Global locating domination in bipartite graphs

Ignacio M. Pelayo

2 Department de Matemàtica Aplicada III
Universitat Politècnica de Catalunya
Barcelona, Catalunya, Spain

24th British Combinatorial Conference
Royal Holloway, University of London

Joint work with Carmen Hernando and Mercè Mora.
A set $D \subset V(G)$ of a graph $G$ is a dominating set if every vertex of $V \setminus D$ has a neighbour in $D$, i.e., for each $u \in V \setminus D$, $N(u) \cap D \neq \emptyset$.

The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$.

$\gamma(P) = 3$, since blue vertices form a minimum dominating set.
A set $D \subset V(G)$ of a graph $G$ is a *dominating set* if every vertex of $V \setminus D$ has a neighbour in $D$, i.e., for each $u \in V \setminus D$, $N(u) \cap D \neq \emptyset$. 

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Let $G = (V, E)$ be a connected graph and $v, w \in V$. 

A vertex $x \in V$ resolves the pair $\{v, w\}$ if $d(x, v) \neq d(x, w)$. 

A set $S \subseteq V$ is a locating set of $G$ if every pair $v, w \in V$ are resolved by some vertex $x \in S$. 

Let $S = \{u_1, \ldots, u_k\}$ be a locating set. The ordered set: 

$$[d(x, u_1), \ldots, d(x, u_k)]$$ 

is the vector of metric coordinates of $x \in V$ with respect to $S$. 

I. M. PELAYO (U.P.C.)
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LOCATING SETS

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A set $D$ of vertices in a graph $G$ is a locating dominating set if it is both locating and dominating. The metric-location-domination number $\eta(G)$ is the minimum cardinality of a locating dominating set of $G$.

Let $S_1, S_2 \subseteq V(G)$. If $S_1$ is dominating and $S_2$ is locating, then $S_1 \cup S_2$ is both locating and dominating. Hence, $\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \gamma(G) + \beta(G)$.
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$$\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \gamma(G) + \beta(G)$$
A set $D$ of vertices in a graph $G$ is a **locating-dominating set**, or simply an **LD-set**, if for every two vertices $u, v \in V(G) \setminus D$, $\emptyset \neq N(u) \cap D \neq N(v) \cap D \neq \emptyset$.

The **location-domination number** $\lambda(G)$ is the minimum cardinality of an LD-set of $G$.

A $\lambda(G)$-code is an LD-set of cardinality $\lambda(G)$.

Every locating-dominating set is both locating and dominating. Hence, $\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \min\{\lambda(G), \gamma(G) + \beta(G)\}$ and both bounds are tight.
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and both bounds are tight.
In all cases, digit 0 means "greater than 1"

\[ \lambda(G) = 3 \], since \( \{ a_1, a_2, a_3 \} \) is a \( \lambda \)-code.
In this example:

\[
\begin{align*}
\beta(G) &= 2 \\
\gamma(G) &= \eta(G) = 3 \\
\lambda(G) &= 4
\end{align*}
\]

\[
\max\{\gamma(G), \beta(G)\} = 3 \leq \eta(G) = 3 \leq \min\{\lambda(G), \gamma(G) + \beta(G)\} = 4
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$G = (V, E)$ is a graph and $\bar{G} = (V, \bar{E})$ is its complementary graph.
Global locating domination

\( G = (V, E) \) is a graph and \( \overline{G} = (V, \overline{E}) \) is its complementary graph.

- A dominating set \( D \subseteq V \) of \( G \) is a \textit{global dominating set} if it is also a dominating set of \( \overline{G} \).

\[ \Rightarrow \text{An LD-set } S \text{ is global iff no vertex } u \in V \text{ satisfies } S \subseteq N_G(u). \]

\[ \Rightarrow \text{A set } S \text{ is a global LD-set of } G \text{ iff it is a global LD-set of } \overline{G}. \]

\[ \Rightarrow \text{Equivalently, an LD-set } S \text{ of } G \text{ is called non-global if there exists a vertex } w \in V \setminus S \text{ such that } S \subseteq N(w). \text{ The vertex } w, \text{ which is necessarily unique, is called the dominating vertex of } S. \]

\[ \Rightarrow \text{If } S \text{ is a non-global LD-set of } G, \text{ then } S + w \text{ is an LD-set of } \overline{G}. \]
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$\Rightarrow$ If $S$ is a non-global LD-set of $G$, then $S + w$ is an LD-set of $\overline{G}$. 
If $G$ contains a global $\lambda$-code, then $\lambda(G) \leq \lambda(G)$.

