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Fate of the phantom dark energy universe in semiclassical gravity. II. Scalar phantom fields

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Quantum corrections coming from massless fields conformally coupled with gravity are studied, in order to see if they can lead to avoidance of the annoying big rip singularity which shows up in a flat Friedmann-Robertson-Walker universe filled with dark energy and modeled by a scalar phantom field. The dynamics of the model are discussed for all values of the two parameters, named $\alpha > 0$ and $\beta < 0$, corresponding to the regularization process. The new results are compared with the ones obtained in [J. Haro J. Amoros, and E. Elizalde, Phys. Rev. D 83, 123528 (2011)] previously, where dark energy was modeled by means of a phantom fluid with equation of state $P = \omega \rho$, with $\omega < -1$.

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I. INTRODUCTION

Recent observations of distant type-Ia supernovae, baryonic acoustic oscillations, anisotropies of the cosmic microwave background radiation, and some other confirm that our Universe expands in an accelerated way [1,2]. In fact, it seems that it is at present in a dark energy phase [3]. A proposal to explain this situation is to assume that the energy density of our Universe is dominated by a phantom scalar field, that is, a model where the energy density and pressure are

$$\rho = -\frac{1}{2} \dot{\phi}^2 + V(\phi)$$

and

$$P = -\frac{1}{2} \dot{\phi}^2 - V(\phi),$$

being the scalar phantom field. In this case, future singularities are bound to appear in a finite time [4]. Actually, they can be classified as follows (for details, see Ref. [5]):

1. type I (big rip): for $t \to t_s$, $a \to \infty$, $\rho \to \infty$, and $|P| \to \infty$
2. type II (sudden): for $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$, and $|P| \to \infty$
3. type III: for $t \to t_s$, $a \to a_s$, $\rho \to 0$, and $|P| \to 0$
4. type IV: for $t \to t_s$, $a \to a_s$, $\rho \to 0$, and $|P| \to \infty$

where $a$ denotes the scale factor and $H$ the Hubble parameter.

These singularities are undoubtedly there in the classical situation when no quantum effects are taken into account, but it seems feasible that, near the singularities, where the curvature has very high values, quantum effects could have the power to drastically modify the behavior of the Universe, yielding a milder singularity or maybe even a nonsingular model.

In this paper we extend the study carried out in Ref. [6] to a dark energy universe modeled by a scalar phantom field.

In fact, we will consider exponential potentials which give rise to a big rip singularity, and introduce quantum corrections in order to avoid these late-time singularities. Specifically, we shall consider the quantum effects due to massless, conformally coupled fields. This is a special, workable case where the quantum vacuum stress tensor—which depends on two regularization parameters, here called $\alpha > 0$ and $\beta < 0$—and the semiclassical Friedmann equation, can be both calculated explicitly.

We will show, analytically and numerically, that quantum effects drastically modify the big rip singularity, rendering it of type III or turning it into a singularity in the contracting phase. In the first case (a type III singularity) the Hubble parameter does not diverge, but the energy density does tend towards infinity. In the other case (a singularity in the contracting phase) the Hubble parameter becomes finite and negative, and the energy density diverges towards minus infinity. What is important to note is that, in both cases, the Hubble parameter remains finite.

The paper is organized as follows. In the next section, using the mathematical theory of dynamical systems, we study some phantom fields driving the universe to a big rip singularity. In Sec. III we introduce the quantum corrections due to a massless conformally coupled field, and we perform an analytic study of the semiclassical Friedmann equations. In Sec. IV a numerical analysis is carried out to check to good approximation the analytic results obtained in the previous section. In Sec. V we analyze the problem in the context of loop quantum cosmology, where it has been stated that quantum corrections do completely avoid the big rip singularity. We will see that the way to obtain these conclusion is in doubt, because they have been got in some places from an incorrectly modified Friedmann equation. In last section we compare the results obtained for a phantom fluid model with those that were derived for a phantom fluid model. The units to be used in the paper are $c = \hbar = M_p = 1$, where $M_p$ is the reduced Planck mass.
II. DARK ENERGY MODELED BY A PHANTOM FIELD

In this section we consider phantom fields, namely $\phi$, minimally coupled with gravity, that is, modeled by an energy density of the kind $\rho = -\frac{1}{2} \dot{\phi}^2 + V(\phi)$ and a pressure $P = -\frac{1}{2} \ddot{\phi}^2 - V(\phi)$. For these fields the Friedmann and conservation equations have a very simple form,

$$H^2 = \frac{1}{3} \left( -\frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \quad 0 = -\ddot{\phi} - 3H \dot{\phi} + \frac{dV}{d\phi},$$

which allow us to perform a detailed qualitative analysis.

