Faltings elliptic curves in twisted $\mathbb{Q}$-isogeny classes

Supplementary Material

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Tables

Introduction

This document contains the tables providing supplementary material for the article *Faltings elliptic curves in twisted* $\mathbb{Q}$*-isogeny classes* written by the same authors and submitted to XXX.

The $\mathbb{Q}$-isogeny classes of elliptic curves are in correspondence with the non-cuspidal rational points of the modular curves $X_0(N)$. For those values of N such that the curve $X_0(N)$ has genus zero, one can parameterize the $\mathbb{Q}$-isogeny classes in terms of the rational values of a Hauptmodul $t = t(\tau)$ generating the function field of modular functions on $\Gamma_0(N)$. The remaining cases correspond to non-cuspidal rational points of $X_0(N)$ for genus ≥ 1 and there are only a finite number of them.

Every $\mathbb{Q}$-isogeny class can be represented by a graph which vertices are the elliptic curves in the class and the edges correspond to the rational prime degree isogenies among them. There are 26 possible types of labeled $\mathbb{Q}$-isogeny graphs:

$$
\begin{aligned}
&L_1\,,\\
&L_2(p) \text{ for } p \text{ in } \{2,3,5,7,11,13,17,19,37,43,67,163\}\,,\\
&L_3(9), L_3(25), L_4, T_4, T_6, T_8\,,\\
&R_4(N) \text{ for } N \text{ in } \{6,10,14,15,21\}\,,\\
&R_6, S_8\,.
\end{aligned}
$$

The subindexs denote the number of vertices of the graph, the letter indicates the shape of the isogeny graph (L line, T tetrahedron, R rectangular, S special), and the level in parenthesis refers to the maximal isogeny degree of a path in the graph (absence of level means isogenies of degree 2 or 3).

In what follows, there is a section for every one of the types of non-trivial $\mathbb{Q}$-isogeny classes. The first subsection entitled *Settings* describes the graph, a Hautpmodul of $X_0(N)$ in the zero genus cases or the rational points otherwise. Then we list the j-invariants along with the signatures (c_4, c_6, Δ) of the elliptic curves in the $\mathbb{Q}$-isogeny classes of that type. The choice of the signatures is *almost* arbitrary, but it fulfills two conditions: on the one hand, the corresponding isogenies are normalized (this is accomplished by using the Velú formulas), and on the other hand the zeroes of the Hauptmodul are also zeroes of the discriminant (in the genus zero cases). Also we describe the action of the automorphisms of $X_0(N)$ that preserve the isogeny graph.

For every type, the second subsection contains the tables displaying the p-adic valuations of: the signatures $\operatorname{sig}_p(E)$, the Weierstrass change u_p giving rise to a p-minimal model of E, the Kodaira symbol $K_p(E)$, and the Pal value $u_p(d)$ for every prime p and every elliptic curve E in the isogeny class. Recall that the Pal value $u_p(d)$ gives the Weierstrass change useful to find a p-minimal model of the quadratic twist E^d for every square-free integer d.

Finally, in every last subsection, we state a proposition that describes the Faltings elliptic curve in the twisted isogenies classes in terms of p-valuations of the rational value of the Hauptmodul (for the genus zero cases) and the square-free integers d. We also display the probability of a vertex in the graph to be the Faltings curve. The last table in each type contains the global information glued from the local information in the previous tables and it provides the data to show the corresponding Proposition, as explained in the article.

1 Type $L_2(2)$

1.1 Settings

Graph

The isogeny graphs of type $L_2(2)$ are given by two 2-isogenous elliptic curves:

$$E_1 \overset{2}{\text{——}} E_2\,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(2)$ parametrize isogeny graphs of type $L_2(2)$. The curve $X_0(2)$ has genus 0 and a hauptmodul for this curve is:

$$t(\tau) = 2^{12}\left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{24}\,.$$

j-invariants

Letting $t = t(\tau)$, one can write

$$\begin{aligned} j(E_1) = j(\tau) \;&= \frac{(t+16)^3}{t}\,, \\ j(E_2) = j(2\tau) &= \frac{(t+256)^3}{t^2}\,. \end{aligned}$$

Signatures

We can (and do) choose Weierstrass equations for (E_1, E_2) in such a way that the isogeny graph is normalized. Their signatures are:

$L_2(2)$	
$c_4(E_1)$	$(t+16)(t+64)$
$c_6(E_1)$	$(t-8)(t+64)^2$
$\Delta(E_1)$	$t(t+64)^3$
$c_4(E_2)$	$(t+64)(t+256)$
$c_6(E_2)$	$(t-512)(t+64)^2$
$\Delta(E_2)$	$t^2(t+64)^3$

Automorphisms

The subgroup of $\operatorname{Aut} X_0(2)$ that fixes the set of vertices of the graph is generated by the Fricke involution of $X_0(2)$, given by $W_2(t) = 2^{12}/t$. With regard to the action of the Fricke involution on the isogeny graph, it can be displayed as follows:

$$W_2\left(E_1 \overset{2}{\text{——}} E_2\right) = \; E_2^{-2t} \overset{2}{\text{——}} E_1^{-2t}\,.$$

1.2 Kodaira symbols, minimal models, and Pal values

Table 1: $L_2(2)$ data for $p \neq 2,3$

$L_2(2)$	$p \neq 2,3$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = m > 0$	E_1	$(0,0,m)$	1	I_m	1	1
	E_2	$(0,0,2m)$	1	I_{2m}	1	1
$v_p(t) = 0$, $v_p(t+64) = 4k$	E_1	$(0,2k,0)$	p^k	I_0	1	1
	E_2	$(0,2k,0)$	p^k	I_0	1	1
$v_p(t) = 0$, $v_p(t+64) = 4k+1$	E_1	$(1,2k+2,3)$	p^k	III	1	1
	E_2	$(1,2k+2,3)$	p^k	III	1	1
$v_p(t) = 0$, $v_p(t+64) = 4k+2$	E_1	$(2,2k+4,6)$	p^k	I_0^*	p	1
	E_2	$(2,2k+4,6)$	p^k	I_0^*	p	1
$v_p(t) = 0$, $v_p(t+64) = 4k+3$	E_1	$(3,2k+6,9)$	p^k	III^*	p	1
	E_2	$(3,2k+6,9)$	p^k	III^*	p	1
$-m = v_p(t) < 0$, m odd	E_1	$(2,3,2m+6)$	$p^{-(m+1)/2}$	I_{2m}^*	p	1
	E_2	$(2,3,m+6)$	$p^{-(m+1)/2}$	I_m^*	p	1
$-m = v_p(t) < 0$, m even	E_1	$(0,0,2m)$	$p^{-m/2}$	I_{2m}	1	1
	E_2	$(0,0,m)$	$p^{-m/2}$	I_m	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod p$	

Table 2: $L_2(2)$ data for $p = 3$

$L_2(2)$	$p=3$					
t	E	$\mathrm{sig}_3(E)$	u_3	$\mathrm{K}_3(E)$	$u_3(d)$	
$v_3(t) = m > 0$	E_1	$(0, 0, m)$	1	I_m	1	1
	E_2	$(0, 0, 2m)$	1	I_{2m}	1	1
$v_p(t) = 0$ $v_p(t + 64) = 4k$	E_1	$(1, 2k + 2, 0)$	3^k	I_0	1	1
	E_2	$(1, 2k + 2, 0)$	3^k	I_0	1	1
$v_3(t) = 0$ $v_3(t + 64) = 4k + 1$	E_1	$(\geq 2, \geq 3, 3)$	3^k	III	1	1
	E_2	$(\geq 2, \geq 3, 3)$	3^k	III	1	1
$v_3(t) = 0$ $v_3(t + 64) = 4k + 2$	E_1	$(3, \geq 6, 6)$	3^k	I_0^*	3	1
	E_2	$(3, \geq 6, 6)$	3^k	I_0^*	3	1
$v_3(t) = 0$ $v_3(t + 64) = 4k + 3$	E_1	$(4, 2k + 8, 9)$	3^k	III^*	3	1
	E_2	$(4, 2k + 8, 9)$	3^k	III^*	3	1
$v_3(t) = -m < 0$ m odd	E_1	$(2, 3, 2m + 6)$	$3^{-(m+1)/2}$	I_{2m}^*	3	1
	E_2	$(2, 3, m + 6)$	$3^{-(m+1)/2}$	I_m^*	3	1
$v_p(t) = -m < 0$ m even	E_1	$(0, 0, 2m)$	$3^{-m/2}$	I_{2m}	1	1
	E_2	$(0, 0, m)$	$3^{-m/2}$	I_m	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod 3$	