If every $\lambda$-code of $G$ is non-global, then $\lambda(G) \leq \lambda(G) + 1$.

$|\lambda(G) - \lambda(G)| \leq 1$

$\lambda(G) \in \{\lambda(G) - 1, \lambda(G), \lambda(G) + 1\}$

$G$ is of type I:

$\lambda(G) - 1 \leq \lambda(G) \leq \lambda(G)$

$G$ is of type II:

$\lambda(G) = \lambda(G) + 1$
⇒ If $G$ contains a global $\lambda$-code, then $\lambda(\overline{G}) \leq \lambda(G)$.
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▷ \( G \) is of type I: \( \lambda(G) - 1 \leq \lambda(\overline{G}) \leq \lambda(G) \)
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▷ $G$ is of **type I**: $\lambda(G) - 1 \leq \lambda(\overline{G}) \leq \lambda(G)$

▷ $G$ is of **type II**: $\lambda(\overline{G}) = \lambda(G) + 1$
Let $S$ be a non-global $\lambda$-code of a graph $G = (V, E)$. Then, $\Rightarrow \text{ecc}(w) \leq 2$. $\Rightarrow G$ is connected. $\Rightarrow \text{rad}(G) \leq 2$. $\Rightarrow \text{diam}(G) \leq 4$. $\Rightarrow \Delta(G) \geq \lambda(G)$. 
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Solving $\lambda(\overline{G}) \leq \lambda(G)$ (Type I)

If $G$ is a graph satisfying at least one of the following conditions, then $\lambda(G) \leq \lambda(G)$.

- $G$ is disconnected.
- $\text{rad}(G) \geq 3$.
- $\text{diam}(G) \geq 5$.
- $\Delta(G) < \lambda(G)$.

Moreover, all conditions are tight.
If $G$ is a graph satisfying at least one of the following conditions, then $\lambda(G) \leq \lambda(\overline{G})$. 

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Moreover, all conditions are tight.
Solving $\lambda(\overline{T}) \leq \lambda(T)$

If $T$ is a tree other than $K_2$, then $\lambda(T) \leq \lambda(T)$.

If $T$ is a tree, then the following statements are equivalent:

- $\text{diam}(T) = 2$.
- $T \sim K_1, n-1$ (i.e., $T$ is a star).
- $T$ is disconnected.
- $\lambda(T) = \lambda(T) = n - 1$.

If $T$ is a tree other than $P_4$, then the following statements are equivalent:

- $\text{diam}(T) = 3$.
- $T \sim K_2(r, s)$ (i.e., $T$ is a double star).
- $\lambda(T) = \lambda(T) - 1 = n - 2$.

⋆ $\lambda(P_n) = \lambda(P_n) - 1$ if $n \in \{5k + 1, 5k + 3\}$, otherwise $\lambda(P_n) = \lambda(P_n)$.

⋆ $\lambda(P_n/2, n/2) = \lambda(P_n/2, n/2) - 1$, $n \neq 6$, $\lambda(P_3, 3) = \lambda(P_3, 3)$.
If $T$ is a tree other than $K_2$, then $\lambda(T) \leq \lambda(T)$. 
If $T$ is a tree other than $K_2$, then $\lambda(\overline{T}) \leq \lambda(T)$.

If $T$ is a tree, then the following statements are equivalent:

- $diam(T) = 2$.
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- $\lambda(\overline{P_n}) = \lambda(P_n) - 1$ if $n \in \{5k + 1, 5k + 3\}$, otherwise $\lambda(\overline{P_n}) = \lambda(P_n)$. 
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- $\lambda(\overline{P_{n/2,n/2}}) = \lambda(P_{n/2,n/2}) - 1$, $n \neq 6$, $\lambda(\overline{P_{3,3}}) = \lambda(P_{3,3})$. 
There are 48 trees of order at most 8, 23 of them s.t. $\lambda(T) = \lambda(T')$. 
Let $G$ be a non-complete graph. If $\lambda(G) = \lambda(G) + 1$, then

* Every $\lambda$-code of $G$ is non-global.