From this system one deduces that $\frac{\dot{\phi}}{\phi} = 3H \ddot{\phi}$, and thus $\dot{H} = \frac{1}{2} \dot{\phi}^2 > 0$, which means that $\ddot{\phi} = H + \dot{H} > 0$, that is, in this model the universe is expanding in an accelerating way.

To analyze the dynamics of the system we consider two specific cases:

1. We start by studying a power law potential, i.e., $V(\phi) = \lambda \phi^n$ with $\lambda > 0$ (note that these kinds of potentials were considered, for the first time in cosmology, in the context of chaotic inflation). The field equation can be written as follows:

$$\frac{d}{dt} \left( \frac{1}{2} \dot{\phi}^2 + \tilde{V}(\phi) \right) = -3H \dot{\phi}^2,$$

where $\tilde{V}(\phi) = -V(\phi)$. This is a dissipative system, and the slow-roll conditions $(V''/V) \ll 1$ and $[V''/|V|] \ll 1$ are satisfied when $|\phi| \gg n$. Then, due to the attractor nature of the slow-roll regime, at late time, the solutions have the same behavior as the slow-roll solution, which satisfies the system

$$H^2 = \frac{1}{3} V(\phi), \quad 0 = -3H \dot{\phi} + \frac{dV}{d\phi}.$$  

Since $\dot{\phi} = 0$ is a critical point, the dynamics of the system decouples for $\dot{\phi} > 0$ and $\dot{\phi} < 0$. For this reason we only consider the domain $\phi > 0$, where the field obeys the equation $\dot{\phi}_s = 2n\sqrt{n/3} \dot{\phi}_s^{n-1}$, whose solution is

$$\phi_s(t) = \left[ \frac{2n(t_0 - t)}{\sqrt{n}} \right]^{\frac{1}{n-1}}, \quad n > 2$$

$$\phi_s(t) = \phi_s(t_0) e^{\sqrt{2}(t_0-t)} \quad n = 2$$

$$\phi_s(t) = \phi_s(t_0) + 2\sqrt{\frac{n}{3}}(t - t_0) \quad n = 1.$$

Evaluating $H_s(t) = \sqrt{\frac{2}{3} \dot{\phi}_s^n(t)}$, one can see that the big rip singularity appears for $n > 2$.

The following remark is in order. This result was obtained in Ref. [7] without a demonstration. To prove, in a rigorous way, the attractor nature of the slow-roll solution, we may use the variables $x = \frac{\dot{\phi}}{\sqrt{6}H}$ and $y = \frac{\dot{\phi}}{\sqrt{6}H}$ [8]. Then, the dynamical equation is

$$\frac{dy}{dx} = -3(1 + y^2 \left[ 1 - \frac{V'}{\sqrt{6}yV} \right]$$

$$= -3(1 + y^2 \left[ 1 - \frac{2n}{6yx} \right])$$

The slow roll solution is the curve $\frac{dy}{dx} = 0$, i.e., $y = \frac{2n}{6x}$, and it is easy to verify that, for large values of $x$, this is the leading term of the solution. Then, since $\frac{dy}{dx} < 0$ above this curve, and $\frac{dy}{dx} > 0$ below it, this definitely proves that the slow-roll solution is an attractor at late times.

