Locating-Dominating Partitions in Graphs

Ignacio M. Pelayo

Universitat Politècnica de Catalunya, Barcelona, Spain

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Joint work with Carmen Hernando and Mercè Mora.
A set \( D \subset V(G) \) of a graph \( G \) is a dominating set if every vertex \( u \) not in \( D \) has at least a neighbor in \( D \), i.e., \( N(u) \cap D \neq \emptyset \).

The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality of a dominating set of \( G \).

A dominating set of cardinality \( \gamma(G) \) is called a \( \gamma \)-code.

\[ \gamma(G) = 2 \] (red vertices form a \( \gamma \)-code).
A set $D \subseteq V(G)$ of a graph $G$ is a **dominating set** if every vertex $u$ not in $D$ has **at least** a neighbor in $D$, i.e., $N(u) \cap D \neq \emptyset$. The **domination number** of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A dominating set of cardinality $\gamma(G)$ is called a $\gamma$-code. $\gamma(G) = 2$ (red vertices form a $\gamma$-code).
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The **domination number** of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. 
A set $D \subset V(G)$ of a graph $G$ is a dominating set if every vertex $u$ not in $D$ has at least a neighbor in $D$, i.e., $N(u) \cap D \neq \emptyset$.

The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$.

A dominating set of cardinality $\gamma(G)$ is called a $\gamma$-code.
A set $D \subset V(G)$ of a graph $G$ is a **dominating set** if every vertex $u$ not in $D$ has **at least** a neighbor in $D$, i.e., $N(u) \cap D \neq \emptyset$.

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A dominating set of cardinality $\gamma(G)$ is called a $\gamma$-**code**.
A set $D \subset V(G)$ of a graph $G$ is a *dominating set* if every vertex $u$ not in $D$ has at least a neighbor in $D$, i.e., $N(u) \cap D \neq \emptyset$.

The *domination number* of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$.

A dominating set of cardinality $\gamma(G)$ is called a $\gamma$-code.

$\gamma(G) = 2$ (red vertices form a $\gamma$-code).
A set $S \subset V(G)$ of a graph $G$ is a metric-locating set if for every pair $v, w \in V$, $d(x, v) \neq d(x, w)$, for some vertex $x \in S$.

The metric dimension of $G$, denoted by $\beta(G)$, is the minimum cardinality of an ML-set of $G$.

A locating set of cardinality $\beta(G)$ is called a metric basis.

$\beta(G) = 3$ (i.e., $S = \{1, 2, 3\}$ is a metric basis).

Note that vertices 222 and 422 are not dominated by $S$. $^2$ML-set for short.
A set $S \subset V(G)$ of a graph $G$ is a **metric-locating set** \footnote{ML-set for short.} if for every pair $v, w \in V$, \[d(x, v) \neq d(x, w),\] for some vertex $x \in S$. 

The **metric dimension** of $G$, denoted by $\beta(G)$, is the minimum cardinality of an ML-set of $G$. A locating set of cardinality $\beta(G)$ is called a **metric basis**. 

For example, $\beta(G) = 3$ (if $S = \{1, 2, 3\}$ is a metric basis). Note that vertices 222 and 422 are not dominated by $S$. 

\footnote{ML-set for short.}
A set $S \subset V(G)$ of a graph $G$ is a *metric-locating set* \(^2\) if for every pair $v, w \in V$, $d(x, v) \neq d(x, w)$, for some vertex $x \in S$.

The *metric dimension* of $G$, denoted by $\beta(G)$, is the minimum cardinality of an ML-set of $G$.

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A set $S \subset V(G)$ of a graph $G$ is a \textit{metric-locating set} \footnote{ML-set for short.} if for every pair $v, w \in V$, $d(x, v) \neq d(x, w)$, for some vertex $x \in S$.

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Note that vertices 222 and 422 are not dominated by $S$.

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$^2$ML-set for short.
A set \( S \subset V \) is an metric-locating-dominating set if it is both dominating and locating.

The \( \eta(G) \) is the minimum cardinality of an MLD-set.

An MLD-set of cardinality \( \eta(G) \) is called an \( \eta \)-code.

\[ \eta(G) = 4 \] (\( S = \{1, 2, 3, 4\} \) is an \( \eta \)-code).