* $\operatorname{rad}(G) \leq 2$ and $\operatorname{diam}(G) \leq 4$.

* If $T$ is a tree other than $K_2$, then $\lambda(T) \leq \lambda(T)$.

What about other bipartite graphs?
Solving $\lambda(\overline{G}) = \lambda(G) + 1$ (Type II)

$\Rightarrow \lambda(\overline{K_n}) = n = \lambda(K_n) + 1$. 
\[ \Rightarrow \lambda(K_n) = n = \lambda(K_n) + 1. \]

\[ \Rightarrow \text{Let } G \text{ be a non-complete graph. If } \lambda(\overline{G}) = \lambda(G) + 1, \text{ then} \]
⇒ \( \lambda(K_n) = n = \lambda(K_n) + 1. \)

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⇒ Let \( G \) be a non-complete graph. If \( \lambda(\overline{G}) = \lambda(G) + 1 \), then

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- \( rad(G) \leq 2 \) and \( diam(G) \leq 4 \).
⇒ \( \lambda(K_n) = n = \lambda(K_n) + 1 \).

⇒ Let \( G \) be a non-complete graph. If \( \lambda(\overline{G}) = \lambda(G) + 1 \), then

* Every \( \lambda \)-code of \( G \) is non-global.

* \( \text{rad}(G) \leq 2 \) and \( \text{diam}(G) \leq 4 \).

* If \( T \) is a tree other than \( K_2 \), then \( \lambda(\overline{T}) \leq \lambda(T) \).
$\Rightarrow \lambda(K_n) = n = \lambda(K_n) + 1.$

$\Rightarrow$ Let $G$ be a non-complete graph. If $\lambda(G) = \lambda(G) + 1$, then

- Every $\lambda$-code of $G$ is non-global.
- $\text{rad}(G) \leq 2$ and $\text{diam}(G) \leq 4$.

- If $T$ is a tree other than $K_2$, then $\lambda(T) \leq \lambda(T)$.

What about other bipartite graphs?
If \( G \sim K_{r,s} \), then \( \lambda(G) = \lambda(G) - 1 \).

If \( G \sim K_r,s \), then \( \lambda(G) = \lambda(G) \).

If \( G \sim C_{2k} \), then \( \lambda(G) - 1 \leq \lambda(G) \).

An \((r,s)\)-graph is a bipartite graph \( G = (V = U \cup W, E) \) order \( n = r + s \) such that \( 2 \leq |U| = r \leq |W| = s \).

Let \( S \) be a \( \lambda \)-code of \( G \). Then, if \( S \cap U \neq \emptyset \) and \( S \cap W \neq \emptyset \), then \( S \) is a global \( \lambda \)-code.

If \( S \cap W = \emptyset \), then \( S = U \).

If \( S \cap U = \emptyset \), then \( S = W \).

If \( r < s \) and \( W \) is a \( \lambda \)-code of \( G \), then \( \lambda(G) \leq \lambda(G) \).

COROLLARY: If \( \lambda(G) = \lambda(G) + 1 \), then \( U \) is the unique \( \lambda \)-code of \( G \).
If \( G \cong K_2(r, s) \), then \( \lambda(\overline{G}) = \lambda(G) - 1 \).
If $G \cong K_2(r, s)$, then $\lambda(\overline{G}) = \lambda(G) - 1$.
If $G \cong K_{r,s}$, then $\lambda(\overline{G}) = \lambda(G)$.
⇒ If $G \cong K_2(r, s)$, then $\lambda(\overline{G}) = \lambda(G) - 1$.
⇒ If $G \cong K_{r,s}$, then $\lambda(\overline{G}) = \lambda(G)$.
⇒ If $G \cong C_{2k}$, then $\lambda(G) - 1 \leq \lambda(\overline{G}) \leq \lambda(G)$. 
⇒ If $G \cong K_2(r, s)$, then $\lambda(\overline{G}) = \lambda(G) - 1$.