Table 3: $L_2(2)$ data for p=2

$L_2(2)$	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=m>11$	E_1	$(6,9,m+6)$	2	I^*_{m-4}	1	2^* or 4^*	1
	E_2	$(6,9,2m-6)$	2^2	I^*_{2m-16}	1	2^* or 4^*	1
$v_2(t)=11$	E_1	$(6,9,17)$	2	I^*_7	1	2	1
	E_2	$(6,9,16)$	2^2	I^*_6	1	2	1
$v_2(t)=10$	E_1	$(6,9,16)$	2	I^*_6	1	2	1
	E_2	$(6,9,14)$	2^2	I^*_4	1	2	1
$v_2(t)=9$	E_1	$(6,9,15)$	2	I^*_5	1	2	1
	E_2	$(6,\geq 9,12)$	2^2	I^*_2	1	2	1
$v_2(t)=8$	E_1	$(6,9,14)$	2	I^*_4	1	2	1
	E_2	$(\geq 7,8,10)$	2^2	I^*_0	1	2	1
$v_2(t)=7$	E_1	$(6,9,13)$	2	I^*_2	1	2	1
	E_2	$(5,7,8)$	2^2	III	1	1	1
$v_2(t)=6$ $v_2(t+64)=4k$ $(t+64)/2^{4k}\equiv 1\,(4)$	E_1	$(4,2k+3,6)$	2^k	II	1	1	1
	E_2	$(6,2k+6,12)$	2^k	I^*_2	1	2	1
$v_2(t)=6$ $v_2(t+64)=4k$ $(t+64)/2^{4k}\equiv 3\,(4)$	E_1	$(4,2k+3,6)$	2^k	III	1	1	1
	E_2	$(6,2k+6,12)$	2^k	I^*_3	1	2	1
$v_2(t)=6$ $v_2(t+64)=4k+1$	E_1	$(5,2k+5,9)$	2^k	III	1	1	1
	E_2	$(7,2k+8,15)$	2^k	III*	1	2	1
$v_2(t)=6$ $v_2(t+64)=4k+2$ $(t+64)/2^{4k+2}\equiv 1\,(4)$	E_1	$(6,2k+7,12)$	2^k	I^*_3	1	2	1
	E_2	$(4,2k+4,6)$	2^{k+1}	III	1	1	1
$v_2(t)=6$ $v_2(t+64)=4k+2$ $(t+64)/2^{4k+2}\equiv 3\,(4)$	E_1	$(6,2k+7,12)$	2^k	I^*_2	1	2	1
	E_2	$(4,2k+4,6)$	2^{k+1}	II	1	1	1
$v_2(t)=6$ $v_2(t+64)=4k+3$	E_1	$(7,2k+9,15)$	2^k	III*	1	2	1
	E_2	$(5,2k+6,9)$	2^{k+1}	III	1	1	1
$v_2(t)=5$	E_1	$(5,7,8)$	2	III	1	1	1
	E_2	$(6,9,13)$	2	I^*_2	1	2	1
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					d (mod 4)		

Table 3: $L_2(2)$ data for p=2 (Continued)

$L_2(2)$	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=4$ $t/2^4 \equiv 1\,(4)$	E_1	$(5,5,4)$	2	III	1	1	1
	E_2	$(4,6,8)$	2	I_1^*	1	1	1
$v_2(t)=4$ $t/2^4 \equiv 3\,(4)$	E_1	$(\geq 6,5,4)$	2	IV	1	1	1
	E_2	$(4,6,8)$	2	IV^*	1	1	1
$v_2(t)=3$	E_1	$(6,\geq 9,12)$	1	I_2^*	1	2	1
	E_2	$(6,9,15)$	1	I_5^*	1	2	1
$v_2(t)=2$ $t/2^2 \equiv 1\,(4)$	E_1	$(4,6,8)$	1	I_0^*	1	1	1
	E_2	$(4,6,10)$	1	I_2^*	1	1	1
$v_2(t)=2$ $t/2^2 \equiv 3\,(4)$	E_1	$(4,6,8)$	1	I_1^*	1	1	1
	E_2	$(4,6,10)$	1	III^*	1	1	1
$v_2(t)=1$	E_1	$(6,9,16)$	2^{-1}	I_6^*	1	2	1
	E_2	$(6,9,17)$	2^{-1}	I_7^*	1	2	1
$v_2(t)=0$ $t \equiv 1\,(4)$	E_1	$(4,6,12)$	2^{-1}	I_4^*	1	1	2
	E_2	$(4,6,12)$	2^{-1}	I_4^*	1	1	2
$v_2(t)=0$ $t \equiv 3\,(4)$	E_1	$(0,0,0)$	1	I_0	1	2^{-1}	2^{-1}
	E_2	$(0,0,0)$	1	I_0	1	2^{-1}	2^{-1}
$v_2(t)=-(2m+1)<0$	E_1	$(6,9,4m+20)$	$2^{-(m+2)}$	I_{4m+10}^*	1	4^* or 2^*	1
	E_2	$(6,9,2m+19)$	$2^{-(m+2)}$	I_{2m+9}^*	1	4^* or 2^*	1
$v_2(t)=-2m<0$ $2^{2m}t \equiv 3\,(4)$	E_1	$(0,0,4m)$	2^{-m}	I_{4m}	1	2^{-1}	2^{-1}
	E_2	$(0,0,2m)$	2^{-m}	I_{2m}	1	2^{-1}	2^{-1}
$v_2(t)=-2m<0$ $2^{2m}t \equiv 1\,(4)$	E_1	$(4,6,12+4m)$	$2^{-(m+1)}$	I_{4m+4}^*	1	1	2
	E_2	$(4,6,12+2m)$	$2^{-(m+1)}$	I_{2m+4}^*	1	1	2
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					d (mod 4)		

Remark (2^* or 4^*): If $v_2(t) > 11$, $t \equiv 1\,(4)$, and $d \equiv 2\,(4)$ then, for E_1, E_2, the value $u_2(d)$ is given by

$$u_2(d) = \begin{cases} 2 \text{ if } d \equiv -2\,(8) \\ 4 \text{ if } d \equiv 2\,(8). \end{cases}$$

.

Remark (4^* or 2^*): If $v_2(t) = -(2m+1) < 0$ and $d \equiv 2\,(4)$ then, for E_1, E_2, the value $u_2(d)$ is given by

- if $t \equiv 1\,(4)$ then $u_2(d) = \begin{cases} 2 \text{ if } d \equiv 2\,(8) \\ 4 \text{ if } d \equiv -2\,(8) \end{cases}$;

- if $t \equiv 3\,(8)$ then $u_2(d) = \begin{cases} 4 \text{ if } d \equiv 2\,(8) \\ 2 \text{ if } d \equiv -2\,(8). \end{cases}$.

1.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

<table>
<tr><th>t</th><th>$[u(E)]$</th><th>$[u(E)(d)]$</th><th>d</th></tr>
<tr><td>$v_2(t) \geq 8$</td><td>$(1:2)$</td><td>$(1:1)$</td><td></td></tr>
<tr><td>$v_2(t) = 7$</td><td rowspan="2">$(1:2)$</td><td>$(1:1)$</td><td>$d \not\equiv 0\,(2)$</td></tr>
<tr><td>$v_2(t) = 6$
$v_2(t+64) \equiv 2, 3\,(4)$</td><td>$(2:1)$</td><td>$d \equiv 0\,(2)$</td></tr>
<tr><td>$v_2(t) = 6$
$v_2(t+64) \equiv 0, 1(4)$</td><td rowspan="2">$(1:1)$</td><td>$(1:1)$</td><td>$d \not\equiv 0\,(2)$</td></tr>
<tr><td>$v_2(t) = 5$</td><td>$(1:2)$</td><td>$d \equiv 0\,(2)$</td></tr>
<tr><td>$v_2(t) \leq 4$</td><td>$(1:1)$</td><td>$(1:1)$</td><td></td></tr>
</table>

The contents of this table are the main ingredients to prove the following result:

Proposition 1. *Let* $E_1 \overset{2}{\text{——}} E_2$ *be a* $\mathbf{Q}$*-isogeny graph of type* $L_2(2)$ *corresponding to a given* t *in* $\mathbf{Q}$, $t \neq 0, -64$ *as above. For every square-free integer* d*, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph* $E_1^d \overset{2}{\text{——}} E_2^d$ *is given by:*

$L_2(2)$	*twisted isogeny graph*	d	*Prob*
$v_2(t) \geq 8$	$E_1^d \longleftarrow E_2^d$		1
$v_2(t) = 7$	$E_1^d \longleftarrow E_2^d$	$d \not\equiv 0\,(2)$	2/3
$v_2(t) = 6$ $v_2(t+64) \equiv 2, 3\,(4)$	$E_1^d \longrightarrow E_2^d$	$d \equiv 0\,(2)$	1/3
$v_2(t) = 6$ $v_2(t+64) \equiv 0, 1(4)$	$E_1^d \longrightarrow E_2^d$	$d \not\equiv 0\,(2)$	2/3
$v_2(t) = 5$	$E_1^d \longleftarrow E_2^d$	$d \equiv 0\,(2)$	1/3
$v_2(t) \leq 4$	$E_1^d \longrightarrow E_2^d$		1

2 Type $L_2(3)$

2.1 Settings

Graph

The isogeny graphs of type $L_2(3)$ are given by two 3-isogenous elliptic curves:

$$E_1 \overset{3}{\textbf{——}} E_3\,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(3)$ parametrize isogeny graphs of type $L_2(3)$. The curve $X_0(3)$ has genus 0 and a hauptmodul for this curve is:

$$t(\tau) = 3^6 \left(\frac{\eta(3\tau)}{\eta(\tau)}\right)^{12}.$$

j-invariants

Letting $t = t(\tau)$, one can write

$$\begin{aligned} j(E_1) = j(\tau) \;\; &= \frac{(t+3)^3(t+27)}{t}\,, \\ j(E_3) = j(3\tau) &= \frac{(t+27)(t+243)^3}{t^3}\,. \end{aligned}$$

Signatures

We can (and do) choose Weierstrass equations for (E_1, E_3) in such a way that the isogeny graph is normalized. Their signatures are:

$L_2(3)$	
$c_4(E_1)$	$(t+3)(t+27)$
$c_6(E_1)$	$(t+27)(t^2+18t-27)$
$\Delta(E_1)$	$t(t+27)^2$
$c_4(E_3)$	$(t+27)(t+243)$
$c_6(E_3)$	$(t+27)(t^2-486t-19683)$
$\Delta(E_3)$	$t^3(t+27)^2$

Automorphisms

The subgroup of $\operatorname{Aut} X_0(3)$ that fixes the set of vertices of the graph is generated by the Fricke involution of $X_0(3)$, given by $W_3(t) = 3^6/t$. With regard to the action of the Fricke involution on the isogeny graph, it can be displayed as follows:

$$W_3\left(E_1 \overset{3}{\textbf{——}} E_3\right) = E_3^{-t} \overset{3}{\textbf{——}} E_1^{-t}\,.$$