\( ^3 \text{MLD-set} \) for short.
A set $S \subset V$ is an *metric-locating-dominating set* if it is both dominating and locating.

An MLD-set of cardinality $\eta(G)$ is called an $\eta$-code.

$\eta(G) = 4$ (since $S = \{1, 2, 3, 4\}$ is an $\eta$-code).
A set $S \subset V$ is an *metric-locating-dominating set* if it is both dominating and locating.

The *MLD number* $\eta(G)$ is the minimum cardinality of an MLD-set.

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A set $S \subseteq V$ is an **metric-locating-dominating set** if it is both **dominating** and **locating**.

The **MLD number** $\eta(G)$ is the minimum cardinality of an MLD-set.

An MLD-set of cardinality $\eta(G)$ is called an **$\eta$-code**.
A set $S \subseteq V$ is an *metric-locating-dominating set* if it is both *dominating* and *locating*.

The *MLD number* $\eta(G)$ is the minimum cardinality of an MLD-set.

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A set $S \subseteq V$ is an **metric-locating-dominating set**\(^3\) if it is both **dominating** and **locating**.

The **MLD number** $\eta(G)$ is the minimum cardinality of an MLD-set.

An MLD-set of cardinality $\eta(G)$ is called an $\eta$-**code**.

$\eta(G) = 4$ ($S = \{1, 2, 3, 4\}$ is an $\eta$-code).

\(^3\text{MLD-set for short.}\)
A set $S \subset V$ is a neighbor-locating-dominating number $4$ if for every two vertices $u, v \in V \setminus S$, $\emptyset \neq N(u) \cap S \neq N(v) \cap S \neq \emptyset$.

The NLD number $\lambda(G)$ is the minimum cardinality of an NLD-set.

An NLD-set of cardinality $\lambda(G)$ is called a $\lambda$-code.

$\lambda(G) = 5$ (the set $\{1, 2, 3, 4, 5\}$ is a $\lambda$-code).
A set $S \subset V$ is a **neighbor-locating-dominating number** if for every two vertices $u, v \in V \setminus S$,

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An NLD-set of cardinality $\lambda(G)$ is called a *\(λ\)-code*.

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A set $S \subset V$ is a \textit{neighbor-locating-dominating number} \footnote{NLD-set for short.} if for every two vertices $u, v \in V \setminus S$,

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The \textit{NLD number} $\lambda(G)$ is the minimum cardinality of an NLD-set.

An NLD-set of cardinality $\lambda(G)$ is called a $\lambda$-\textit{code}.

\[\lambda(G) = 5\] (\(S = \{1, 2, 3, 4, 5\}\) is a $\lambda$-code).
$\max\{\gamma, \beta\} \leq \eta \leq \min\{\gamma + \beta, \lambda\}$
A partition $\Pi = \{S_1, \ldots, S_k\}$ of $V$ dominates $G$ if, for every $i \in \{1, \ldots, k\}$, for every vertex $v \in S_i$ and for some $j \in \{1, \ldots, k\}$, $d(v, S_j) = 1$.

The partition domination number $\gamma_p(G)$ is the minimum cardinality of a dominating partition of $G$.

A dominating partition of cardinality $\gamma_p(G)$ is called a $\gamma_p$-partition.
A partition $\Pi = \{S_1, \ldots, S_k\}$ of $V$ dominates $G$ if, for every $i \in \{1, \ldots, k\}$, for every vertex $v \in S_i$ and for some $j \in \{1, \ldots, k\}$, $d(v, S_j) = 1$.

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A dominating partition of cardinality $\gamma_p(G)$ is called a $\gamma_p$-partition.

$\gamma_p(G) = 2$ (TRUE for every graph).
A partition $\Pi = \{S_1, \ldots, S_k\}$ is an ML-partition if, for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$, $d(u, S_j) \neq d(v, S_j)$.

The partition dimension $\beta_p(G)$ is the minimum cardinality of a metric-locating partition of $G$.