⇒ If $G \cong K_{r,s}$, then $\lambda(\overline{G}) = \lambda(G)$.

⇒ If $G \cong C_{2k}$, then $\lambda(G) - 1 \leq \lambda(\overline{G}) \leq \lambda(G)$.

▶ An $(r, s)$-graph is a bipartite graph $G = (V = U \cup W, E)$ order $n = r + s$ such that $2 \leq |U| = r \leq |W| = s$. 
⇒ If $G \cong K_2(r, s)$, then $\lambda(\overline{G}) = \lambda(G) - 1$.
⇒ If $G \cong K_{r,s}$, then $\lambda(\overline{G}) = \lambda(G)$.
⇒ If $G \cong C_{2k}$, then $\lambda(G) - 1 \leq \lambda(\overline{G}) \leq \lambda(G)$.

▷ An $(r, s)$-graph is a bipartite graph $G = (V = U \cup W, E)$ order $n = r + s$ such that $2 \leq |U| = r \leq |W| = s$.

⇒ Let $S$ be a $\lambda$-code of $G$. Then,
  - If $S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$, then $S$ is a global $\lambda$-code.
  - If $S \cap W = \emptyset$, then $S = U$.
  - If $S \cap U = \emptyset$, then $S = W$. 
⇒ If \( G \cong K_2(r, s) \), then \( \lambda(\overline{G}) = \lambda(G) - 1 \).
⇒ If \( G \cong K_{r,s} \), then \( \lambda(\overline{G}) = \lambda(G) \).
⇒ If \( G \cong C_{2k} \), then \( \lambda(G) - 1 \leq \lambda(\overline{G}) \leq \lambda(G) \).

▷ An \((r, s)\)-graph is a bipartite graph \( G = (V = U \cup W, E) \) order \( n = r + s \) such that \( 2 \leq |U| = r \leq |W| = s \).

⇒ Let \( S \) be a \( \lambda \)-code of \( G \). Then,
  - If \( S \cap U \neq \emptyset \) and \( S \cap W \neq \emptyset \), then \( S \) is a global \( \lambda \)-code.
  - If \( S \cap W = \emptyset \), then \( S = U \).
  - If \( S \cap U = \emptyset \), then \( S = W \).

⇒ If \( r < s \) and \( W \) is a \( \lambda \)-code of \( G \), then \( \lambda(\overline{G}) \leq \lambda(G) \).
⇒ If $G \cong K_2(r, s)$, then $\lambda(\overline{G}) = \lambda(G) - 1$.
⇒ If $G \cong K_{r,s}$, then $\lambda(\overline{G}) = \lambda(G)$.
⇒ If $G \cong C_{2k}$, then $\lambda(G) - 1 \leq \lambda(\overline{G}) \leq \lambda(G)$.

▷ An $(r, s)$-graph is a bipartite graph $G = (V = U \cup W, E)$ order $n = r + s$ such that $2 \leq |U| = r \leq |W| = s$.

⇒ Let $S$ be a $\lambda$-code of $G$. Then,

- If $S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$, then $S$ is a global $\lambda$-code.
- If $S \cap W = \emptyset$, then $S = U$.
- If $S \cap U = \emptyset$, then $S = W$.

⇒ If $r < s$ and $W$ is a $\lambda$-code of $G$, then $\lambda(\overline{G}) \leq \lambda(G)$.

COROLLARY: If $\lambda(\overline{G}) = \lambda(G) + 1$, then $U$ is the unique $\lambda$-code of $G$. 
\[ G = (V = U \cup W, E), \quad n = r + s, \quad \text{with} \quad 1 \leq |U| = r \leq |W| = s. \]
\[ G = (V = U \cup W, E), \ n = r + s, \ \text{with} \ 1 \leq |U| = r \leq |W| = s. \]

\[ \Rightarrow \text{If} \ 1 \leq r \leq 3, \ \text{then} \ \lambda(G) \leq \lambda(G), \ \text{unless} \ (r, s) \in \{(3, 6), (3, 7)\}. \]

\[ \begin{align*}
(r, s) &= (3, 6) \\
(r, s) &= (3, 7)
\end{align*} \]

In both cases \( \lambda(G) = 4 = \lambda(G) + 1 \)
\[ G = (V = U \cup W, E), \; n = r + s, \text{ with } 1 \leq |U| = r \leq |W| = s. \]

⇒ If \( 1 \leq r \leq 3 \), then \( \lambda(\overline{G}) \leq \lambda(G) \), unless \( (r, s) \in \{(3, 6), (3, 7)\} \).

