Locating domination in graphs

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$G = (V, E)$ is a simple finite connected graph.

- A set $D$ of vertices in $G$ is a *dominating set* if, for every $u \in V(G) \setminus D$:
  \[ N(u) \cap D \neq \emptyset \]

- The *domination number* of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$.

- A set $D = \{x_1, \ldots, x_k\}$ is a *locating set* if, for every pair $u, v \in V(G)$,
  \[ (d(u, x_1), \ldots, d(u, x_k)) \neq (d(v, x_1), \ldots, d(v, x_k)). \]

- The *metric dimension* (also called the *location number*) $\beta(G)$ is the minimum cardinality of a locating set of $G$. 

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- The metric dimension (also called the location number) \( \beta(G) \) is the minimum cardinality of a locating set of \( G \).
A set $D$ of vertices in a graph $G$ is a \textit{locating dominating set} if it is both locating and dominating.

The \textit{metric-location-domination number} $\eta(G)$ is the minimum cardinality of a locating dominating set of $G$.

Let $S_1, S_2 \subseteq V(G)$. If $S_1$ is dominating and $S_2$ is locating, then $S_1 \cup S_2$ is both locating and dominating. Hence,

$$\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \gamma(G) + \beta(G)$$

Given three positive integers $a, b, c$ verifying that $\max\{a, b\} \leq c \leq a + b$, there always exists a graph $G$ such that $\gamma(G) = a, \beta(G) = b$ and $\eta(G) = c$, except for the case $1 = b < a < c = a + 1$. 
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\( \eta(G) = 2 \), since \( \{a, b\} \) is a minimum locating dominating set
In this example: \( \max\{\gamma(G), \beta(G)\} = 3 \leq \eta(G) = 4 \leq \gamma(G) + \beta(G) = 5 \)
A set $D$ of vertices in a graph $G$ is a \textit{locating-dominating set} if for every two vertices $u, v \in V(G) \setminus D$,

\[ \emptyset \neq N[u] \cap D \neq N[v] \cap D \neq \emptyset. \]

The \textit{location-domination number} $\lambda(G)$ is the minimum cardinality of a locating-dominating set of $G$.

Every locating-dominating set is both locating and dominating. Hence,

\[ \max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \min\{\lambda(G), \gamma(G) + \beta(G)\} \]

and both bounds are tight.

Realization theorem? Not yet.
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Realization theorem? Not yet.
In all cases, digit 0 means "greater than 1"

\[ \lambda(G) = 3, \text{ since } \{a_1, a_2, a_3\} \text{ is a minimum locating-dominating set} \]
In this example:

\[ \max\{\gamma(G), \beta(G)\} = 3 \leq \eta(G) = 3 \leq \min\{\lambda(G), \gamma(G) + \beta(G)\} = 4 \]
<table>
<thead>
<tr>
<th>$G$</th>
<th>$\gamma$</th>
<th>$\beta$</th>
<th>$\eta$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_n, n &gt; 3$</td>
<td>$\left\lceil \frac{n}{3} \right\rceil$</td>
<td>1</td>
<td>$\left\lceil \frac{n}{3} \right\rceil$</td>
<td>$\left\lceil \frac{2n}{5} \right\rceil$</td>
</tr>
<tr>
<td>$C_n, n &gt; 6$</td>
<td>$\left\lceil \frac{n}{3} \right\rceil$</td>
<td>2</td>
<td>$\left\lceil \frac{n}{3} \right\rceil$</td>
<td>$\left\lceil \frac{2n}{5} \right\rceil$</td>
</tr>
<tr>
<td>$K_n, n &gt; 1$</td>
<td>1</td>
<td>$n - 1$</td>
<td>$n - 1$</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$K_{1,n-1}, n &gt; 2$</td>
<td>1</td>
<td>$n - 2$</td>
<td>$n - 1$</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$K_{r,n-r}, n - r \geq r &gt; 1$</td>
<td>2</td>
<td>$n - 2$</td>
<td>$n - 2$</td>
<td>$n - 2$</td>
</tr>
<tr>
<td>$W_{1,n-1}, n &gt; 7$</td>
<td>1</td>
<td>$\left\lfloor \frac{2n}{5} \right\rfloor$</td>
<td>$\left\lfloor \frac{2n-2}{5} \right\rfloor$</td>
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</tbody>
</table>

Domination parameters of some basic families
$G$ is a graph of order $n$, diameter $D \geq 2$, location number $\beta$, metric-location-domination number $\eta$ and location-domination number $\lambda$.