$\tau(G) \leq \beta_p(G) \leq \beta(G) + 1$ ($\tau(G)$ is the twin number of $G$).
A partition $\Pi = \{S_1, \ldots, S_k\}$ is an **ML-partition** if, for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

$$d(u, S_j) \neq d(v, S_j)$$

The partition dimension $\beta_p(G)$ is the minimum cardinality of a metric-locating partition of $G$. $\tau(G) \leq \beta_p(G) \leq \beta(G) + 1$ (where $\tau(G)$ is the twin number of $G$).
A partition \( \Pi = \{S_1, \ldots, S_k\} \) is an **ML-partition** if, for every \( i \in \{1, \ldots, k\} \), every pair \( u, v \in S_i \) and some \( j \in \{1, \ldots, k\} \),

\[
    d(u, S_j) \neq d(v, S_j)
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The **partition dimension** \( \beta_p(G) \) is the minimum cardinality of a metric-locating partition of \( G \).
A partition $\Pi = \{S_1, \ldots, S_k\}$ is an ML-partition if, for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

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($\tau(G)$ is the twin number of $G$).
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\[ d(u, S_j) \neq d(v, S_j) \]

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A partition $\Pi = \{S_1, \ldots, S_k\}$ is an \textit{ML-partition} if, for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

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The \textit{partition dimension} $\beta_p(G)$ is the minimum cardinality of a metric-locating partition of $G$.

$$\tau(G) \leq \beta_p(G) \leq \beta(G) + 1 \quad (\tau(G) \text{ is the } \text{twin number} \text{ of } G).$$

$$\beta_p(G) = \tau(G) = 3 \quad (\Pi = \{A, B, C\} \text{ is a } \beta_p\text{-partition}).$$
A partition $\Pi = \{S_1, \ldots, S_k\}$ is an ML-partition if, for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

\[ d(u, S_j) \neq d(v, S_j) \]

The partition dimension $\beta_p(G)$ is the minimum cardinality of a metric-locating partition of $G$.

\[ \tau(G) \leq \beta_p(G) \leq \beta(G) + 1 \]

($\tau(G)$ is the twin number of $G$).

\[ \beta_p(G) = \tau(G) = 3 \]

($\Pi = \{A, B, C\}$ is a $\beta_p$-partition).

$\Pi$ is not dominating ($022$ is an internal vertex of part $A$).
A partition \( \Pi = \{ S_1, \ldots, S_k \} \) of \( V \) is an MLD-partition of \( G \) if it is both a dominating and a metric-locating partition of \( G \).

The partition metric-location-domination number \( \eta_p(G) \) is the minimum cardinality of an MLD-partition of \( G \).

\[ \beta_p(G) \leq \eta_p(G) \leq \eta(G) + 1 \]

\( \eta_p(G) = 4 \) (\( \{ A, B, C, D \} \) is an \( \eta_p \)-partition).

I. M. Pelayo (U.P.C.)

Locating-Dominating Partitions in Graphs

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A partition $\Pi = \{S_1, \ldots, S_k\}$ of $V$ is an \textit{MLD-partition} of $G$ if it is both a \textbf{dominating} and a \textbf{metric-locating} partition of $G$. The \textit{partition metric-location-domination number} $\eta_p(G)$ is the minimum cardinality of an MLD-partition of $G$. It holds that $\beta_p(G) \leq \eta_p(G) \leq \eta(G) + 1$. For instance, $\eta_p(G) = 4$ if $\{A, B, C, D\}$ is an $\eta_p$-partition of $G$. 
A partition $\Pi = \{S_1, \ldots, S_k\}$ of $V$ is an **MLD-partition** of $G$ if it is both a **dominating** and a **metric-locating** partition of $G$.

The **partition metric-location-domination number** $\eta_p(G)$ is the minimum cardinality of an MLD-partition of $G$. 

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▷ A partition \( \Pi = \{ S_1, \ldots, S_k \} \) of \( V \) is an \textit{MLD-partition} of \( G \) if it is both a \textit{dominating} and a \textit{metric-locating} partition of \( G \).

▷ The \textit{partition metric-location-domination number} \( \eta_p(G) \) is the minimum cardinality of an MLD-partition of \( G \).

\[ \beta_p(G) \leq \eta_p(G) \leq \eta(G) + 1 \]
A partition $\Pi = \{S_1, \ldots, S_k\}$ of $V$ is an MLD-partition of $G$ if it is both a **dominating** and a **metric-locating** partition of $G$.

The *partition metric-location-domination number* $\eta_p(G)$ is the minimum cardinality of an MLD-partition of $G$.

$$\beta_p(G) \leq \eta_p(G) \leq \eta(G) + 1$$
A partition \( \Pi = \{ S_1, \ldots, S_k \} \) of \( V \) is an **MLD-partition** of \( G \) if it is both a **dominating** and a **metric-locating** partition of \( G \).