⇒ If \( 4 \leq r \leq s \): If \( \lambda(\overline{G}) = \lambda(G) + 1 \), then \( U \) is the unique \( \lambda \)-code of \( G \).
\[ G = (V = U \cup W, E), \ n = r + s, \text{ with } 1 \leq |U| = r \leq |W| = s. \]
⇒ If \( 1 \leq r \leq 3 \), then \( \lambda(\overline{G}) \leq \lambda(G) \), unless \( (r, s) \in \{(3, 6), (3, 7)\} \).
⇒ 4 \leq r \leq s : \text{If } \lambda(\overline{G}) = \lambda(G) + 1, \text{ then } U \text{ is the unique } \lambda\text{-code of } G.

- Let \( G \) be a bipartite graph such that \( U \) is its unique \( \lambda \)-code.
\( G = (V = U \cup W, E), n = r + s, \) with \( 1 \leq |U| = r \leq |W| = s. \)

\( \Rightarrow \) If \( 1 \leq r \leq 3, \) then \( \lambda(\overline{G}) \leq \lambda(G), \) unless \( (r, s) \in \{(3, 6), (3, 7)\}. \)

\( \Rightarrow 4 \leq r \leq s : \) If \( \lambda(\overline{G}) = \lambda(G) + 1, \) then \( U \) is the unique \( \lambda \)-code of \( G. \)

Let \( G \) be a bipartite graph such that \( U \) is its unique \( \lambda \)-code.

\( \Rightarrow s \leq 2^r - 1. \)
Bipartite graphs of type II

- \( G = (V = U \cup W, E), \ n = r + s, \) with \( 1 \leq |U| = r \leq |W| = s. \)

⇒ If \( 1 \leq r \leq 3, \) then \( \lambda(\overline{G}) \leq \lambda(G), \) unless \( (r, s) \in \{(3, 6), (3, 7)\}. \)

⇒ \( 4 \leq r \leq s : \) If \( \lambda(\overline{G}) = \lambda(G) + 1, \) then \( U \) is the unique \( \lambda \)-code of \( G. \)

- Let \( G \) be a bipartite graph such that \( U \) is its unique \( \lambda \)-code.

⇒ \( s \leq 2^r - 1. \)

⇒ If \( U \) is non-global and \( 2^{r-1} + 2 \leq s, \) then \( \lambda(\overline{G}) = \lambda(G) + 1. \)


G = (V = U ∪ W, E), n = r + s, with 1 ≤ |U| = r ≤ |W| = s.

⇒ If 1 ≤ r ≤ 3, then \( \lambda(\overline{G}) \leq \lambda(G) \), unless \((r, s) \in \{(3, 6), (3, 7)\} \).

⇒ 4 ≤ r ≤ s: If \( \lambda(\overline{G}) = \lambda(G) + 1 \), then \( U \) is the unique \( \lambda \)-code of \( G \).

Let \( G \) be a bipartite graph such that \( U \) is its unique \( \lambda \)-code.

⇒ \( s \leq 2^r - 1 \).

⇒ If \( U \) is non-global and \( 2^{r-1} + 2 \leq s \), then \( \lambda(\overline{G}) = \lambda(G) + 1 \).
THEOREM: If \(2r + 1 \leq s \leq 2^r - 1\), then there exists a bipartite \((r, s)\)-graph \(H_{r,s}\) such that \(\lambda(H_{r,s}) = \lambda(H_{r,s}) + 1\).
THEOREM: If $2r + 1 \leq s \leq 2^r - 1$, then there exists a bipartite $(r, s)$-graph $H_{r,s}$ such that $\lambda(H_{r,s}) = \lambda(H_{r,s}) + 1$.

Sketch of proof:
THEOREM: If $2r + 1 \leq s \leq 2^r - 1$, then there exists a bipartite \((r, s)\)-graph \(H_{r,s}\) such that \(\lambda(H_{r,s}) = \lambda(H_{r,s}) + 1\).

