EXTREMAL GRAPH THEORY FOR
METRIC DIMENSION AND DIAMETER

CARMEN HERNANDO, MERCÈ MORA, IGNACIO M. PELAYO, CARLOS SEARA,
AND DAVID R. WOOD

ABSTRACT. A set of vertices $S$ resolves a connected graph $G$ if every vertex is uniquely
determined by its vector of distances to the vertices in $S$. The metric dimension of $G$ is
the minimum cardinality of a resolving set of $G$. Let $G_{\beta,D}$ be the set of graphs with metric
dimension $\beta$ and diameter $D$. It is well-known that the minimum order of a graph in $G_{\beta,D}$ is exactly $\beta + D$. The first contribution of this paper is to characterise the graphs in $G_{\beta,D}$ with order $\beta + D$ for all values of $\beta$ and $D$. Such a characterisation was previously only known for $D \leq 2$ or $\beta \leq 1$. The second contribution is to determine the maximum order of a graph in $G_{\beta,D}$ for all values of $D$ and $\beta$. Only a weak upper bound was previously known.

1. INTRODUCTION

Let $G$ be a connected graph of order $n$. A vertex $x \in V(G)$ resolves a pair of vertices $v, w \in V(G)$ if $\text{dist}(v, x) \neq \text{dist}(w, x)$. A set of vertices $S \subseteq V(G)$ resolves $G$, and $S$ is a resolving set of $G$, if every pair of distinct vertices of $G$ are resolved by some vertex in $S$. Informally, $S$ resolves $G$ if every vertex is uniquely determined by its vector of distances to the vertices in $S$.

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1Graphs in this paper are finite, undirected, and simple. The vertex set and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$. For vertices $v, w \in V(G)$, we write $v \sim w$ if $vw \in E(G)$, and $v \not\sim w$ if $vw \not\in E(G)$. For $S \subseteq V(G)$, let $G[S]$ be the subgraph of $G$ induced by $S$. That is, $V(G[S]) = S$ and $E(G[S]) = \{vw \in E(G) : v \in S, w \in S\}$. For $S \subseteq V(G)$, let $G \setminus S$ be the graph $G[V(G) \setminus S]$. For $v \in V(G)$, let $G \setminus v$ be the graph $G \setminus \{v\}$. Suppose that $G$ is connected. The distance between vertices $v, w \in V(G)$, denoted by $\text{dist}_G(v, w)$, is the length (that is, the number of edges) in a shortest path between $v$ and $w$ in $G$. The eccentricity of a vertex $v$ in $G$ is $\text{ecc}_G(v) := \max\{\text{dist}_G(v, w) : w \in V(G)\}$. We drop the subscript $G$ from these notations if the graph $G$ is clear from the context. The diameter of $G$ is $\text{diam}(G) := \max\{\text{dist}(v, w) : v, w \in V(G)\} = \max\{\text{ecc}(v) : v \in V(G)\}$. For integers $a \leq b$, let $[a, b] := \{a, a+1, \ldots, b\}$.

2It will be convenient to also use the following definitions for a connected graph $G$. A vertex $x \in V(G)$ resolves a set of vertices $T \subseteq V(G)$ if $x$ resolves every pair of distinct vertices in $T$. A set of vertices $S \subseteq V(G)$ resolves a set of vertices $T \subseteq V(G)$ if for every pair of distinct vertices $v, w \in T$, there exists a vertex $x \in S$ that resolves $v, w$.