2.2 Kodaira symbols, minimal models, and Pal values

Table 4: $L_2(3)$ data for $p \neq 2, 3$

$L_2(3)$	$p \neq 2,3$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = m > 0$	E_1	$(0,0,m)$	1	I_m	1	1
	E_3	$(0,0,3m)$	1	I_{3m}	1	1
$v_p(t) = 0$, $v_p(t+27) = 6k$	E_1	$(2k,0,0)$	p^k	I_0	1	1
	E_3	$(2k,0,0)$	p^k	I_0	1	1
$v_p(t) = 0$, $v_p(t+27) = 6k+1$	E_1	$(1+2k,1,2)$	p^k	II	1	1
	E_3	$(1+2k,1,2)$	p^k	II	1	1
$v_p(t) = 0$, $v_p(t+27) = 6k+2$	E_1	$(2+2k,2,4)$	p^k	IV	1	1
	E_3	$(2+2k,2,4)$	p^k	IV	1	1
$v_p(t) = 0$, $v_p(t+27) = 6k+3$	E_1	$(3+2k,3,6)$	p^k	I_0^*	p	1
	E_3	$(3+2k,3,6)$	p^k	I_0^*	p	1
$v_p(t) = 0$, $v_p(t+27) = 6k+4$	E_1	$(4+2k,4,8)$	p^k	IV^*	p	1
	E_3	$(4+2k,4,8)$	p^k	IV^*	p	1
$v_p(t) = 0$, $v_p(t+27) = 6k+5$	E_1	$(5+2k,5,10)$	p^k	II^*	p	1
	E_3	$(5+2k,5,10)$	p^k	II^*	p	1
$v_p(t) = -m < 0$, m even	E_1	$(0,0,3m)$	$p^{-m/2}$	I_{3m}	1	1
	E_3	$(0,0,m)$	$p^{-m/2}$	I_m	1	1
$v_p(t) = -m < 0$, m odd	E_1	$(2,3,3m+6)$	$p^{-(m+1)/2}$	I_{3m}^*	p	1
	E_3	$(2,3,m+6)$	$p^{-(m+1)/2}$	I_m^*	p	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 5: $L_2(3)$ data for $p = 3$

$L_2(3)$	$p=3$					
t	E	$\mathrm{sig}_3(E)$	u_3	$\mathrm{K}_3(E)$	$u_3(d)$	
$v_3(t) = m \geq 6$	E_1	$(0, 0, m-6)$	3	I_{m-6}	1	1
	E_3	$(0, 0, 3(m-6))$	3^2	$\mathrm{I}_{3(m-6)}$	1	1
$v_3(t) = 5$	E_1	$(4, 6, 11)$	1	II^*	3	1
	E_3	$(\geq 4, 6, 9)$	3	IV^*	3	1
$v_3(t) = 4$	E_1	$(4, 6, 10)$	1	IV^*	3	1
	E_3	$(3, 5, 6)$	3	IV	1	1
$v_3(t) = 3$ $v_3(t+27) = 6k+3$ $(t+27)/3^{6k+3} \equiv 4, 5\,(9)$	E_1	$(2k+4, 6, 9)$	3^k	III^*	3	1
	E_3	$(2k+2, 3, 3)$	3^{k+1}	III	1	1
$v_3(t) = 3$ $v_3(t+27) = 6k+3$ $(t+27)/3^{6k+3} \not\equiv 4, 5\,(9)$	E_1	$(2k+4, 6, 9)$	3^k	IV^*	3	1
	E_3	$(2k+2, 3, 3)$	3^{k+1}	II	1	1
$v_3(t) = 3$ $v_3(t+27) = 6k+4$	E_1	$(2k+5, 7, 11)$	3^k	IV^*	3	1
	E_3	$(2k+3, 4, 5)$	3^{k+1}	II	1	1
$v_3(t) = 3$ $v_3(t+27) = 6k+5$	E_1	$(2k+6, 8, 13)$	3^k	II^*	3	1
	E_3	$(2k+4, 5, 7)$	3^{k+1}	IV	1	1
$v_3(t) = 3$ $v_3(t+27) = 6k+6$ $(t+27)/3^{6k+6} \equiv 4, 5\,(9)$	E_1	$(2k+3, 3, 3)$	3^{k+1}	II	1	1
	E_3	$(2k+5, 6, 9)$	3^{k+1}	III^*	3	1
$v_3(t) = 3$ $v_3(t+27) = 6k+6$ $(t+27)/3^{6k+6} \not\equiv 4, 5\,(9)$	E_1	$(2k+3, 3, 3)$	3^{k+1}	II	1	1
	E_3	$(2k+5, 6, 9)$	3^{k+1}	IV^*	3	1
$v_3(t) = 3$ $v_3(t+27) = 6k+7$	E_1	$(2k+4, 4, 5)$	3^{k+1}	II	1	1
	E_3	$(2k+6, 7, 11)$	3^{k+1}	IV^*	3	1
$v_3(t) = 3$ $v_3(t+27) = 6k+8$	E_1	$(2k+5, 5, 7)$	3^{k+1}	IV	1	1
	E_3	$(2k+7, 8, 13)$	3^{k+1}	II^*	3	1
$v_3(t) = 2$	E_1	$(3, 5, 6)$	1	IV	1	1
	E_3	$(4, 6, 10)$	1	IV^*	3	1
$v_3(t) = 1$	E_1	$(\geq 2, 3, 3)$	1	II	1	1
	E_3	$(2, 3, 5)$	1	IV	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod 3$	

Table 5: $L_2(3)$ data for $p = 3$ (Continued)

$L_2(3)$	$p = 3$					
t	E	$\mathrm{sig}_3(E)$	u_3	$\mathrm{K}_3(E)$	$u_3(d)$	
$v_3(t) = 0$	E_1	$(0, 0, 0)$	1	I_0	1	1
	E_3	$(0, 0, 0)$	1	I_0	1	1
$-m = v_3(t) < 0$ m even	E_1	$(0, 0, 3m)$	$3^{-m/2}$	I_{3m}	1	1
	E_3	$(0, 0, m)$	$3^{-m/2}$	I_m	1	1
$-m = v_3(t) < 0$ m odd	E_1	$(2, 3, 3m + 6)$	$3^{-(m+1)/2}$	I^*_{3m}	3	1
	E_3	$(2, 3, m + 6)$	$3^{-(m+1)/2}$	I^*_m	3	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod 3$	

Table 6: $L_2(3)$ data for p=2

$L_2(3)$	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=m\geq 2$	E_1	$(0,0,m)$	1	I_m	1	2^{-1}	2^{-1}
	E_3	$(0,0,3m)$	1	I_{3m}	1	2^{-1}	2^{-1}
$v_2(t)=1$	E_1	$(4,6,13)$	2^{-1}	I_5^*	1	1	2
	E_3	$(4,6,15)$	2^{-1}	I_7^*	1	1	2
$v_2(t)=0$ $v_2(t+27)=1$	E_1	$(6,9,14)$	2^{-1}	I_4^*	1	2	1
	E_3	$(6,9,14)$	2^{-1}	I_4^*	1	2	1
$v_2(t)=0$ $v_2(t+27)=6k>1$ $(t+27)/2^{6k}\equiv 1\,(4)$	E_1	$(2k+7,9,12)$	2^{k-1}	II^*	1	2	2
	E_3	$(2k+7,9,12)$	2^{k-1}	II^*	1	2	2
$v_2(t)=0$ $v_2(t+27)=6k>1$ $(t+27)/2^{6k}\equiv 3\,(4)$	E_1	$(2k+3,3,0)$	2^k	I_0	1	1	2^{-1}
	E_3	$(2k+3,3,0)$	2^k	I_0	1	1	2^{-1}
$v_2(t)=0$ $v_2(t+27)=6k+1>1$	E_1	$(2k+8,10,14)$	2^{k-1}	II^*	1	2	1
	E_3	$(2k+8,10,14)$	2^{k-1}	II^*	1	2	1
$v_2(t)=0$ $v_2(t+27)=6k+2$ $(t+27)/2^{6k+2}\equiv 1\,(4)$	E_1	$(\geq 4,5,4)$	2^k	II	1	1	1
	E_3	$(\geq 4,5,4)$	2^k	II	1	1	1
$v_2(t)=0$ $v_2(t+27)=6k+2$ $(t+27)/2^{6k+2}\equiv 3\,(4)$	E_1	$(\geq 4,5,4)$	2^k	IV	1	1	1
	E_3	$(\geq 4,5,4)$	2^k	IV	1	1	1
$v_2(t)=0$ $v_2(t+27)=6k+3$	E_1	$(2k+6,6,6)$	2^k	II	1	1^* or 2^*	1
	E_3	$(2k+6,6,6)$	2^k	II	1	1^* or 2^*	1
$v_2(t)=0$ $v_2(t+27)=6k+4$ $(t+27)/2^{6k+4}\equiv 1\,(4)$	E_1	$(2k+7,7,8)$	2^k	I_0^*	1	1	1
	E_3	$(2k+7,7,8)$	2^k	I_0^*	1	1	1
$v_2(t)=0$ $v_2(t+27)=6k+4$ $(t+27)/2^{6k+4}\equiv 3\,(4)$	E_1	$(2k+7,7,8)$	2^k	IV^*	1	1	1
	E_3	$(2k+7,7,8)$	2^k	IV^*	1	1	1
$v_2(t)=0$ $v_2(t+27)=6k+5$	E_1	$(2k+8,8,10)$	2^k	I_0^*	1	2	1
	E_3	$(2k+8,8,10)$	2^k	I_0^*	1	2	1
$v_2(t)=-m<0$ m odd	E_1	$(6,9,3m+18)$	$2^{-(m+3)/2}$	I_{3m+8}^*	1	2	1
	E_3	$(6,9,m+18)$	$2^{-(m+3)/2}$	I_{m+8}^*	1	2	1
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					d (mod 4)		

Continued on next page

Table 6: $L_2(3)$ data for p=2 (Continued)