2. Another specific example which could be studied analytically is given by the potential $V(\phi) = V_0 e^{-2\phi/\phi_0}$ (being $V_0$ and $\phi_0$, two constant parameters). Then, with the change of function $\phi = \phi_0 \ln(\psi)$ [now $\psi$ belongs in the domain $(0, \infty)$], the system becomes

$$H^2 = \frac{1}{3} \psi^2 \left( -\frac{\phi_0^2}{2} \dot{\psi}^2 + V_0 \right)$$

$$0 = -\phi_0 (\dot{\psi} \dot{\psi} - \dot{\phi}^2) - 3H \phi_0 \dot{\psi} \dot{\phi} - 2 \frac{V_0}{\phi_0}.$$  

The system decouples in the expanding phase ($H > 0$) and in the contracting one ($H < 0$) because only a sign can be chosen in the square root of $H$ in the Friedmann equation [the first equation in (6)]. Since nowadays we are in the expanding phase, it seems natural to consider only this one, in which, the system can be written as follows:

$$\phi_0 \ddot{\psi} + \frac{2}{\phi_0} \left( -\frac{\phi_0^2}{2} \dot{\psi}^2 + V_0 \right)$$

$$+ \sqrt{3} \phi_0 \dot{\psi} \sqrt{-\frac{\phi_0^2}{2} \dot{\psi}^2 + V_0} = 0.$$  

and dividing this equation by $\dot{\phi}$—i.e., using the variable $\phi$ as a time, one obtains the equation

$$\frac{d\dot{\psi}}{d\phi} = -\frac{2}{\phi_0^2} \left( -\frac{\phi_0^2}{2} \dot{\psi}^2 + V_0 \right)$$

$$- \frac{\sqrt{3}}{\phi_0} \sqrt{-\frac{\phi_0^2}{2} \dot{\psi}^2 + V_0} = F(\dot{\psi}).$$  

This is an autonomous first-order differential equation, therefore, it can be completely studied just through the sign of the function $F$. From the Friedmann equation, one can see that the domain of $F$ is the interval $[0, \sqrt{2V_0}/\phi_0]$; the zeros of $F$ are the
points $\pm \sqrt{2V_{0}/\phi_0}$ and $-\sqrt{2V_{0}/\phi_0} - 1/\sqrt{1+\phi_0^2}$. $F$ has a vertical asymptotic at zero. Finally, $F$ is positive in the interval $(-\sqrt{2V_{0}/\phi_0} - 1/\sqrt{1+\phi_0^2},0)$ and negative in $(-\sqrt{2V_{0}/\phi_0}, -\sqrt{2V_{0}/\phi_0} \times -1/\sqrt{1+\phi_0^2}) \cup (0, \sqrt{2V_{0}/\phi_0})$.

This all means that, using $\phi$ as time, the critical points $\sqrt{2V_{0}/\phi_0}$ and $-\sqrt{2V_{0}/\phi_0} - 1/\sqrt{1+\phi_0^2}$ are repellers and the critical points 0 and $-\sqrt{2V_{0}/\phi_0}$ are attractors (see Fig. 1).

However, in the domain $\psi < 0$, when the time $\phi$ increases, the cosmic time $t$ decreases, and vice versa, which means that, in terms of the cosmic time, the critical point $-\sqrt{2V_{0}/\phi_0} - 1/\sqrt{1+\phi_0^2}$ is a global attractor while the other two critical points $\pm \sqrt{2V_{0}/\phi_0}$ are repellers (see Fig. 2).

In terms of the field $\phi$, the critical points obtained above are

$$\psi = -\sqrt{2V_{0}/\phi_0} \frac{1}{\sqrt{1+\phi_0^2}} \implies \phi(t)$$

$$= \phi_0 \ln \left( \frac{t_s - t}{\phi_0 \sqrt{1+\phi_0^2}} \right);$$

$$H(t) = \phi_0^2/2 / (t_s - t);$$

$$\psi = \pm \sqrt{2V_{0}/\phi_0} \implies \phi(t) = \phi_0 \ln \left( \frac{t_s - t}{\phi_0 \sqrt{2V_{0}/\phi_0}} \right);$$

$$H(t) = 0,$$

where $t_s$ is an arbitrary constant. Then, since $-\sqrt{2V_{0}/\phi_0} - 1/\sqrt{1+\phi_0^2}$ is a global attractor, it follows that, except for the solutions $\psi = \pm \sqrt{2V_{0}/\phi_0}$, all the others have a big rip singularity [in Fig. 3 we show the phase portrait of the system in the coordinates ($\psi, \psi$)].