The **partition metric-location-domination number** \( \eta_p(G) \) is the minimum cardinality of an MLD-partition of \( G \).

- \( \beta_p(G) \leq \eta_p(G) \leq \eta(G) + 1 \)

\[ \eta_p(G) = 4 \quad (\{ A, B, C, D \} \text{ is an } \eta_p\text{-partition}). \]
A partition $\Pi = \{S_1, \ldots, S_k\}$ of $V$ is an **MLD-partition** of $G$ if it is both a **dominating** and a **metric-locating** partition of $G$.

The **partition metric-location-domination number** $\eta_p(G)$ is the minimum cardinality of an MLD-partition of $G$.

- $\beta_p(G) \leq \eta_p(G) \leq \eta(G) + 1$

- $\eta_p(G) = 4$ ($\{A, B, C, D\}$ is an $\eta_p$-partition).

- $\eta_p(G) \leq \beta_p(G) + 1$
A partition \( \Pi = \{ S_1, \ldots, S_k \} \) is an NLD-partition of \( G \) if for every \( i \in \{ 1, \ldots, k \} \), every pair \( u, v \in S_i \) and some \( j \in \{ 1, \ldots, k \} \),
\[
d(u, S_j) = 1 \text{ and } d(v, S_j) > 1.
\]

The partition neighbor-location-domination number \( \lambda_p(G) \) is the minimum cardinality of an NLD-partition of \( G \).

\[ \beta_p(P_{10}) = 2, \quad \eta_p(P_{10}) = 3, \quad \lambda_p(P_{10}) = 4 \]

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A partition $\Pi = \{S_1, \ldots, S_k\}$ is an \textit{NLD-partition} of $G$ if for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

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A partition $\Pi = \{S_1, \ldots, S_k\}$ is an **NLD-partition** of $G$ if for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

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A partition $\Pi = \{S_1, \ldots, S_k\}$ is an **NLD-partition** of $G$ if for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

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$$\beta_p(G) \leq \eta_p(G) \leq \lambda_p(G) \leq \lambda(G) + 1$$
A partition $\Pi = \{S_1, \ldots, S_k\}$ is an \textit{NLD-partition} of $G$ if for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

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The \textit{partition neighbor-location-domination number} $\lambda_p(G)$ is the minimum cardinality of an NLD-partition of $G$.

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The \textit{partition neighbor-location-domination number} $\lambda_p(G)$ is the minimum cardinality of an NLD-partition of $G$.

- $\beta_p(G) \leq \eta_p(G) \leq \lambda_p(G) \leq \lambda(G) + 1$

- $\lambda_p(G) = 4$, (\{A, B, C, D\} is a $\lambda_p$-partition).
A partition $\Pi = \{S_1, \ldots, S_k\}$ is an \textit{NLD-partition} of $G$ if for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$, $d(u, S_j) = 1$ and $d(v, S_j) > 1$.

The \textit{partition neighbor-location-domination number} $\lambda_p(G)$ is the minimum cardinality of an NLD-partition of $G$.

$\beta_p(G) \leq \eta_p(G) \leq \lambda_p(G) \leq \lambda(G) + 1$

$\lambda_p(G) = 4$, ($\{A, B, C, D\}$ is a $\lambda_p$-partition).

$\beta_p(P_{10}) = 2$, $\eta_p(P_{10}) = 3$, $\lambda_p(P_{10}) = 4$
\[ \gamma \leq \eta \]
\[ \beta_p + 1 \leq \eta_p \leq \lambda_p \]
\[ \beta + 1 \leq \eta + 1 \leq \lambda + 1 \]
\[ \gamma + \beta + 1 \]
\( \gamma_p = 2 \triangle S \gamma \rightarrow \{ S, V \setminus S \} \gamma_p \)-code \( \Rightarrow \{ \{ v_1 \}, \ldots, \{ v_k \} \}, V \setminus S \} \gamma_p \)-partition.

\( \beta_p \leq \beta_p + 1 \triangle S = \{ v_1, \ldots, v_k \} \) ML-set \( \Rightarrow \{ \{ v_1 \}, \ldots, \{ v_k \}\}, V \setminus S \} \gamma_p \)-partition.