$L_2(3)$	$p = 2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t) = -m < 0$ m even $2^m t \equiv 1\,(4)$	E_1	$(4, 6, 3m + 12)$	$2^{-(m+2)/2}$	I^*_{3m+4}	1	1	2
	E_3	$(4, 6, m + 12)$	$2^{-(m+2)/2}$	I^*_{m+4}	1	1	2
$v_2(t) = -m < 0$ m even $2^m t \equiv 3\,(4)$	E_1	$(0, 0, 3m)$	$2^{-m/2}$	I_{3m}	1	2^{-1}	2^{-1}
	E_3	$(0, 0, m)$	$2^{-m/2}$	I_m	1	2^{-1}	2^{-1}
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					d (mod 4)		

Remark (1^* or 2^*): If $v_2(t) = 0$, $t \equiv 1\,(4)$, $v_2(t + 27) = 6k + 3$, and $d \equiv 2\,(4)$ then, for E_1, E_3, the value $u_2(d)$ is given by

$$u_2(d) = \begin{cases} 1 \text{ if } d \equiv 2\,(8) \\ 2 \text{ if } d \equiv -2\,(8). \end{cases}$$

2.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

t	$[u(E)]$	$[u(E)(d)]$	d	Prob
$v_3(t) \geq 5$	$(1:3)$	$(1:1)$		$(0,1)$
$v_3(t) = 4$	$(1:3)$	$(1:1)$	$d \not\equiv 0\,(3)$	$\left(\frac{1}{4}, \frac{3}{4}\right)$
$v_3(t) = 3$ $v_3(t+27) \equiv 3,4,5\,(6)$		$(3:1)$	$d \equiv 0\,(3)$	
$v_3(t) = 3$ $v_3(t+27) \equiv 0,1,2\,(6)$	$(1:1)$	$(1:1)$	$d \not\equiv 0\,(3)$	$\left(\frac{3}{4}, \frac{1}{4}\right)$
$v_3(t) = 2$		$(1:3)$	$d \equiv 0\,(3)$	
$v_3(t) \leq 1$	$(1:1)$	$(1:1)$		$(1,0)$

This table is the main ingredient to prove the following result:

Proposition 2. *Let* $E_1 \overset{3}{\text{———}} E_3$ *be a* $\mathbf{Q}$*-isogeny graph of type* $L_2(3)$ *corresponding to a given* t *in* $\mathbf{Q}^*$, $t \neq -27$. *For every square-free integer* d*, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph* $E_1^d \overset{3}{\text{———}} E_3^d$ *is given by:*

$L_2(3)$	*twisted isogeny graph*	d	*Prob*
$v_3(t) \geq 5$	$E_1^d \longleftarrow E_3^d$		1
$v_3(t) = 4$	$E_1^d \longleftarrow E_3^d$	$d \not\equiv 0\,(3)$	$3/4$
$v_3(t) = 3$ $v_3(t+27) \equiv 3,4,5\,(6)$	$E_1^d \longrightarrow E_3^d$	$d \equiv 0\,(3)$	$1/4$
$v_3(t) = 3$ $v_3(t+27) \equiv 0,1,2\,(6)$	$E_1^d \longrightarrow E_3^d$	$d \not\equiv 0\,(3)$	$3/4$
$v_3(t) = 2$	$E_1^d \longleftarrow E_3^d$	$d \equiv 0\,(3)$	$1/4$
$v_3(t) \leq 1$	$E_1^d \longrightarrow E_3^d$		1

3 Type $L_2(5)$

3.1 Settings

Graph

The isogeny graphs of type $L_2(5)$ are given by two 5-isogenous elliptic curves:

$$E_1 \overset{5}{\text{———}} E_5 \,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(5)$ parametrize isogeny graphs of type $L_2(5)$. The curve $X_0(5)$ has genus 0 and a hauptmodul for this curve is:

$$t(\tau) = 5^3 \left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^6 .$$

j-invariants

Letting $t = t(\tau)$, one can write

$$\begin{aligned} j(E_1) = j(\tau) \; &= \frac{\left(t^2 + 10t + 5\right)^3}{t} \,, \\ j(E_5) = j(5\tau) &= \frac{\left(t^2 + 250t + 3125\right)^3}{t^5} \,. \end{aligned}$$

Signatures

We can (and do) choose Weierstrass equations for (E_1, E_5) in such a way that the isogeny graph is normalized. Their signatures are:

	$L_2(5)$
$c_4(E_1)$	$(t^2 + 10t + 5)(t^2 + 22t + 125)$
$c_6(E_1)$	$(t^2 + 4t - 1)(t^2 + 22t + 125)^2$
$\Delta(E_1)$	$t(t^2 + 22t + 125)^3$
$c_4(E_5)$	$(t^2 + 22t + 125)(t^2 + 250t + 3125)$
$c_6(E_5)$	$(t^2 - 500t - 15625)(t^2 + 22t + 125)^2$
$\Delta(E_5)$	$t^5(t^2 + 22t + 125)^3$

Automorphisms

The subgroup of $\operatorname{Aut} X_0(5)$ that fixes the set of vertices of the graph is generated by the Fricke involution of $X_0(5)$, given by $W_5(t) = 5^3/t$. With regard to the action of the Fricke involution on the isogeny graph, it can be displayed as follows:

$$W_5 \left(E_1 \overset{5}{\text{———}} E_5\right) = \; E_5^{-1} \overset{5}{\text{———}} E_1^{-1} \,.$$

3.2 Kodaira symbols, minimal models, and Pal values

Table 7: $L_2(5)$ data for $p \neq 2, 5$

$L_2(5)$	$p \neq 2, 5$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = m > 0$	E_1	$(0, 0, m)$	1	I_m	1	1
	E_5	$(0, 0, 5m)$	1	I_{5m}	1	1
$v_p(t) = 0$ $v_p(t^2 + 22\,t + 125) = 4m > 0$	E_1	$(0, 2m, 0)$	p^m	I_0	1	1
	E_5	$(0, 2m, 0)$	p^m	I_0	1	1
$v_p(t) = 0$ $v_p(t^2 + 22\,t + 125) = 4m + 1 > 0$	E_1	$(1, 2m + 2, 3)$	p^m	III	1	1
	E_5	$(1, 2m + 2, 3)$	p^m	III	1	1
$v_p(t) = 0$ $v_p(t^2 + 22\,t + 125) = 4m + 2 > 0$	E_1	$(2, 2m + 4, 6)$	p^m	I_0^*	p	1
	E_5	$(2, 2m + 4, 6)$	p^m	I_0^*	p	1
$v_p(t) = 0$ $v_p(t^2 + 22\,t + 125) = 4m + 3 > 0$	E_1	$(3, 2m + 6, 9)$	p^m	III^*	p	1
	E_5	$(3, 2m + 6, 9)$	p^m	III^*	p	1
$v_p(t) = -m < 0$	E_1	$(0, 0, 5m)$	p^{-m}	I_{5m}	1	1
	E_5	$(0, 0, m)$	p^{-m}	I_m	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

The polynomial $t^2+22t+125$ factors in $\mathbb{Q}_5[t]$ as $(t-\alpha_1)(t-\alpha_2)$ with:

$$\alpha_1 = 3+4\cdot 5^2+2\cdot 5^3+5^4+4\cdot 5^5+2\cdot 5^7+5^8+5^9+5^{12}+3\cdot 5^{13}+3\cdot 5^{14}+5^{15}+O(5^{17})$$
$$\alpha_2 = 2\cdot 5^3+3\cdot 5^4+4\cdot 5^6+2\cdot 5^7+3\cdot 5^8+3\cdot 5^9+4\cdot 5^{10}+4\cdot 5^{11}+3\cdot 5^{12}+5^{13}+5^{14}+O(5^{15})$$

Table 8: $L_2(5)$ data for $p=5$

$L_2(5)$	$p=5$					
t	E	$\mathrm{sig}_5(E)$	u_5	$\mathrm{K}_5(E)$	$u_5(d)$	
$v_5(t)=m>3$	E_1	$(0,0,m-3)$	5	I_{m-3}	1	1
	E_5	$(0,0,5(m-3))$	5^2	$\mathrm{I}_{5(m-3)}$	1	1
$v_5(t)=3$, $v_5(t^2+22t+125)=4m+3>0$	E_1	$(0,\geq 0,0)$	5^{m+1}	I_0	1	1
	E_5	$(0,\geq 0,0)$	5^{m+2}	I_0	1	1
$v_5(t)=3$, $v_5(t^2+22t+125)=4m+4>0$	E_1	$(1,\geq 2,3)$	5^{m+1}	III	1	1
	E_5	$(1,\geq 2,3)$	5^{m+2}	III	1	1
$v_5(t)=3$, $v_5(t^2+22t+125)=4m+5>0$	E_1	$(2,\geq 4,6)$	5^{m+1}	I_0^*	5	1
	E_5	$(2,\geq 4,6)$	5^{m+2}	I_0^*	5	1
$v_5(t)=3$, $v_5(t^2+22t+125)=4m+6>0$	E_1	$(3,\geq 6,9)$	5^{m+1}	III^*	5	1
	E_5	$(3,\geq 6,9)$	5^{m+2}	III^*	5	1
$v_5(t)=2$	E_1	$(3,4,8)$	1	IV^*	5	1
	E_5	$(2,2,4)$	5	IV	1	1
$v_5(t)=1$	E_1	$(2,2,4)$	1	IV	1	1
	E_5	$(3,4,8)$	1	IV^*	5	1
$v_5(t)=0$, $t\not\equiv 3\,(5)$	E_1	$(0,0,0)$	1	I_0	1	1
	E_5	$(0,0,0)$	1	I_0	1	1
$v_5(t)=0$, $v_5(t^2+22t+125)=4m$	E_1	$(0,\geq 0,0)$	5^m	I_0	1	1
	E_5	$(0,\geq 0,0)$	5^m	I_0	1	1
$v_5(t)=0$, $v_5(t^2+22t+125)=4m+1$	E_1	$(1,\geq 2,3)$	5^m	III	1	1
	E_5	$(1,\geq 2,3)$	5^m	III	1	1
$v_5(t)=0$, $v_5(t^2+22t+125)=4m+2$	E_1	$(2,\geq 4,6)$	5^m	I_0^*	5	1
	E_5	$(2,\geq 4,6)$	5^m	I_0^*	5	1
$v_5(t)=0$, $v_5(t^2+22t+125)=4m+3$	E_1	$(3,\geq 6,9)$	5^m	III^*	5	1
	E_5	$(3,\geq 6,9)$	5^m	III^*	5	1
$v_5(t)=-m<0$	E_1	$(0,0,5m)$	5^{-m}	I_{5m}	1	1
	E_5	$(0,0,m)$	5^{-m}	I_m	1	1
					$d\equiv 0$	$d\not\equiv 0$
					$d \pmod 5$	

