $\Delta = 3$, $\gamma_1^1(P) = \gamma_1^2(P) = 4$, $\gamma_1^3(P) = \gamma_1(P) = 3$. 

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Quasiperfect Domination in Trees
A dominating set \( D \subset V(G) \) of a graph \( G \) is a \( k \)-quasiperfect dominating set if every vertex of \( V \setminus D \) has at most \( k \) neighbours in \( D \), i.e., for each \( u \in V \setminus D \), \( 1 \leq |N(u) \cap D| \leq k \).
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The $k$-quasiperfect perfect domination number, denoted $\gamma_{1k}(G)$, is the minimum cardinality of a $k$-quasiperfect dominating set of $G$. 
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Note that $n = 10$, $\Delta = 3$, $\gamma_{11}(P) = \gamma_{12}(P) = 4$, $\gamma_{13}(P) = \gamma(P) = 3$. 
A QP-chain is called SHORT if \(1 \leq |\Gamma(G)| \leq 2\). For example:

\[ \gamma_{11}(G) \geq \gamma_{12}(G) = \ldots = \gamma(G) \]
\( G \) is a graph of order \( n \) and maximum degree \( \Delta \).
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- **QP-sequence** of $G$: $\Gamma(G) = \{\gamma_{1_i}(G)\}_{i=1}^{\Delta}$

- A **QP-chain** is called **short** if $1 \leq |\Gamma(G)| \leq 2$. For example: $\gamma_{11}(G) \geq \gamma_{12}(G) = \ldots = \gamma(G)$

- A short QP-chain is called **constant** if $\gamma_{11}(G) = \gamma_{12}(G) = \ldots = \gamma(G)$
- $G$ is a graph of order $n$ and maximum degree $\Delta$.
- **QP-sequence** of $G$: $\Gamma(G) = \{\gamma_{1i}(G)\}_{i=1}^{\Delta}$
- **QP-chain** of $G$:

\[
\gamma_{11}(G) \geq \gamma_{12}(G) \geq \ldots \geq \gamma_{1\Delta}(G) = \gamma(G)
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**QP-sequence** of \( G \):

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A QP-chain is called *SHORT* if \( 1 \leq |\Gamma(G)| \leq 2 \). For example:

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\gamma_{11}(G) \geq \gamma_{12}(G) = \ldots = \gamma(G)
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A short QP-chain is called *CONSTANT* if

\[
\gamma_{11}(G) = \gamma_{12}(G) = \ldots = \gamma(G)
\]
1. QP-dominating codes and the QP-chain

2. Short QP-chains

3. Trees
### QP-chain of some basic graph families

<table>
<thead>
<tr>
<th>$G$</th>
<th>$P_n$</th>
<th>$C_n$</th>
<th>$K_n$</th>
<th>$K_{1,n-1}$</th>
<th>$K_{p,n-p}$</th>
<th>$W_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta(G)$</td>
<td>2</td>
<td>2</td>
<td>$n-1$</td>
<td>$n-1$</td>
<td>$n-p$</td>
<td>$n-1$</td>
</tr>
<tr>
<td>$\gamma_{11}(G)$</td>
<td>$\left\lceil \frac{n}{3} \right\rceil$</td>
<td>$\left\lfloor \frac{2n}{3} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$\gamma_{12}(G)$</td>
<td>$\left\lceil \frac{n}{3} \right\rceil$</td>
<td>$\left\lfloor \frac{n}{3} \right\rfloor$</td>
<td>1</td>
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<td>2</td>
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</table>

All of these graph families have a constant QP-chain, except cycles $C_{3k+2}$ whose QP-chain is short.
More graph families with a short QP-chain
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\[ \Delta = n - 1 \text{. Every graph satisfies: } \gamma_{11} = \gamma = 1. \]
More graph families with a short QP-chain

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Every graph satisfies: \( \gamma_{11} = \gamma = 1 \).

\( \{u\} \) is a \( \gamma_{11} \)-code of \( G = K_1 \lor H \).
More graph families with a short QP-chain
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\[ \Delta = n - 2 \]. Every graph satisfies: \( \gamma_{12}(G) = \gamma(G) = 2 \). Moreover:
More graph families with a short QP-chain

1. $\Delta = n - 2$. Every graph satisfies: $\gamma_{12}(G) = \gamma(G) = 2$. Moreover:

2. If $n \geq 6$ and $2 \leq k \leq n$, then there exists a graph $G$ of order $n$ s.t. $\Delta(G) = n - 2$, $\gamma_{11}(G) = k$. 
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- If $n \geq 6$ and $2 \leq k \leq n$, then there exists a graph $G$ of order $n$ s.t. $\Delta(G) = n - 2$, $\gamma_{11}(G) = k$.