- $\beta + D \leq n \leq (\lceil \frac{2D}{3} \rceil + 1)^\beta + \beta \sum_{i=1}^{\lceil D/3 \rceil} (2i - 1)^{\beta - 1}$
- If $G \neq K_{1,n-1}$, then $\eta + \lceil \frac{2D}{3} \rceil \leq n \leq \eta + \eta \cdot 3^{\eta - 1}$
- $\lambda + \lceil \frac{3D+1}{5} \rceil \leq n \leq \lambda + 2^\lambda - 1$

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\[ \eta(G) = 1 \iff \lambda(G) = 1 \iff G = P_2 \]

\[ \lambda(G) = 2 \implies \eta(G) = 2. \text{ [converse false]} \]

There are 16 graphs s.t. \( \lambda = 2 \) (notice that \( \lambda = 2 \implies n \leq 5 \))
There are 51 graphs satisfying $\eta = 2$

- $\eta = 2 \Rightarrow n \leq 8$
- If $\{u, v\}$ is an $\eta$-set, then $d(u, v) \leq 2$.
- Every graph verifying $\beta \leq 2$ can be embedded into the strong grid.
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SOLVING $\eta = \lambda$

- $\eta(G) = n - 1 \iff \lambda(G) = n - 1$

- $\lambda(G) = n - 1 \iff G = K_n$ or $G = K_1, n-1$

- $\lambda(G) = n - 2 \iff \eta(G) = n - 2$

- $\lambda(G) = n - 2 \iff G \in F_1 \cup \cdots \cup F_7$, where
  
  $F_1 = \{K_{r,s} : 2 \leq r \leq s\}$,
  
  $F_2 = \{K_{r} + \overline{K}_{s} : 2 \leq r \leq s\}$, etc.

- $\eta(G) = n - 3 \Rightarrow \lambda(G) = n - 3$ [converse false]

- If $D = 2$, then $\lambda(G) = \eta(G)$ [for $D \geq 3$, false]
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     etc.
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  \item If \( D = 2 \), then \( \lambda(G) = \eta(G) \) \([\text{for } D \geq 3, \text{false}]\)
\end{itemize}
$n = 6, \ D = 3, \ n - 4 = 2 = \eta(G) < \lambda(G) = 3 = n - 3$
• $\eta(K_m \Box K_n) = \lambda(K_m \Box K_n)$, since $\text{diam}(K_m \Box K_n) = 2$.

• $\beta(K_m \Box K_n) \leq \eta(K_m \Box K_n) \leq \beta(K_m \Box K_n) + 1$.

• For $m, n \geq 2$, a dominating set $S$ resolves $K_n \Box K_m$ iff
  - there is at most one empty row and at most one empty column;
  - there is at most one lonely vertex.

$\implies$ If $2m - 1 < n$, then $\lambda(K_m \Box K_n) = \eta(K_m \Box K_n) = \beta(K_m \Box K_n) = n - 1$

$\implies$ If $m \leq n \leq 2m - 1$, then

$$\lambda(K_n \Box K_m) = \begin{cases} \left\lfloor \frac{2}{3}(n + m - 1) \right\rfloor + 1 & \text{if } n + m = 3k + 2 \\ \left\lfloor \frac{2}{3}(n + m - 1) \right\rfloor & \text{otherwise} \end{cases}$$
• $\eta(K_m \square K_n) = \lambda(K_m \square K_n)$, since $\text{diam}(K_m \square K_n) = 2$.

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$$\lambda(K_n \square K_m) = \begin{cases} \lfloor \frac{2}{3}(n + m - 1) \rfloor + 1 & \text{if } n + m = 3k + 2 \\ \lfloor \frac{2}{3}(n + m - 1) \rfloor & \text{otherwise} \end{cases}$$
• \( \eta(K_m \square K_n) = \lambda(K_m \square K_n) \), since \( \text{diam}(K_m \square K_n) = 2 \).

• \( \beta(K_m \square K_n) \leq \eta(K_m \square K_n) \leq \beta(K_m \square K_n) + 1 \).

• For \( m, n \geq 2 \), a dominating set \( S \) resolves \( K_n \square K_m \) iff
  1. there is at most one empty row and at most one empty column;
  2. there is at most one lonely vertex.

\[ \implies \text{If } 2m - 1 < n, \text{ then } \lambda(K_m \square K_n) = \eta(K_m \square K_n) = \beta(K_m \square K_n) = n - 1 \]

\[ \implies \text{If } m \leq n \leq 2m - 1, \text{ then } \lambda(K_n \square K_m) = \begin{cases} \left\lfloor \frac{2}{3}(n + m - 1) \right\rfloor + 1 & \text{if } n + m = 3k + 2 \\ \left\lfloor \frac{2}{3}(n + m - 1) \right\rfloor & \text{otherwise} \end{cases} \]
• $\eta(K_m \Box K_n) = \lambda(K_m \Box K_n)$, since $diam(K_m \Box K_n) = 2$.

• $\beta(K_m \Box K_n) \leq \eta(K_m \Box K_n) \leq \beta(K_m \Box K_n) + 1$.

• For $m, n \geq 2$, a dominating set $S$ resolves $K_n \Box K_m$ iff
  1. there is at most one empty row and at most one empty column;
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$\implies$ If $2m - 1 < n$, then $\lambda(K_m \Box K_n) = \eta(K_m \Box K_n) = \beta(K_m \Box K_n) = n - 1$

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$$\lambda(K_n \Box K_m) = \left\{ \begin{array}{ll} \left\lfloor \frac{2}{3}(n + m - 1) \right\rfloor + 1 & \text{if } n + m = 3k + 2 \\ \left\lfloor \frac{2}{3}(n + m - 1) \right\rfloor & \text{otherwise} \end{array} \right.$$
• $\eta(K_m \square K_n) = \lambda(K_m \square K_n)$, since $\text{diam}(K_m \square K_n) = 2$.

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