Table 9: $L_2(5)$ data for p=2

$L_2(5)$	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=m\geq 1$	E_1	$(0,0,m)$	1	I_m	1	2^{-1}	2^{-1}
	E_5	$(0,0,5m)$	1	I_{5m}	1	2^{-1}	2^{-1}
$v_2(t)=0$, $t\equiv 1\,(4)$	E_1	$(6,6,6)$	1	II	1	1^* or 2^*	1
	E_5	$(6,6,6)$	1	II	1	1^* or 2^*	1
$v_2(t)=0$, $t\equiv 3\,(4)$	E_1	$(5,8,9)$	1	III	1	1	1
	E_5	$(5,8,9)$	1	III	1	1	1
$v_2(t)=-m<0$	E_1	$(4,6,5m+12)$	$2^{-(m+1)}$	I^*_{5m+4}	1	1	2
	E_5	$(4,6,m+12)$	$2^{-(m+1)}$	I^*_{m+4}	1	1	2
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					$d \pmod 4$		

Remark (1^* or 2^*): If $v_2(t)=0$, $t\equiv 1\,(4)$, and $d\equiv 2\,(4)$ then, for E_1,E_5, the value $u_2(d)$ is given by

- if $t\equiv 1\,(8)$ then $u_2(d)=\begin{cases}1 \text{ if } d\equiv -2\,(8)\\ 2 \text{ if } d\equiv 2\,(8);\end{cases}$
- if $t\equiv 5\,(8)$ then $u_2(d)=\begin{cases}2 \text{ if } d\equiv -2\,(8)\\ 1 \text{ if } d\equiv 2\,(8).\end{cases}$

3.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

t	$[u(E)]$	$[u(E)(d)]$	d	Prob
$v_5(t) \geq 3$	$(1:5)$	$(1:1)$		$(0,1)$
$v_5(t) = 2$	$(1:5)$	$(1:1)$	$d \not\equiv 0\,(5)$	$\left(\frac{5}{6}, \frac{1}{6}\right)$
		$(5:1)$	$d \equiv 0\,(5)$	
$v_5(t) = 1$	$(1:1)$	$(1:1)$	$d \not\equiv 0\,(5)$	$\left(\frac{1}{6}, \frac{5}{6}\right)$
		$(1:5)$	$d \equiv 0\,(5)$	
$v_5(t) \leq 0$	$(1:1)$	$(1:1)$		$(1,0)$

Proposition 3. *Let* $E_1 \overset{5}{\text{———}} E_5$ *be a* $\mathbf{Q}$*-isogeny graph of type* $L_2(5)$ *corresponding to a given* t *in* $\mathbf{Q}^*$*. For every square-free integer* d*, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph* $E_1^d \overset{5}{\text{———}} E_5^d$ *is given by:*

$L_2(5)$	*twisted isogeny graph*	d	*Prob*
$v_5(t) \geq 3$	$E_1^d \longleftarrow E_5^d$		1
$v_5(t) = 2$	$E_1^d \longleftarrow E_5^d$	$d \not\equiv 0\,(5)$	5/6
	$E_1^d \longrightarrow E_5^d$	$d \equiv 0\,(5)$	1/6
$v_5(t) = 1$	$E_1^d \longrightarrow E_5^d$	$d \not\equiv 0\,(5)$	5/6
	$E_1^d \longleftarrow E_5^d$	$d \equiv 0\,(5)$	1/6
$v_5(t) \leq 0$	$E_1^d \longrightarrow E_5^d$		1

4 Type $L_2(7)$

4.1 Settings

Graph

The isogeny graphs of type $L_2(7)$ are given by two 7-isogenous elliptic curves:

$$E_1 \overset{7}{\text{———}} E_7\,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(7)$ parametrize isogeny graphs of type $L_2(7)$. The curve $X_0(7)$ has genus 0 and a hauptmodul for this curve is:

$$t(\tau) = 7^2\left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^4 .$$

j-invariants

Letting $t = t(\tau)$, one can write

$$\begin{aligned} j(E_1) = j(\tau) &= \frac{\left(t^2+5\,t+1\right)^3\left(t^2+13\,t+49\right)}{t}\,, \\ j(E_7) = j(7\tau) &= \frac{\left(t^2+13\,t+49\right)\left(t^2+245\,t+2401\right)^3}{t^7}\,. \end{aligned}$$

Signatures

We can (and do) choose Weierstrass equations for (E_1, E_7) in such a way that the isogeny graph is normalized. Their signatures are:

	$L_2(7)$
$c_4(E_1)$	$(t^2+5\,t+1)(t^2+13\,t+49)$
$c_6(E_1)$	$(t^2+13\,t+49)(t^4+14\,t^3+63\,t^2+70\,t-7)$
$\Delta(E_1)$	$t(t^2+13\,t+49)^2$
$c_4(E_7)$	$(t^2+13\,t+49)(t^2+245\,t+2401)$
$c_6(E_7)$	$(t^2+13\,t+49)(t^4-490\,t^3-21609\,t^2-235298\,t-823543)$
$\Delta(E_7)$	$t^7(t^2+13\,t+49)^2$

Automorphisms

The subgroup of $\operatorname{Aut} X_0(7)$ that fixes the set of vertices of the graph is generated by the Fricke involution of $X_0(7)$, given by $W_7(t) = 7^2/t$. With regard to the action of the Fricke involution on the isogeny graph, it can be displayed as follows:

$$W_7\,(E_1 \overset{7}{\text{———}} E_7) = E_7^{-7} \overset{7}{\text{———}} E_1^{-7}\,.$$

4.2 Kodaira symbols, minimal models, and Pal values

Table 10: $L_2(7)$ data for $p \neq 2, 3, 7$

$L_2(7)$	$p \neq 2,3,7$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = m > 0$	E_1	$(0,0,m)$	1	I_m	1	1
	E_7	$(0,0,7m)$	1	I_{7m}	1	1
$v_p(t) = 0$, $v_p(t^2+13t+49) = 6m > 0$	E_1	$(2m,0,0)$	p^m	I_0	1	1
	E_7	$(2m,0,0)$	p^m	I_0	1	1
$v_p(t) = 0$, $v_p(t^2+13\,t+49) = 6m+1 > 0$	E_1	$(2m+1,1,2)$	p^m	II	1	1
	E_7	$(2m+1,1,2)$	p^m	II	1	1
$v_p(t) = 0$, $v_p(t^2+13\,t+49) = 6m+2 > 0$	E_1	$(2m+2,2,4)$	p^m	IV	1	1
	E_7	$(2m+2,2,4)$	p^m	IV	1	1
$v_p(t) = 0$, $v_p(t^2+13\,t+49) = 6m+3 > 0$	E_1	$(2m+3,3,6)$	p^m	I_0^*	p	1
	E_7	$(2m+3,3,6)$	p^m	I_0^*	p	1
$v_p(t) = 0$, $v_p(t^2+13\,t+49) = 6m+4 > 0$	E_1	$(2m+4,4,8)$	p^m	IV^*	p	1
	E_7	$(2m+4,4,8)$	p^m	IV^*	p	1
$v_p(t) = 0$, $v_p(t^2+13\,t+49) = 6m+5 > 0$	E_1	$(2m+5,5,10)$	p^m	II^*	p	1
	E_7	$(2m+5,5,10)$	p^m	II^*	p	1
$v_p(t) = -m < 0$	E_1	$(0,0,7m)$	p^{-m}	I_{7m}	1	1
	E_7	$(0,0,m)$	p^{-m}	I_m	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod p$	

Table 11: $L_2(7)$ data for $p = 3$

$L_2(7)$	$p=3$					
t	E	$\mathrm{sig}_3(E)$	u_3	$\mathrm{K}_3(E)$	$u_3(d)$	
$v_3(t) = m > 0$	E_1	$(0, 0, m)$	1	I_m	1	1
	E_7	$(0, 0, 7m)$	1	I_{7m}	1	1
$v_3(t) = 0$ $t \equiv 1, 4\,(9)$	E_1	$(2, 3, 4)$	1	II	1	1
	E_7	$(2, 3, 4)$	1	II	1	1
$v_3(t) = 0$ $t \equiv 16, 25\,(27)$	E_1	$(3, 5, 6)$	1	IV	1	1
	E_7	$(3, 5, 6)$	1	IV	1	1
$v_3(t) = 0$ $t \equiv 7\,(27)$	E_1	$(3, \geq 6, 6)$	1	I_0^*	3	1
	E_7	$(3, \geq 6, 6)$	1	I_0^*	3	1
$v_3(t) = -m < 0$	E_1	$(0, 0, 7m)$	3^{-m}	I_{7m}	1	1
	E_7	$(0, 0, m)$	3^{-m}	I_m	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod 3$	