- Case $k = n$: Take $G = P_{n-2} \vee \bar{K}_2$

\[ \{u, w\} \text{ is a } \gamma\text{-code and the unique } \gamma_{11}\text{-set is the whole graph.} \]
More graph families with a short QP-chain
More graph families with a short QP-chain

\[ \Delta = n - 3 \].

Every graph satisfies: \( \gamma_{12}(G) = \gamma(G) \in \{2, 3\} \).

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More graph families with a short QP-chain

- \[ \Delta = n - 3 \]. Every graph satisfies: \( \gamma_{12}(G) = \gamma(G) \in \{2, 3\} \).

Moreover:

- If \( n \geq 7 \) and \( 2 \leq k \leq n \), then there exists a graph \( G \) of order \( n \) and \( \gamma(G) = 2 \) s.t. \( \Delta(G) = n - 3 \), \( \gamma_{11}(G) = k \).
More graph families with a short QP-chain

- $\Delta = n - 3$. Every graph satisfies: $\gamma_{12}(G) = \gamma(G) \in \{2, 3\}$. Moreover:

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- If $n \geq 9$ and $3 \leq k \leq n$, then there exists a graph $G$ of order $n$ and $\gamma(G) = 3$ s.t. $\Delta(G) = n - 3$, $\gamma_{11}(G) = k$. 
More graph families with a short QP-chain
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\[ P_4 \text{-free graphs} \]. Every cograph satisfies: \( \gamma_{12}(G) = \gamma(G) \).
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More graph families with a short QP-chain

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Moreover:
- $\gamma_{11}(G) \in \{1, 2, n\}$
- $\gamma_{11}(G) = 2 \iff G$ is as in Figure.

Figure: $H = H_1 \vee H_2$, $N_G(u_1) = V(H_1) + u_2$, $N_G(u_2) = V(H_2) + u_1$. 
More graph families with a short QP-chain

Let $h, k, n$ be integers such that $2 \leq h \leq k$ and $h \leq \frac{n}{2}$. Then, there exists a claw-free graph $G$ s.t. $\left[\gamma(G), \gamma_1(G), |V(G)|\right] = [h, k, n]$ if at least one of the following conditions holds:

1. $4 \leq n \leq 7$: $3h + k < 2n$ or $[h, k, n] \in \{[2, 6, 6], [3, 3, 6]\}$ (converse true).
2. $h + k \leq n$.
3. $3h + k + 1 \leq 2n$.

Open Problem: $3 \leq h < k$, $2n \leq 3h + k$.
More graph families with a short QP-chain

\[ K_{1,3} \]-free graphs. Every claw-free graph satisfies: \( \gamma_{12}(G) = \gamma(G) \).

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More graph families with a short QP-chain

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More graph families with a short QP-chain

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1. QP-dominating codes and the QP-chain

2. Short QP-chains

3. Trees
If $T$ is a tree, then $\gamma_1^k(T)$ can be computed in linear time.

For $k \geq 1$:

$$\gamma_1^k(T) \leq \gamma(T) + \left\lceil \gamma(T) \frac{k}{k} \right\rceil - 1$$

Sketch of proof:

Take $S$, a $\gamma$-code of $T$. Assume $S$ is not a $\gamma_1^k$-set of $T$.

Let $r$ be the number of the components of $T[S]$: $\gamma(T) \geq r > k$.

Every $v \not\in S$ has at most one neighbor in each component of $T[S]$.

Take $x_0 \not\in S$ with at least $k + 1$ neighbors in $S$.

Take $S_1 = S + x_0$. $T[S_1]$ has at most $r - k$ components.

If $S_1$ is not a $\gamma_1^k$-set of $T$, goto $\Delta \Delta$.