We should here remark that, if one considers the contracting phase, following the same analysis one obtains that the orbit $\psi = \sqrt{2V_{0}/\phi_0} - 1/\sqrt{1+\phi_0^2}$ is a repeller and that the orbits $\psi = \pm \sqrt{2V_{0}/\phi_0}$ are global attractors.

**III. QUANTUM CORRECTIONS**

In this section we will study, in detail, the change in the dynamics in the model $V(\phi) = V_0 e^{-2\phi/\phi_0}$ when one takes into account quantum effects. Let us note that we have chosen the potential $V(\phi) = V_0 e^{-2\phi/\phi_0}$ just for simplicity, but that exactly the same kind of analysis to be carried out in this section could be made for power law potentials or other.

Here we only consider quantum effects due to massless conformally coupled fields because only these kind of fields quantum effects can be computed analytically. In fact, it is well-known that for a massless, conformally coupled field, the anomalous trace is given by [5,6]

$$T_{\text{vac}} = \alpha \Box R - \frac{\beta}{2} G,$$

where $\alpha$ and $\beta$ are positive constants.
with \( R = 6(\dot{H} + 2H^2) \) the scalar curvature and \( G = 24H^2(\dot{H} + H^2) \) the Gauss-Bonnet curvature invariant.

The coefficients, \( \alpha \) and \( \beta \), coming from dimensional regularization are [9]

\[
\alpha = \frac{1}{2880\pi^2} (N_0 + 6N_{1/2} + 12N_1) > 0,
\]

\[
\beta = -\frac{1}{2880\pi^2} (N_0 + 11N_{1/2} + 62N_1) < 0,
\]

(11)

being \( N_0 \) the number of scalar fields, \( N_{1/2} \) the number of four-component neutrinos, and \( N_1 \) the number of electromagnetic fields, respectively.

In terms of the Hubble parameter, Eq. (10) is [10]

\[
T_{\text{vac}} = 6\alpha(\ddot{H} + 12H^2\dot{H} + 7H\dot{H} + 4H^3) - 12\beta(H^4 + H^2\dot{H}).
\]

(12)

With the trace anomaly being \( T_{\text{vac}} = \rho_{\text{vac}} - 3P_{\text{vac}} \) and, inserting (12) into the conservation equation, \( \dot{\rho}_{\text{vac}} + 3H(\rho_{\text{vac}} + P_{\text{vac}}) = 0 \), the modified energy density reads

\[
\rho_{\text{vac}} = 6\alpha(3H^2\dot{H} + H\ddot{H} - \frac{1}{2}\dot{H}^2) - 3\beta H^4,
\]

(13)

and the semiclassical Friedmann equation becomes

\[
\dot{H}^2 = \frac{\rho + \rho_{\text{vac}}}{3},
\]

(14)

with \( \rho = -\frac{1}{2} \dot{\phi}^2 + V_0 e^{-2\phi/\phi_0} \). The following remark is in order. Here, it is here important to understand that only quantum corrections due to massless conformally coupled fields give rise to the semiclassical Friedmann equation, because only in that case is it possible to obtain a local expression of the trace anomaly (see, Chap. 9 of Ref. [11] for details).

Using the dimensionless variables \( \tilde{t} = H^{-1/2}t, \tilde{H} = H/H_{\text{vac}}, \tilde{Y} = \dot{H}/H^2, \tilde{\psi} = \psi, \text{ and } \tilde{\rho} = \rho/\phi_{\text{vac}}^2 \), the semiclassical Friedmann equation and the conservation equation can be written as an autonomous system

\[
\tilde{H}' = \tilde{\phi}'^2 - 3\tilde{H} \tilde{\psi} - 2\tilde{V}_0/\phi_0^2 \tilde{\psi}^2,
\]

(15)

where \( \tilde{t}' \) denotes derivative with respect to the time \( \tilde{t} \), and we have defined the new parameters \( \tilde{V}_0 = V_0/H^3_\text{vac} \) and \( \phi_{0}\text{vac} = \phi_0 \).

What we see at first sight from this system is that it does not have any critical point. It is also easy to show that the energy density \( \tilde{\rho} \) evolves in accordance with the equation

\[
\tilde{\rho}' = \tilde{H}\tilde{\phi}_0^2 \tilde{\phi}'/\phi_0^2,
\]

which means that the energy density increases in the expanding phase and decreases in the contracting one.