\( \eta_p \leq \beta_p + 1 \triangle \Pi = \{ S_1, \ldots, S_k \} \) ML-partition.

\( \triangle 1 \leq i \leq k : B_i = \{ w \in S_i : N(w) \subseteq S_i \} \) (internal vertices of \( S_i \)).

\( \triangle 1 \leq i \leq k : C_i = \{ w \in B_i : d(w, V \setminus S) \) is even \}.

\( \Delta D = C_1 \cup \ldots \cup C_k \).

\( \Delta \Pi' = \{ S_1, \ldots, S_k, D \} \) MLD-partition.
\[ \gamma_p = 2 \]
\[ \gamma_p = 2 \]

\[ S \gamma\text{-code} \Rightarrow \{ S, V \setminus S \} \gamma_p\text{-partition}. \]
\[ \gamma_p = 2 \]

\[ S \gamma \text{-code } \Rightarrow \{S, V \setminus S\} \gamma_p \text{-partition.} \]

\[ \beta_p \leq \beta + 1 \]
\[ \gamma_p = 2 \]

- \( S \gamma \text{-code} \Rightarrow \{ S, V \setminus S \} \gamma_p \text{-partition.} \)

\[ \beta_p \leq \beta + 1 \]

- \( S = \{ v_1, \ldots, v_k \} \) ML-set \( \Rightarrow \) \( \{ \{ v_1 \}, \ldots, \{ v_k \}, V \setminus S \} \) ML-partition.
\( \gamma_p = 2 \)

- \( S \) \( \gamma \)-code \( \Rightarrow \{ S, V \setminus S \} \gamma_p \)-partition.

\( \beta_p \leq \beta + 1 \)

- \( S = \{ v_1, \ldots, v_k \} \) ML-set \( \Rightarrow \{ \{ v_1 \}, \ldots, \{ v_k \}, V \setminus S \} \) ML-partition.

\( \eta_p \leq \beta_p + 1 \)
\( \gamma_p = 2 \)

- \( S \gamma \)-code \( \Rightarrow \{ S, V \setminus S \} \gamma_p \)-partition.

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- \( \Pi = \{ S_1, \ldots, S_k \} \) ML-partition.
\( \gamma_p = 2 \)

- \( S \) \( \gamma \)-code \( \Rightarrow \{ S, V \setminus S \} \) \( \gamma_p \)-partition.

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- \( \Pi = \{ S_1, \ldots, S_k \} \) ML-partition.

- \( 1 \leq i \leq k: B_i = \{ w \in S_i : N(w) \subseteq S_i \} \) (internal vertices of \( S_i \)).
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\( \Delta = \)
\( \gamma_p = 2 \)

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\( D = C_1 \cup \ldots \cup C_k. \)
\[ \gamma_p = 2 \]

\[ S \gamma \text{-code} \Rightarrow \{ S, V \setminus S \} \gamma_p \text{-partition.} \]

\[ \beta_p \leq \beta + 1 \]

\[ S = \{ v_1, \ldots, v_k \} \text{ ML-set} \Rightarrow \{ \{ v_1 \}, \ldots, \{ v_k \}, V \setminus S \} \text{ ML-partition.} \]

\[ \eta_p \leq \beta_p + 1 \]

\[ \Pi = \{ S_1, \ldots, S_k \} \text{ ML-partition.} \]

\[ 1 \leq i \leq k: B_i = \{ w \in S_i : N(w) \subseteq S_i \} \text{ (internal vertices of } S_i). \]

\[ 1 \leq i \leq k: C_i = \{ w \in B_i : d(w, V \setminus S) \text{ is even} \}. \]

\[ D = C_1 \cup \ldots \cup C_k. \]

\[ \Pi' = \{ S_1, \ldots, S_k, D \} \text{ MLD-partition.} \]
\[ \beta_p = n : K_n. \]

\[ \eta_p = n \iff \lambda_p = n : K_n, K_1, n - 1, K_{n - 2} \lor K_2, K_1 \lor (K_1 + K_{n - 2}). \]

\[ \beta_p = n - 1 : \{H_i\}_{15}^{15} = 1. \]

\[ \eta_p = n - 2 \iff \lambda_p = n - 2 : \{H_i\}_{17}^{17} = 1. \]
\( \beta_p = n: K_n. \)
\( \beta_p = n: K_n. \)