The polynomial $t^2 + 13t + 49$ factors over $\mathbb{Q}_7[t]$ as $(t - \alpha_1)(t - \alpha_2)$ with:

$$\alpha_1 = 1 + 5 \cdot 7 + 5 \cdot 7^2 + 4 \cdot 7^3 + 7^4 + 6 \cdot 7^5 + 4 \cdot 7^6 + 7^7 + 6 \cdot 7^8 + 3 \cdot 7^9 + 6 \cdot 7^{10} + 6 \cdot 7^{11} + O(7^{12})$$
$$\alpha_2 = 7^2 + 2 \cdot 7^3 + 5 \cdot 7^4 + 2 \cdot 7^6 + 5 \cdot 7^7 + 3 \cdot 7^9 + 4 \cdot 7^{12} + 5 \cdot 7^{13} + 2 \cdot 7^{14} + O(7^{15})$$

Table 12: $L_2(7)$ data for $p = 7$

$L_2(7)$	$p = 7$					
t	E	$\mathrm{sig}_7(E)$	u_7	$\mathrm{K}_7(E)$	$u_7(d)$	
$v_7(t) = m \geq 3$	E_1	$(2, 3, m+4)$	1	I^*_{m-2}	7	1
	E_7	$(2, 3, 7m-8)$	7	I^*_{7m-14}	7	1
$v_7(t) = 2$	E_1	$(\geq 1, 1, 2)$	7^m	II	1	1
$v_7(t^2 + 13t + 49) = 6m \geq 2$	E_7	$(\geq 1, 1, 2)$	7^{m+1}	II	1	1
$v_7(t) = 2$	E_1	$(\geq 3, 2, 4)$	7^m	IV	1	1
$v_7(t^2 + 13t + 49) = 6m + 1 \geq 2$	E_7	$(\geq 3, 2, 4)$	7^{m+1}	IV	1	1
$v_7(t) = 2$	E_1	$(\geq 2, 3, 6)$	7^m	I^*_0	7	1
$v_7(t^2 + 13t + 49) = 6m + 2$	E_7	$(\geq 2, 3, 6)$	7^{m+1}	I^*_0	7	1
$v_7(t) = 2$	E_1	$(\geq 3, 4, 8)$	7^m	IV^*	7	1
$v_7(t^2 + 13t + 49) = 6m + 3$	E_7	$(\geq 3, 4, 8)$	7^{m+1}	IV^*	7	1
$v_7(t) = 2$	E_1	$(\geq 4, 5, 10)$	7^m	II^*	7	1
$v_7(t^2 + 13t + 49) = 6m + 4$	E_7	$(\geq 4, 5, 10)$	7^{m+1}	II^*	7	1
$v_7(t) = 2$	E_1	$(\geq 1, 0, 0)$	7^{m+1}	I_0	1	1
$v_7(t^2 + 13t + 49) = 6m + 5$	E_7	$(\geq 1, 0, 0)$	7^{m+2}	I_0	1	1
$v_7(t) = 1$	E_1	$(1, 2, 3)$	1	III	1	1
	E_7	$(3, 5, 9)$	1	III^*	7	1
$v_7(t) = 0$	E_1	$(\geq 1, 0, 0)$	7^m	I_0	1	1
$v_7(t^2 + 13t + 49) = 6m \geq 0$	E_7	$(\geq 1, 0, 0)$	7^m	I_0	1	1
$v_7(t) = 0$	E_1	$(\geq 2, 1, 2)$	7^m	II	1	1
$v_7(t^2 + 13t + 49) = 6m + 1$	E_7	$(\geq 2, 1, 2)$	7^m	II	1	1
$v_7(t) = 0$	E_1	$(\geq 3, 2, 4)$	7^m	IV	1	1
$v_7(t^2 + 13t + 49) = 6m + 2$	E_7	$(\geq 3, 2, 4)$	7^m	IV	1	1
$v_7(t) = 0$	E_1	$(\geq 3, 3, 6)$	7^m	I^*_0	7	1
$v_7(t^2 + 13t + 49) = 6m + 3$	E_7	$(\geq 3, 3, 6)$	7^m	I^*_0	7	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod 7$	

Continued on next page

Table 12: $L_2(7)$ data for $p = 7$ (Continued)

$L_2(7)$	$p = 7$					
t	E	$\mathrm{sig}_7(E)$	u_7	$\mathrm{K}_7(E)$	$u_7(d)$	
$v_7(t) = 0$ $v_7(t^2 + 13t + 49) = 6m + 4$	E_1	$(\geq 4, 4, 8)$	7^m	IV^*	7	1
	E_7	$(\geq 4, 4, 8)$	7^m	IV^*	7	1
$v_7(t) = 0$ $v_7(t^2 + 13t + 49) = 6m + 5$	E_1	$(\geq 5, 5, 10)$	7^m	II^*	7	1
	E_7	$(\geq 5, 5, 10)$	7^m	II^*	7	1
$v_7(t) = -m < 0$	E_1	$(0, 0, 7m)$	7^{-m}	I_{7m}	1	1
	E_7	$(0, 0, m)$	7^{-m}	I_m	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod 7$	

Table 13: $L_2(7)$ data for p=2

$L_2(7)$	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=m\geq 2$	E_1	$(4,6,m+12)$	2^{-1}	I^*_{m+4}	1	1	2
	E_7	$(4,6,7m+12)$	2^{-1}	I^*_{7m+4}	1	1	2
$v_2(t)=1$	E_1	$(0,0,1)$	1	I_1	1	2^{-1}	2^{-1}
	E_7	$(0,0,7)$	1	I_7	1	2^{-1}	2^{-1}
$v_2(t)=0$ $t\equiv 1\,(4)$	E_1	$(0,0,0)$	1	I_0	1	2^{-1}	2^{-1}
	E_7	$(0,0,0)$	1	I_0	1	2^{-1}	2^{-1}
$v_2(t)=0$ $t\equiv 3\,(4)$	E_1	$(4,6,12)$	2^{-1}	I^*_4	1	1	2
	E_7	$(4,6,12)$	2^{-1}	I^*_4	1	1	2
$v_2(t)=-1$	E_1	$(0,0,7)$	2^{-1}	I_7	1	2^{-1}	2^{-1}
	E_7	$(0,0,1)$	2^{-1}	I_1	1	2^{-1}	2^{-1}
$v_2(t)=-m\leq -2$	E_1	$(4,6,7m+12)$	$2^{-(m+1)}$	I^*_{7m+4}	1	1	2
	E_7	$(4,6,m+12)$	$2^{-(m+1)}$	I^*_{m+4}	1	1	2
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					d (mod 4)		

4.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

t	$[u(E)]$	$[u(E)(d)]$	d	Prob
$v_7(t) \geq 2$	$(1:7)$	$(1:1)$		$(0,1)$
$v_7(t) = 1$	$(1:1)$	$(1:1)$	$d \not\equiv 0\,(7)$	$\left(\frac{7}{8}, \frac{1}{8}\right)$
		$(1:7)$	$d \equiv 0\,(7)$	
$v_7(t) \leq 0$	$(1:1)$	$(1:1)$		$(1,0)$

The contents of this table are the main ingredients to prove the following result:

Proposition 4. *Let* $E_1 \stackrel{7}{\text{———}} E_7$ *be a* $\mathbf{Q}$*-isogeny graph of type* $L_2(7)$ *corresponding to a given* t *in* $\mathbf{Q}^*$*. For every square-free integer* d*, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph* $E_1^d \stackrel{7}{\text{———}} E_7^d$ *is given by:*

$L_2(7)$	*twisted isogeny graph*	d	*Prob*
$v_7(t) \geq 2$	$E_1^d \longleftarrow E_7^d$		1
$v_7(t) = 1$	$E_1^d \longrightarrow E_7^d$	$d \not\equiv 0\,(7)$	7/8
	$E_1^d \longleftarrow E_7^d$	$d \equiv 0\,(7)$	1/8
$v_7(t) \leq 0$	$E_1^d \longrightarrow E_7^d$		1

5 Type $L_2(11)$

5.1 Settings

Graph

The isogeny graphs of type $L_2(11)$ are given by two 11-isogenous elliptic curves:

$$E_1 \overset{11}{\longrightarrow} E_{11}\,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(11)$ parametrize isogeny graphs of type $L_2(11)$. The modular curve $X_0(11)$ is an elliptic curve of rank 0 over the rationals. More precisely, we can choose the Weierstrass model $y^2+y = x^3-x^2-10\,x-20$ for $X_0(11)$. The j-forgetful map $j\colon X_0(11) \to X_0(1)$ is given by

$$j = \frac{P(x) + y\,Q(x)}{(-17x^2 - xy - 243x - 105y + 859)^3}$$

with

$$\begin{aligned} P(x) = \;& x^8 + 160170x^7 + 22013817x^6 - 1234891244x^5 + 18403682346x^4 - 145947253957x^3 \\ & + 1422949497947x^2 + 5880426893238x + 7325611514413\,, \\ Q(x) = \;& -692x^6 - 12510792x^5 + 815793738x^4 - 17947463042x^3 + 112966993208x^2 \\ & + 491634446704x - 468196759663\,. \end{aligned}$$

j-invariants

One has

$$X_0(11)(\mathbb{Q}) = \{(0:1:0), (5:-6:1), (5:5:1), (16:-61:1), (16:60:1)\}$$

and thus:

$$j((0:1:0)) = \infty\,, \quad j((16:-61:1)) = \infty\,,$$
$$j((16:60:1)) = -11^2\,, \quad j((5:-6:1)) = -11\cdot 131^3\,, \quad j((5:5:1)) = -2^{15}\,.$$

These rational points are: two rational cusps $(\infty) = (0:1:0)$, $(0) = (16:-61:1)$, one rational CM point $(5:5:1)$ that corresponds to $\tau_b = \frac{1}{2} + \frac{\sqrt{-11}}{2\cdot 11} \in \mathbb{H}$ and two non-cuspidal non-CM points $(16:60:1)$ and $(5:-6:1)$ that correspond to $\tau_a = 0.5 + 0.09227...i$, and $\tau_a' = 0.5 + 0.24630...i \in \mathbb{H}$. That is

$$j(\tau_b) = -2^{15}, \qquad j(\tau_a) = -11^2, \qquad j(\tau_a') = -11\cdot 131^3,$$

and we have $j(11\tau_b) = j(\tau_b)$ and $j(11\tau_a) = j(\tau_a')$.