Note also that bouncing solutions (solutions that cross the plane \( \tilde{H} = 0 \), i.e., that come from the expanding phase to the contracting one or vice versa) could appear when one considers the semiclassical equations. Effectively, from the second equation of the system (15) one deduces that when \( \tilde{H} = 0 \), the other variables must satisfy \( \tilde{\rho}' = -\tilde{\rho} \), and this condition is satisfied because \( \alpha > 0 \) and \( \beta < 0 \).

Now, we look for future singular solutions of the system with the following behavior near the singularity (see Refs. [12,13]). Since \( \rho = \frac{1}{\phi_{\text{vac}}^2} (\frac{\phi_{\text{vac}}^2}{4} \tilde{\psi}^2 + V_0) \) when \( \tilde{\psi} \to 0 \) one has \( \rho \to \pm \infty \), therefore we will choose singular solutions of the form

\[
\tilde{\psi}(\tilde{t}) = A(\tilde{r}_s - \tilde{t}) + B(\tilde{r}_s - \tilde{t})^2 + O((\tilde{r}_s - \tilde{t})^3),
\]

(16)

where \( A \) and \( B \) are some constants. Inserting these functions into the conservation equation [the last equation of (15)], one obtains

\[
A = \frac{\sqrt{2V_0}}{\phi_0}, \quad B = -\frac{3}{2} \tilde{H}A,
\]

(17)

and since \( B \) and \( A \) are constants we deduce that \( \tilde{H} \) has the form \( \tilde{H}(\tilde{t}) = \tilde{H}_0 + \delta \tilde{H}(\tilde{t}) \), where \( \tilde{H}_0 \) is a constant and thus, the second equation of (16) behaves \( B = -\frac{3}{2} \tilde{H}_0A \).

Finally, inserting them in the semiclassical Friedmann equation [the second equation of (15)] and retaining the leading terms, one gets

\[
\delta \mathcal{A}^\prime(\tilde{t}) = \frac{B}{2\alpha} \frac{\phi_{\text{vac}}^2}{\tilde{r}_s} \ln((\tilde{r}_s - \tilde{t})/T),
\]

(18)

where \( T \) is an integration constant.

We here observe that, when we introduce quantum corrections, the big rip singularity, for \( \tilde{H}_0 > 0 \), is transformed into a type III singularity, because as \( \tilde{t} \to \tilde{r}_s \) one has \( \tilde{H} \to \tilde{H}_0 \to \infty \), and \( |\tilde{P}| \to \infty \). And, when \( \tilde{H}_0 < 0 \), one gets \( \tilde{H} \to \tilde{H}_0 \) (contracting phase), \( \rho \to -\infty \) and \( |\tilde{P}| \to \infty \).

In order to qualitatively study the system it is quite convenient, as in Refs. [14,15], to perform the variable change \( \tilde{\rho} \equiv \sqrt{\tilde{H}} \). After what, the semiclassical Friedmann and conservation equations become

\[
\tilde{\rho}' = -\partial_{\tilde{t}}W(\tilde{\rho}, \tilde{\rho}) - 3\epsilon \tilde{\rho}^2(\tilde{\rho}')^2, \quad \tilde{\rho}' = \tilde{H}\tilde{\phi}_0^2 \tilde{\phi}'/\phi_0^2,
\]

(19)

where \( W(\tilde{\rho}, \tilde{\rho}) = \frac{\phi_{\text{vac}}^2}{8\epsilon}(\tilde{\rho}^2(1 - \frac{1}{2} \tilde{\rho}^2) + \tilde{\rho}^2) \), and \( \epsilon \equiv \text{sign}(\tilde{H}) \).