\( \eta_p = n \iff \lambda_p = n: K_n, K_{1,n-1}. \)
\[ \beta_p = n : K_n. \]

\[ \eta_p = n \iff \lambda_p = n : K_n, K_{1,n-1}. \]

\[ \beta_p = n-1 : K_{1,n-1}, K_{n-2} \lor \overline{K_2}, K_1 \lor (K_1 + K_{n-2}). \]
\( \beta_p = n: K_n. \)
\( \eta_p = n \iff \lambda_p = n: K_n, K_{1,n-1}. \)

\( \beta_p = n - 1: K_{1,n-1}, K_{n-2} \lor \overline{K_2}, K_1 \lor (K_1 + K_{n-2}). \)
\( \eta_p = n - 1 \iff \lambda_p = n - 1: K_{n-2} \lor \overline{K_2}, K_1 \lor (K_1 + K_{n-2}). \)
\( \beta_p = n: K_n. \)

\( \eta_p = n \iff \lambda_p = n: K_n, K_{1,n-1}. \)

\( \beta_p = n - 1: K_{1,n-1}, K_{n-2} \lor \overline{K_2}, K_1 \lor (K_1 + K_{n-2}). \)

\( \eta_p = n - 1 \iff \lambda_p = n - 1: K_{n-2} \lor \overline{K_2}, K_1 \lor (K_1 + K_{n-2}). \)

\( \beta_p = n - 2: \left\{ H_i \right\}_{i=1}^{15}. \)
• $\beta_p = n$: $K_n$.

• $\eta_p = n \iff \lambda_p = n$: $K_n, K_{1,n-1}$.

• $\beta_p = n - 1$: $K_{1,n-1}, K_{n-2} \lor \overline{K_2}, K_1 \lor (K_1 + K_{n-2})$.

• $\eta_p = n - 1 \iff \lambda_p = n - 1$: $K_{n-2} \lor \overline{K_2}, K_1 \lor (K_1 + K_{n-2})$.

• $\beta_p = n - 2$: $\{H_i\}_{i=1}^{15}$.

▷ *Discrete Math.*, 308 (2008), 5026–5031. *(wrong result: 22 families)*
\[ \beta_p = n: K_n. \]
\[ \eta_p = n \Leftrightarrow \lambda_p = n: K_n, K_{1,n-1}. \]

\[ \beta_p = n - 1: K_{1,n-1}, K_{n-2} \lor \overline{K_2}, K_1 \lor (K_1 + K_{n-2}). \]
\[ \eta_p = n - 1 \Leftrightarrow \lambda_p = n - 1: K_{n-2} \lor \overline{K_2}, K_1 \lor (K_1 + K_{n-2}). \]

\[ \beta_p = n - 2: \left\{ H_i \right\}_{i=1}^{15}. \]
\[ \eta_p = n - 2 \Leftrightarrow \lambda_p = n - 2: \left\{ H_i \right\}_{i=1}^{17}. \]

\[ \text{Discrete Math., 308 (2008), 5026–5031. (Wrong result: 22 families)} \]
A partition $\Pi = \{S_1, \ldots, S_k\}$ is a coloring partition of $G$, if for every $i \in \{1, \ldots, k\}$, $S_i$ is an independent set.

$\gamma \xrightarrow{\rightarrow} \gamma_i \xrightarrow{\rightarrow} \chi$ chromatic number

$\beta \xrightarrow{\rightarrow} \beta_i \leq \eta \xrightarrow{\rightarrow} \eta_i \leq \chi_{ML}$ ML-chromatic number

$\lambda \xrightarrow{\rightarrow} \lambda_i \leq \chi_{NL}$ NL-chromatic number

If $G$ is a graph of order $n \geq 3$ and diameter $d \geq 2$, then $n \leq k \cdot \left[ 2^k - 1 - 1 \right]$, where $k \in \{\eta_i, \chi_{ML}\}$.

Also called *stable partition* or *proper coloring*.
A partition $\Pi = \{S_1, \ldots, S_k\}$ is a coloring partition\(^5\) of $G$, if for every $i \in \{1, \ldots, k\}$, $S_i$ is an independent set.