After at most $j = \left\lceil \frac{r - k}{k} \right\rceil$ iterations, $S_j$ is a $\gamma_1^k$-set of $T$.

$$\gamma_1^k(T) \leq |S_j| = |S| + j \leq \gamma(T) + \left\lceil \gamma(T) - \frac{k}{k} \right\rceil \leq \gamma(T) + \left\lceil \gamma(T) - k \right\rceil$$
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- Take $S$, a $\gamma$-code of $T$. Assume $S$ is not a $\gamma_{1k}$-set of $T$.
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- Take $x_0 \notin S$ with at least $k + 1$ neighbors in $S$. 

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- Every $v \notin S$ has at most one neighbor in each component of $T[S]$.
- Take $x_0 \notin S$ with at least $k + 1$ neighbors in $S$.
- Take $S_1 = S + x_0$. $T[S_1]$ has at most $r - k$ components.
- If $S_1$ is not a $\gamma_{1k}$-set of $T$, goto $\triangleright\triangleright$.
- After at most $j = \left\lceil \frac{r-k}{k} \right\rceil$ iterations, $S_j$ is a $\gamma_{1k}$-set of $T$. 
If $T$ is a tree, then $\gamma_{1k}(T)$ can be computed in linear time.

$k \geq 1:\ \ \ \ \ \gamma_{1k}(T) \leq \gamma(T) + \left\lceil \frac{\gamma(T)}{k} \right\rceil - 1$

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- After at most $j = \left\lceil \frac{r-k}{k} \right\rceil$ iterations, $S_j$ is a $\gamma_{1k}$-set of $T$.
- $\gamma_{1k}(T) \leq |S_j| = |S| + j \leq \gamma(T) + \left\lceil \frac{r-k}{k} \right\rceil \leq \gamma(T) + \left\lceil \frac{\gamma(T)-k}{k} \right\rceil$
$T$ is a tree.

$\gamma_1(T) \leq \gamma(T) + \lceil \gamma(T) \rceil - 1$ is tight.

Sketch of proof:
If $k \geq \gamma(T)$, then $\gamma_1(T) = \gamma(T)$.
If $k < \gamma(T) = a = q \cdot k + r$, with $1 \leq r \leq k$, then take this tree:
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![Tree Diagram]

**Figure:** Squared vertices are a $\gamma$-code and black vertices are a $\gamma_{1k}$-code.
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Sketch of proof:

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Figure: Squared vertices are a $\gamma$-code and black vertices are a $\gamma_{1k}$-code

\[
\gamma(T) + \left\lceil \frac{\gamma(T)}{k} \right\rceil - 1 = a + \left\lceil \frac{q \cdot k + r}{k} \right\rceil - 1 = a + q + 1 - 1 = q \cdot k + r + q = \gamma_{1k}(T)
\]
A tree $T$ satisfies $\gamma(T) \leq \gamma_{11}(T) \leq 2\gamma(T) - 1$ iff:

1. The set $S$ of strong support vertices of $T$ is an independent dominating set, and
2. Every vertex of any component $C$ of $T - (S \cup L)$ has exactly one neighbor in $S$ except one vertex that has two neighbors in $S$. 

$\gamma(T)$ is the domination number, $\gamma_{11}(T)$ is the quasiperfect domination number, and $\gamma(T)$ is the upper domination number.
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Two realization theorems

Let $a, b, n$ be integers such that $2 \leq a \leq b \leq 2a - 1$ and $n > 2b$:

 большим

There exists a tree $T$ of order $n$ s.t. $\gamma(T) = a$ and $\gamma_1(T) = b$.

There exists a tree with maximum degree $\Delta$ satisfying each one of the $2\Delta - 1$ possible combinations of the QP-chain.
Two realization theorems

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\[ \gamma_{11}(T) \geq \gamma_{12}(T) \geq \cdots \geq \gamma_{1\Delta}(T) = \gamma(T) \]
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\( \Delta \geq 3 \):
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Two tree families with a short QP-chain

★ A *caterpillar* is a tree s.t. the removal of all its leaves gives rise to a path.
★ A *k-ary tree* is a rooted tree such that each vertex has at most $k$ children.
★ A *full k-ary tree* is a $k$-ary tree such that all vertices that are not leaves have exactly $k$ children.
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If $T$ is a caterpillar, then:

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▷ If $T$ is full $k$-ary tree, then:

\[ n - \ell(T) = \gamma_{11}(T) = \cdots = \gamma_{1,k-1}(T) > \gamma_{1,k}(T) = \gamma_{1,k+1}(T) = \gamma(T) \]


Bibliography

GRACIAS/OBRIGADO/THANKS