For positive values of \( \tilde{\rho} \), the potential \( W \) (Fig. 3 of Ref. [15]), has a unique zero, at \( \tilde{\rho}_0 = (3/2)^{1/4} \times (1 + \sqrt{1 + 4\tilde{\rho}})^{1/4} \), and two critical points at \( \tilde{\rho}_\pm = (1 + \sqrt{1 + 4\tilde{\rho}})^{1/4} \), \( \tilde{\rho}_+ \). Thus, for \( \tilde{\rho} > 1/4 \) there are no critical points, being the potential strictly increasing, from \( -\infty \) to \( \infty \). For \( \tilde{\rho} < 1/4 \), the potential satisfies \( W(0) = -\infty \) and \( W(\infty) = \infty \), and exhibits a relative maximum at \( \tilde{\rho}_- \) and a relative minimum at \( \tilde{\rho}_+ \) (a hollow one). For very
small values of \( \rho \), at \( \tilde{p}_- \) one has \( \ddot{H} \equiv \ddot{\rho} \), that is, the system is close to the Friedmann phase and, at \( \tilde{p}_+ \) one has \( |\ddot{H}| \equiv 1 \), that is, the system is close to the de Sitter phase. On the other hand, for negative values of \( \tilde{p} \), the potential only has a critical point at \( \tilde{p}_+ \), and satisfies \( W(0) = W(\infty) = \infty \).

Now, assume that, initially, the system has an energy density which is positive, and that it is in the expanding phase (what does happen nowadays). Then, since in the expanding phase the energy density increases, this means that the slope of the potential is more steep, and thus the system can evolve to the contracting phase (a bouncing solution). When it enters that phase, the energy density decreases and even it could be negative; if so, the system is confined in the decreasing phase because the potential satisfies \( W(0) = W(\infty) = \infty \).

It is very important to stress here that the system cannot remain in the expanding phase all the time, due to the form of the potential and also because of the fact that the energy density is increasing in this phase. Three different situations may occur:

1. The system may develop a singularity in a finite time (type III singularity). This comes from Eq. (18).
2. The system may enter in the decreasing phase (the Universe bounces) and the energy density becomes negative, and then the system cannot abandon this phase. In this situation the energy density could by \( -\infty \) in a finite time.
3. The system may bounce infinitely many times (an oscillating universe).

This is all one can say by analytically studying the system. What we will do in next section is to perform a corresponding numerical study, which will show that only the first two situations are actually possible.

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**FIG. 4 (color online).** Three different simulations, for the values \( \beta/\alpha = -0.5 \), \( \beta/\alpha = -1 \), and \( \beta/\alpha = -10 \), respectively. Red points (bright points) mean initial conditions driving to a type III singularity. Blue points (dark points), initial conditions driving to a singularity in the contracting phase.
IV. NUMERICAL ANALYSIS

In this section we numerically integrate the system (15), assuming that initially the system is in the Friedmann phase, that is, at time $\tilde{t} = 0$ the variables $(H(0), Y(0), \dot{\phi}(0), \ddot{\phi}(0))$ satisfy the constrains

$$
H^2(0) = \frac{1}{3} \left( -\tilde{Y}(0) + \frac{\tilde{V}_0}{\tilde{\psi}^2(0)} \right); \quad \tilde{Y}(0) = \frac{\tilde{\phi}_0^2}{2} \tilde{\psi}^2(0).
$$

This means that the initial conditions depend on two variables. Next, to perform our calculations we choose as variables $(\tilde{\psi}(0), \tilde{\phi}(0))$, and also the following values for the parameters: $\tilde{V}_0 = 24$ and $\tilde{\phi}_0 = 4/\sqrt{3}$. Note that, from Eq. (20) with our choice of parameters the variable $\tilde{\psi}(0)$ belongs to the interval $[-3, 3]$ and $\tilde{\phi}(0)$ belongs to $(0, \infty)$.

In Fig. 4 we plot three simulations for different values of $\beta/\alpha$ (the system (15) depends on this quotient), the first being for $\beta/\alpha = -0.5$, the second for $\beta/\alpha = -1$, and the last one for $\beta/\alpha = -10$. The blue color means initial conditions which drive to a singularity of the form given by Eqs. (16)–(18) in the contracting phase, that is, the Hubble parameter is negative and the energy density diverges to minus infinity (the Universe is initially in the expanding phase then it bounces and enters in the contracting one where it develops the singularity). On the other hand, the red color means initial conditions which drive to a type III singularity (not that in that case it does not bounce).