Sketch of proof:

- \(G_r\) is the \((r, 2r + 1)\)-graph such that \(U = \{1, \ldots, r\}\), \(W = \{[1], [2], [3], [12], [13], [23], [24], [34], \ldots, [2r], [3r], [12 \ldots r]\}\) and \(N(j) = \{w \in W : j \in w\}\).
THEOREM: If $2r + 1 \leq s \leq 2^r - 1$, then there exists a bipartite $(r, s)$-graph $H_{r,s}$ such that $\lambda(H_{r,s}) = \lambda(H_{r,s}) + 1$.

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$G_3$ and $G_5$ illustrate the structure of these graphs.
THEOREM: If $2r + 1 \leq s \leq 2^r - 1$, then there exists a bipartite $(r, s)$-graph $H_{r,s}$ such that $\lambda(H_{r,s}) = \lambda(H_{r,s}) + 1$.

Sketch of proof:

- $G_r$ is the $(r, 2r + 1)$-graph such that $U = \{1, \ldots, r\}$, $W = \{[1], [2], [3], [12], [13], [23], [24], [34], \ldots, [2r], [3r], [12 \ldots r]\}$ and $N(j) = \{w \in W : j \in w\}$.

- Prove that $U$ is its unique LD-set, that it is non-global, and that $\lambda(G_r) = \lambda(G_r) + 1$. 
THEOREM: If \( 2r + 1 \leq s \leq 2^{r} - 1 \), then there exists a bipartite \((r, s)\)-graph \( H_{r,s} \) such that \( \lambda(\overline{H_{r,s}}) = \lambda(H_{r,s}) + 1 \).

Sketch of proof:

- \( G_{r} \) is the \((r, 2r + 1)\)-graph such that \( U = \{1, \ldots, r\} \), \( W = \{[1], [2], [3], [12], [13], [23], [24], [34], \ldots, [2r], [3r], [12 \ldots r]\} \) and \( N(j) = \{w \in W : j \in w\} \).

- Prove that \( U \) is its unique LD-set, that it is non-global, and that \( \lambda(\overline{G_{r}}) = \lambda(G_{r}) + 1 \).

- \( H_{r,s} \) is any \((r, s)\)-graph such that \( U = \{1, \ldots, r\} \), \( \{[1], [2], [3], [12], [13], [23], [24], [34], \ldots, [2r], [3r], [12 \ldots r]\} \subseteq W \), \( N(j) = \{w \in W : j \in w\} \) and \( U \) is an LD-set.
THEOREM: If $2r + 1 \leq s \leq 2^r - 1$, then there exists a bipartite $(r, s)$-graph $H_{r,s}$ such that $\lambda(\overline{H}_{r,s}) = \lambda(H_{r,s}) + 1$.

Sketch of proof:

- $G_r$ is the $(r, 2r + 1)$-graph such that $U = \{1, \ldots, r\}$, $W = \{[1], [2], [3], [12], [13], [23], [24], [34], \ldots, [2r], [3r], [12 \ldots r]\}$ and $N(j) = \{w \in W : j \in w\}$.

- Prove that $U$ is its unique LD-set, that it is non-global, and that $\lambda(\overline{G}_r) = \lambda(G_r) + 1$.

- $H_{r,s}$ is any $(r, s)$-graph such that $U = \{1, \ldots, r\}$, $\{[1], [2], [3], [12], [13], [23], [24], [34], \ldots, [2r], [3r], [12 \ldots r]\} \subseteq W$, $N(j) = \{w \in W : j \in w\}$ and $U$ is an LD-set.

- Prove that $U$ is its unique LD-set, that it is non-global, and that $\lambda(\overline{H}_{r,s}) = \lambda(H_{r,s}) + 1$. 
CONJECTURE 1:

For every bipartite \((r, r)\)-graph \(G\), \(\lambda(\overline{G}) \leq \lambda(G)\).

CONJECTURE 2:

For every bipartite \((r, s)\)-graph \(G\), if \(r \leq s \leq 2r\), then \(\lambda(\overline{G}) \leq \lambda(G)\).