\(^5\)also called stable partition or proper coloring.
A partition $\Pi = \{S_1, \ldots, S_k\}$ is a *coloring partition*\(^5\) of $G$, if for every $i \in \{1, \ldots, k\}$, $S_i$ is an independent set.

\[\begin{array}{ccl}
\gamma_p & \mapsto & \gamma^i_p = \chi \\
\beta_p & \mapsto & \beta^i_p \\
\eta_p & \mapsto & \eta^i_p = \chi_{\text{ML}} \\
\lambda_p & \mapsto & \lambda^i_p = \chi_{\text{NL}}
\end{array}\]

\(^5\)also called *stable partition* or *proper coloring*. 
A partition $\Pi = \{S_1, \ldots, S_k\}$ is a \textit{coloring partition} \footnote{also called \textit{stable partition} or \textit{proper coloring}.} of $G$, if for every $i \in \{1, \ldots, k\}$, $S_i$ is an independent set.

\[
\begin{align*}
\gamma_p & \mapsto \gamma^i_p = \chi \quad \text{chromatic number} \\
\beta_p & \mapsto \beta^i_p \\
\eta_p & \mapsto \eta^i_p = \chi_{ML} \quad \text{ML-chromatic number} \\
\lambda_p & \mapsto \lambda^i_p = \chi_{NL} \quad \text{NL-chromatic number}
\end{align*}
\]

If $G$ is a graph of order $n \geq 3$ and diameter $d \geq 2$, then

\[
n \leq k \cdot \left[2^k - 1 - 1\right], \quad \text{where } k \in \{\eta_p, \chi_{ML}\}.
\]

\[
n \leq k \cdot \left[2^{k-1} - 1\right], \quad \text{where } k \in \{\lambda_p, \chi_{NL}\}.
\]
A partition \( \Pi = \{S_1, \ldots, S_k\} \) is a \textit{coloring partition} \(^5\) of \( G \), if for every \( i \in \{1, \ldots, k\} \), \( S_i \) is an independent set.

![Diagram]

\[\begin{align*}
\gamma_p & \mapsto \gamma^i_p = \chi & \text{chromatic number} \\
\beta_p & \mapsto \beta^i_p \\
\eta_p & \mapsto \eta^i_p = \chi_{\text{ML}} & \text{ML-chromatic number} \\
\lambda_p & \mapsto \lambda^i_p = \chi_{\text{NL}} & \text{NL-chromatic number}
\end{align*}\]

If \( G \) is a graph of order \( n \geq 3 \) and diameter \( d \geq 2 \), then

\[n \leq k \cdot \left[ d^{k-1} - (d - 1)^{k-1} \right], \text{ where } k \in \{\eta_p, \chi_{\text{ML}}\}.
\]

\(^5\)also called \textit{stable partition} or \textit{proper coloring}.
A partition \( \Pi = \{S_1, \ldots, S_k\} \) is a *coloring partition*\(^5\) of \( G \), if for every \( i \in \{1, \ldots, k\} \), \( S_i \) is an independent set.

\[
\begin{align*}
\gamma_p & \mapsto \gamma_p^i = \chi \quad \text{chromatic number} \\
\beta_p & \mapsto \beta_p^i \\
\eta_p & \mapsto \eta_p^i = \chi_{ML} \quad \text{ML-chromatic number} \\
\lambda_p & \mapsto \lambda_p^i = \chi_{NL} \quad \text{NL-chromatic number}
\end{align*}
\]

If \( G \) is a graph of order \( n \geq 3 \) and diameter \( d \geq 2 \), then

\[
\begin{align*}
n & \leq k \cdot \left[ d^{k-1} - (d - 1)^{k-1} \right], \text{ where } k \in \{\eta_p, \chi_{ML}\}. \\
n & \leq k \cdot \left[ 2^{k-1} - 1 \right], \text{ where } k \in \{\lambda_p, \chi_{NL}\}.
\end{align*}
\]

\(^{5}\) also called *stable partition* or *proper coloring.*
\[ n = 13 + 3 = 16 \]
\[ m = 29 \]
\[ D = 5 \]
\[ \delta = 1 \]
\[ \Delta = 9 \]
$\chi = 4$
\[ \chi_{NL} = 5 \]
Google, Yahoo: arxiv pelayo locating partition

GRACIAS
Thank you
Danke!!