In Fig. 5 we have integrated the system (15) for $\beta/\alpha = -10$, and we show the evolution of the Hubble parameter and of the energy density. In the first two plots the initial conditions are taken in the blue region of Fig. 4, giving a bouncing universe that evolves, at late times, in the contracting phase with an energy density which diverges at late times. The last two plots correspond to initial conditions taken in the red region of Fig. 4, and they show a type III singularity.

V. PHANTOM FIELDS IN LOOP QUANTUM COSMOLOGY

For the flat Friedmann-Robertson-Walker spacetime, Einstein’s theory is obtained from the Lagrangian $\mathcal{L} = \frac{1}{2} Ra^3 + \mathcal{L}_{\text{matter}}$, where $a$ denotes the scalar factor and $\mathcal{L}_{\text{matter}} = a^3 P = a^3 (-\frac{1}{2} \dot{\phi}^2 - V(\phi))$. This Lagrangian can be written as follows: $\mathcal{L} = 3(\frac{d(a^2\dot{a})}{dt} - \dot{a}^2 a) + a^3 P$. The conjugate momentum of the scale
factor is then given by \( p = \frac{\delta L}{\delta \dot{\phi}} = -6\dot{\alpha}a \), and thus the Hamiltonian is

\[
\mathcal{H}_E = \dot{a}p + a^3 \frac{\delta P}{\delta \phi} - L_E = -\frac{p^2}{12a} + a^3 \rho = -3H^2a^3 + a^3 \rho. \tag{21}
\]

On the other hand, in loop cosmology the following effective Hamiltonian, which captures the underlying loop quantum dynamics, is considered [16–18]

\[
\mathcal{H}_{LQC} = -3V \sin^2(\lambda \beta) \frac{V}{\gamma^2 \lambda^2} + V \rho, \tag{22}
\]

where \( \gamma \) is the Barbero-Immirzi parameter and \( \lambda \) is a parameter with dimensions of length, which is determined by invoking the quantum nature of the geometry, that is, through identification of its square with the minimum eigenvalue of the area operator in LQG, which gives as a result \( \lambda = \sqrt{\frac{3}{4\pi} \frac{\hbar c}{2\mu}} \) (see Ref. [18]). Here \( V \) is the physical volume \( V = a^3 \) and \( \beta \) is canonically conjugated to \( V \), and satisfies \( \{\beta, V\} = \frac{\gamma}{2} \), where \( \{,\} \) is the Poisson bracket.

The Hamiltonian constraint is then given by \( \frac{\sin^2(\lambda \beta)}{\gamma^2 \lambda^2} = \frac{\rho}{3} \), and the Hamiltonian equation yields the identity

\[
\mathcal{V} = \{\mathcal{V}, \mathcal{H}_{LQC}\} = -\frac{\gamma}{2} \frac{\partial \mathcal{H}_{LQC}}{\partial \beta} \Longleftrightarrow H = \frac{\sin(2\lambda \beta)}{2 \gamma \lambda} \Longleftrightarrow \beta = \frac{1}{2 \lambda} \arcsin(2\gamma \lambda H). \tag{23}
\]

Writing this last equation as \( H^2 = \frac{\sin^2(\lambda \beta)}{\gamma^2 \lambda^2} (1 - \sin^2(\lambda \beta)) \), and using the Hamiltonian constraint \( \mathcal{H}_{LQC} = 0 \Longleftrightarrow \frac{\sin^2(\lambda \beta)}{\gamma^2 \lambda^2} = \frac{\rho}{3} \), one gets the following modified Friedmann equation in loop quantum cosmology:

\[
H^2 = \frac{\rho}{3} \left(1 - \frac{\rho}{\rho_c}\right) = \frac{H^2}{\rho_c/12} + \frac{(\rho - \rho_c)^2}{\rho_c^2/4} = 1, \tag{24}
\]

being \( \rho_c \equiv \frac{3}{\gamma \lambda^2} \). This equation, together with the conservation equation \( -\dot{\phi} - 3H \dot{\phi} + \frac{\delta V}{\delta \phi} = 0 \), determine the dynamics of the Universe in loop cosmology.