Signatures

We can (and do) choose Weierstrass equations in such a way that the isogeny graphs are normalized:

E	Minimal Weierstrass model	$j(E)$	label
E_{1_a}	$y^2 + xy + y = x^3 + x^2 - 30x - 76$	$-11 \cdot 131^3$	`121a1`
E_{11_a}	$y^2 + xy + y = x^3 + x^2 - 305x + 7888$	-11^2	`121a2`
E_{1_b}	$y^2 + y = x^3 - x^2 - 7x + 10$	-2^{15}	`121b1`
E_{11_b}	$y^2 + y = x^3 - x^2 - 887x - 10143$	-2^{15}	`121b2`

Their signatures are:

E	E_{1_a}	E_{11_a}
$c_4(E)$	$11 \cdot 131$	11^4
$c_6(E)$	$11 \cdot 4973$	$-11^5 \cdot 43$
$\Delta(E)$	-11^2	-11^{10}

E	E_{1_b}	E_{11_b}
$c_4(E)$	$2^5 \cdot 11$	$2^5 \cdot 11^3$
$c_6(E)$	$-2^3 \cdot 7 \cdot 11^2$	$2^3 \cdot 7 \cdot 11^5$
$\Delta(E)$	-11^3	-11^9

For $k \in \{a, b\}$, we have that the Faltings curve (circled) in the graph is

$$E_{1_k} \longrightarrow E_{11_k}$$

Note that any $\mathbb{Q}$-isogeny class of elliptic curves of type $L_2(11)$ is isomorphic to $E_{1_k} \overset{11}{\text{——}} E_{11_k}$ for some $k \in \{a, b\}$. Note also that E_{1_b} and E_{11_b} have complex multiplication by the ring of integers of $\mathbf{Q}(\sqrt{-11})$ and $E_{11_b} = E_{1_b}^{-11}$.

5.2 Kodaira symbols, minimal models, and Pal values

There is only one bad reduction prime $p = 11$.

$p = 11$				
E	$\mathrm{sig}_{11}(E)$	$\mathrm{K}_{11}(E)$	$u_{11}(d)$	
E_{1_a}	$(1, 1, 2)$	II	1	1
E_{11_a}	$(4, 5, 10)$	II*	11	1
E_{1_b}	$(1, 2, 3)$	III	1	1
E_{11_b}	$(3, 5, 9)$	III*	11	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{11}$	

5.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

$[u(E)(d)]$	d	Prob
$(1:1)$	$d \not\equiv 0\,(11)$	$\left(\frac{11}{12}, \frac{1}{12}\right)$
$(1:11)$	$d \equiv 0\,(11)$	

This table is the main ingredient to prove the following result:

Proposition 5. *Let $k \in \{a, b\}$. For every square-free integer d, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph* $E^d_{1_k} \stackrel{11}{\text{———}} E^d_{11_k}$ *is given by:*

twisted isogeny graph	*condition*	*Prob*
$E^d_{1_k} \longrightarrow E^d_{11_k}$	$d \not\equiv 0\,(11)$	$11/12$
$E^d_{1_k} \longrightarrow E^d_{11_k}$	$d \equiv 0\,(11)$	$1/12$

6 Type $L_2(13)$

6.1 Settings

Graph

The isogeny graphs of type $L_2(13)$ are given by two 13-isogenous elliptic curves:

$$E_1 \xrightarrow{\;13\;} E_{13}\,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(13)$ parametrize isogeny graphs of type $L_2(13)$. The curve $X_0(13)$ has genus 0 and a hauptmodul for this curve is:

$$t(\tau) = 13\left(\frac{\eta(13\tau)}{\eta(\tau)}\right)^2 .$$

j-invariants

Letting $t = t(\tau)$, one can write

$$\begin{aligned} j(E_1) = j(\tau) \quad &= \frac{\left(t^2+5t+13\right)\left(t^4+7t^3+20t^2+19t+1\right)^3}{t}\,, \\ j(E_{13}) = j(13\tau) &= \frac{\left(t^2+5t+13\right)\left(t^4+247t^3+3380t^2+15379t+28561\right)^3}{t^{13}}\,. \end{aligned}$$

Signatures

We can (and do) choose Weierstrass equations for (E_1, E_{13}) in such a way that the isogeny graph is normalized. Their signatures are:

	$L_2(13)$
$c_4(E_1)$	$(t^2+5t+13)(t^2+6t+13)(t^4+7t^3+20t^2+19t+1)$
$c_6(E_1)$	$(t^2+5t+13)(t^2+6t+13)^2(t^6+10t^5+46t^4+108t^3+122t^2+38t-1)$
$\Delta(E_1)$	$t(t^2+5t+13)^2(t^2+6t+13)^3$
$c_4(E_{13})$	$(t^2+5t+13)(t^2+6t+13)(t^4+247t^3+3380t^2+15379t+28561)$
$c_6(E_{13})$	$(t^2+5t+13)(t^2+6t+13)^2(t^6-494t^5-20618t^4-237276t^3-1313806t^2-3712930t-4826809)$
$\Delta(E_{13})$	$t^{13}(t^2+5t+13)^2(t^2+6t+13)^3$

Automorphism

The Fricke involution of $X_0(13)$ is given by $W_{13}(t) = 13/t$. With regard to the action of the Fricke involution on the isogeny graph, it can be described as:

$$W_{13}(E_1 \xrightarrow{\;13\;} E_{13}) \;=\; E_{13}^{-13} \xrightarrow{\;13\;} E_1^{-13}\,.$$

6.2 Kodaira symbols, minimal models, and Pal values

Table 14: $L_2(13)$ data for $p \neq 2, 3, 13$

$L_2(13)$	$p \neq 2, 3, 13$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = m > 0$	E_1	$(0, 0, m)$	1	I_m	1	1
	E_{13}	$(0, 0, 13m)$	1	I_{13m}	1	1
$v_p(t) = 0$, $v_p(t^2 + 5t + 13) = 6m$	E_1	$(2m, 0, 0)$	p^m	I_0	1	1
	E_{13}	$(2m, 0, 0)$	p^m	I_0	1	1
$v_p(t) = 0$, $v_p(t^2 + 5t + 13) = 6m + 1$	E_1	$(2m + 1, 1, 2)$	p^m	II	1	1
	E_{13}	$(2m + 1, 1, 2)$	p^m	II	1	1
$v_p(t) = 0$, $v_p(t^2 + 5t + 13) = 6m + 2$	E_1	$(2m + 2, 2, 4)$	p^m	IV	1	1
	E_{13}	$(2m + 2, 2, 4)$	p^m	IV	1	1
$v_p(t) = 0$, $v_p(t^2 + 5t + 13) = 6m + 3$	E_1	$(2m + 3, 3, 6)$	p^m	I_0^*	p	1
	E_{13}	$(2m + 3, 3, 6)$	p^m	I_0^*	p	1
$v_{13}(t) = 0$, $v_p(t^2 + 5t + 13) = 6m + 4$	E_1	$(2m + 4, 4, 8)$	p^m	IV^*	p	1
	E_{13}	$(2m + 4, 4, 8)$	p^m	IV^*	p	1
$v_p(t) = 0$, $v_p(t^2 + 5t + 13) = 6m + 5$	E_1	$(2m + 5, 5, 10)$	p^m	II^*	p	1
	E_{13}	$(2m + 5, 5, 10)$	p^m	II^*	p	1
$v_p(t) = 0$, $v_p(t^2 + 6t + 13) = 4m$	E_1	$(0, 2m, 0)$	p^m	I_0	1	1
	E_{13}	$(0, 2m, 0)$	p^m	I_0	1	1
$v_p(t) = 0$, $v_p(t^2 + 6t + 13) = 4m + 1$	E_1	$(1, 2m + 2, 3)$	p^m	III	1	1
	E_{13}	$(1, 2m + 2, 3)$	p^m	III	1	1
$v_p(t) = 0$, $v_p(t^2 + 6t + 13) = 4m + 2$	E_1	$(2, 2m + 4, 6)$	p^m	I_0^*	p	1
	E_{13}	$(2, 2m + 4, 6)$	p^m	I_0^*	p	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod p$	

Table 14: $L_2(13)$ data for $p \neq 2, 3, 13$ (Continued)

$L_2(13)$	$p \neq 2, 3, 13$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = 0$ $v_p(t^2 + 6t + 13) = 4m + 3$	E_1	$(3, 2m + 6, 9)$	p^m	III^*	p	1
	E_{13}	$(3, 2m + 6, 9)$	p^m	III^*	p	1
$v_p(t) = -m < 0$	E_1	$(0, 0, 13m)$	p^{-2m}	I_{13m}	1	1
	E_{13}	$(0, 0, m)$	p^{-2m}	I_m	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod p$	

Table 15: $L_2(13)$ data for $p = 3$

$L_2(13)$	$p=3$					
t	E	$\mathrm{sig}_3(E)$	u_3	$\mathrm{K}_3(E)$	$u_3(d)$	
$v_3(t)=m>0$	E_1	$(0,0,m)$	1	I_m	1	1
	E_{13}	$(0,0,13m)$	1	I_{13m}	1	1
$v_3(t)=0$ $t\equiv 1\,(3)$	E_1	$(1,\geq 3,0)$	1	I_0	1	1
	E_{13}	$(1,\geq 3,0)$	1	I_0	1	1
$v_3(t)=0$ $t\equiv 5,8\,(9)$	E_1	$(2,3,4)$	1	II	1	1
	E_{13}	$(2,3,4)$	1	II	1	1
$v_3(t)=0$ $t\equiv 2,20\,(27)$	E_1	$(3,5,6)$	1	IV	1	1
	E_{13}	$(3,5,6)$	1	IV	1	1
$v_3(t)=0$ $t\equiv 11\,(27)$	E_1	$(3,\geq 6,6)$	1	I_0^*	3	1
	E_{13}	$(3,\geq 6,6)$	1	I_0^*	3	1
$v_3(t)=-m<0$	E_1	$(0,0,13m)$	3^{-2m}	I_{13m}	1	1
	E_{13}	$(0,0,m)$	3^{-2m}	I_m	1	1
					$d\equiv 0$	$d\not\equiv 0$
					$d \pmod 3$	