From the equation for the ellipse, one can easily check that the Hubble parameter belongs to the interval \([\rho_c/12, \rho_c/12]\), and the energy density, \( \rho \), to \([0, \rho_c]\), which means that there is no big rip. In fact, an exhaustive study of the potential \( V = V_0 e^{-2\phi/\phi_0} \) was performed in Refs. [19,20].

But here a problem appears. It is well known that the current cosmological theories are built up from two invariants: the scalar curvature \( R = 6(H + 2H^2) \) and the Gauss-Bonnet curvature invariant \( G = 24H^2(H + H^2) \). For example, in Gauss-Bonnet gravity [21] the Lagrangian \( \mathcal{L}_{MG} = a^3 f(R, G) + a^3 P \) is used, and semiclassical gravity, when one takes into account the quantum effects due to a massless conformally coupled field (see for instance Ref. [6]), is based on the trace anomaly \( T_{\text{vac}} = \alpha \Box R - \frac{\mu}{\lambda} G \) (being \( \alpha > 0 \) and \( \beta < 0 \), two renormalization coefficients). However, from the Legendre transformation

\[
\mathcal{H}_{LQC} = -\frac{2}{\gamma} \mathcal{V} \beta + \mathcal{V} \frac{\delta P}{\delta \phi} \mathcal{H} - L_{LQC}, \tag{25}
\]

one gets, in terms of the standard variables, the following Lagrangian:

\[
\mathcal{L}_{LQC} = -\frac{3a^3 H}{\gamma \lambda} \arcsin(2\lambda \gamma H)
+ \frac{3a^3}{2\gamma^2 \lambda^2} \left(1 - \sqrt{1 - 4\gamma^2 \lambda^2 H^2}\right) + a^3 P, \tag{26}
\]

which is not invariant. This is in disagreement with one of the main principles of general relativity.

From these observations, one can notice that the modified Friedmann equation (24) does not stand in this form because it has been obtained assuming that (22) is the Hamiltonian of the system, which is in contradiction with the invariance of general relativity. We then conclude that the results obtained from this modified Friedmann equation need deep revision (for more details, see Ref. [22]).

VI. DISCUSSION AND COMPARISON WITH THE PHANTOM FLUID MODEL

In Ref. [6] we studied, in detail, the case of a phantom fluid modeled by the EoS \( p = \omega \rho \) with \( \omega < -1 \). We showed that, in the case \(-1 \leq \omega < 0\), there exists a one parameter family of solutions which evolves into the contracting Friedmann phase at late times and only a particular solution asymptotically converging towards the contracting de Sitter universe. All the other solutions enter into the contracting phase and become singular at finite time, satisfying \( \lim_{t \to t_c} H(t) = -\infty \) and \( \lim_{t \to t_c} \rho(t) = 0 \). On the other hand, we also showed in Ref. [6] that for \(-1 > \omega > 0\) almost all solutions describe a universe bouncing infinitely many times (an oscillating universe).

In the present paper, by studying a phantom field we have proven, both analytically and numerically, that all solutions are singular. Some of them display type III singularities and the others are singular in the contracting phase, satisfying \( H(t) \to H(t_c) < 0 \), \( \rho(t) \to -\infty \), and \( P(t) \to \infty \), when \( t \to t_c \).

The difference between these two situations comes from the fact that, for a phantom fluid, when one considers the dynamics in \( \mathbb{R}^3 \) using the coordinates \((H, H, \rho)\), the manifold \( \rho = 0 \) is invariant. More precisely, the half plane \( \rho = 0 \) with \( H > 0 \) is a repeller, whereas when \( H < 0 \) it is an attractor. This means that, at late future time, all the solutions go toward this half plane. Moreover, in the contracting phase there is a critical point \((-\sqrt{-1/\beta}, 0, 0)\) (the contracting de Sitter universe), which restricted to the plane \( \rho = 0 \) is a
repeller. This means that only a solution tends asymptotically towards this point, while all the other escape towards infinity in finite time (this was proven in Ref. [6]). The same does not happen for a phantom field, where the manifold $\rho = 0$ is not invariant and, eventually, the system will cross this manifold. In other words, it will acquire negative energy density and then it cannot leave the decreasing phase, becoming singular at finite time, as we have shown once more, both numerically and analytically.

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