Table 16: $L_2(13)$ data for $p = 13$

$L_2(13)$	$p = 13$					
t	E	$\mathrm{sig}_{13}(E)$	u_{13}	$\mathrm{K}_{13}(E)$	$u_{13}(d)$	
$v_{13}(t) = m \geq 2$	E_1	$(2, 3, m+5)$	1	I^*_{m-1}	13	1
	E_{13}	$(2, 3, 13m-7)$	13	$\mathrm{I}^*_{13(m-1)}$	13	1
$v_{13}(t) = 1$ $t/13 \not\equiv 2, 5\,(13)$	E_1	$(2, 3, 6)$	1	I^*_0	13	1
	E_{13}	$(2, 3, 6)$	13	I^*_0	13	1
$v_{13}(t) = 1$ $t/13 \equiv 2\,(13)$ $v_{13}(t^2+6t+13) = 4m$	E_1	$(1, \geq 3, 3)$	13^m	III	1	1
	E_{13}	$(1, \geq 3, 3)$	13^{m+1}	III	1	1
$v_{13}(t) = 1$ $t/13 \equiv 2\,(13)$ $v_{13}(t^2+6t+13) = 4m+1$	E_1	$(2, \geq 3, 6)$	13^m	I^*_0	13	1
	E_{13}	$(2, \geq 4, 6)$	13^{m+1}	I^*_0	13	1
$v_{13}(t) = 1$ $t/13 \equiv 2\,(13)$ $v_{13}(t^2+6t+13) = 4m+2$	E_1	$(3, \geq 5, 9)$	13^m	III^*	13	1
	E_{13}	$(3, \geq 6, 9)$	13^{m+1}	III^*	13	1
$v_{13}(t) = 1$ $t/13 \equiv 2\,(13)$ $v_{13}(t^2+6t+13) = 4m+3$	E_1	$(0, \geq 1, 0)$	13^{m+1}	I_0	1	1
	E_{13}	$(0, \geq 2, 0)$	13^{m+2}	I_0	1	1
$v_{13}(t) = 1$ $t/13 \equiv 5\,(13)$ $v_{13}(t^2+5t+13) = 6m$	E_1	$(\geq 2, 2, 4)$	13^m	IV	1	1
	E_{13}	$(\geq 2, 2, 4)$	13^{m+1}	IV	1	1
$v_{13}(t) = 1$ $t/13 \equiv 5\,(13)$ $v_{13}(t^2+5t+13) = 6m+1$	E_1	$(\geq 2, 3, 6)$	13^m	I^*_0	13	1
	E_{13}	$(\geq 3, 3, 6)$	13^{m+1}	I^*_0	13	1
$v_{13}(t) = 1$ $t/13 \equiv 5\,(13)$ $v_{13}(t^2+5t+13) = 6m+2$	E_1	$(\geq 3, 4, 8)$	13^m	IV^*	13	1
	E_{13}	$(\geq 4, 4, 8)$	13^{m+1}	IV^*	13	1
$v_{13}(t) = 1$ $t/13 \equiv 5\,(13)$ $v_{13}(t^2+5t+13) = 6m+3$	E_1	$(\geq 4, 5, 10)$	13^m	II^*	13	1
	E_{13}	$(\geq 5, 5, 10)$	13^{m+1}	II^*	13	1
$v_{13}(t) = 1$ $t/13 \equiv 5\,(13)$ $v_{13}(t^2+5t+13) = 6m+4$	E_1	$(\geq 0, 0, 0)$	13^{m+1}	I_0	1	1
	E_{13}	$(\geq 2, 0, 0)$	13^{m+2}	I_0	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{13}$	

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Table 16: $L_2(13)$ data for $p = 13$ (Continued)

$L_2(13)$	$p = 13$					
t	E	$\mathrm{sig}_{13}(E)$	u_{13}	$\mathrm{K}_{13}(E)$	$u_{13}(d)$	
$v_{13}(t) = 1$ $t/13 \equiv 5\,(13)$ $v_{13}(t^2 + 5t + 13) = 6m + 5$	E_1	$(\geq 2, 1, 2)$	13^{m+1}	II	1	1
	E_{13}	$(\geq 3, 1, 2)$	13^{m+2}	II	1	1
$v_{13}(t) = 0$ $t \not\equiv 7, 8\,(13)$	E_1	$(0, 0, 0)$	1	I_0	1	1
	E_1	$(0, 0, 0)$	1	I_0	1	1
$v_{13}(t) = 0$ $t \equiv 7\,(13)$ $v_{13}(t^2 + 6t + 13) = 4m$	E_1	$(0, \geq 1, 0)$	13^m	I_0	1	1
	E_{13}	$(0, \geq 0, 0)$	13^m	I_0	1	1
$v_{13}(t) = 0$ $t \equiv 7\,(13)$ $v_{13}(t^2 + 6t + 13) = 4m + 1$	E_1	$(1, \geq 3, 3)$	13^m	III	1	1
	E_{13}	$(1, \geq 2, 3)$	13^m	III	1	1
$v_{13}(t) = 0$ $t \equiv 7\,(13)$ $v_{13}(t^2 + 6t + 13) = 4m + 2$	E_1	$(2, \geq 5, 6)$	13^m	I_0^*	13	1
	E_{13}	$(2, \geq 4, 6)$	13^m	I_0^*	13	1
$v_{13}(t) = 0$ $t \equiv 7\,(13)$ $v_{13}(t^2 + 6t + 13) = 4m + 3$	E_1	$(3, \geq 7, 9)$	13^m	III^*	13	1
	E_{13}	$(3, \geq 6, 9)$	13^m	III^*	13	1
$v_{13}(t) = 0$ $t/13 \equiv 8\,(13)$ $v_{13}(t^2 + 5t + 13) = 6m$	E_1	$(\geq 1, 0, 0)$	13^m	I_0	1	1
	E_{13}	$(\geq 0, 0, 0)$	13^m	I_0	1	1
$v_{13}(t) = 0$ $t/13 \equiv 8\,(13)$ $v_{13}(t^2 + 5t + 13) = 6m + 1$	E_1	$(\geq 2, 1, 2)$	13^m	II	1	1
	E_{13}	$(\geq 1, 1, 2)$	13^m	II	1	1
$v_{13}(t) = 0$ $t \equiv 8\,(13)$ $v_{13}(t^2 + 5t + 13) = 6m + 2$	E_1	$(\geq 3, 2, 4)$	13^m	IV	1	1
	E_{13}	$(\geq 2, 2, 4)$	13^m	IV	1	1
$v_{13}(t) = 0$ $t \equiv 8\,(13)$ $v_{13}(t^2 + 5t + 13) = 6m + 3$	E_1	$(\geq 4, 3, 6)$	13^m	I_0^*	13	1
	E_1	$(\geq 3, 3, 6)$	13^m	I_0^*	13	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{13}$	

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Table 16: $L_2(13)$ data for $p = 13$ (Continued)

$L_2(13)$	$p = 13$					
t	E	$\mathrm{sig}_{13}(E)$	u_{13}	$\mathrm{K}_{13}(E)$	$u_{13}(d)$	
$v_{13}(t) = 0$ $t \equiv 8\,(13)$ $v_{13}(t^2 + 5t + 13) = 6m + 4$	E_1	$(\geq 5, 4, 8)$	13^m	IV^*	13	1
	E_{13}	$(\geq 4, 4, 8)$	13^m	IV^*	13	1
$v_{13}(t) = 0$ $t \equiv 8\,(13)$ $v_{13}(t^2 + 5t + 13) = 6m + 5$	E_1	$(\geq 6, 5, 10)$	13^m	II^*	13	1
	E_{13}	$(\geq 5, 5, 10)$	13^m	II^*	13	1
$v_{13}(t) = -m < 0$	E_1	$(0, 0, 13m)$	13^{-2m}	I_{13m}	1	1
	E_{13}	$(0, 0, m)$	13^{-2m}	I_m	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{13}$	

Table 17: $L_2(13)$ data for p=2

$L_2(13)$	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=m\geq 2$	E_1	$(0,0,m)$	1	I_m	1	2^{-1}	2^{-1}
	E_{13}	$(0,0,13m)$	1	I_{13m}	1	2^{-1}	2^{-1}
$v_2(t)=1$	E_1	$(4,6,13)$	2^{-1}	I_5^*	1	1	2
	E_{13}	$(4,6,25)$	2^{-1}	I_{17}^*	1	1	2
$v_2(t)=0$ $t\equiv 1\,(4)$	E_1	$(6,6,6)$	1	II	1	1^* or 2^*	1
	E_{13}	$(6,6,6)$	1	II	1	1^* or 2^*	1
$v_2(t)=0$ $t\equiv 3\,(4)$	E_1	$(5,8,9)$	1	III	1	1	1
	E_{13}	$(5,8,9)$	1	III	1	1	1
$v_2(t)=-1$	E_1	$(0,0,13)$	2^{-2}	I_{13}	1	2^{-1}	2^{-1}
	E_{13}	$(0,0,1)$	2^{-2}	I_1	1	2^{-1}	2^{-1}
$v_2(t)=-m\leq -2$	E_1	$(4,6,13m+12)$	$2^{-(2m+1)}$	I_{13m+4}^*	1	1	2
	E_{13}	$(4,6,m+12)$	$2^{-(2m+1)}$	I_{m+4}^*	1	1	2
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					d (mod 4)		

Remark (1^* or 2^*): If $v_2(t)=0$, $t\equiv 1\,(4)$ and $d\equiv 2\,(4)$ then, for E_1,E_{13}, the value $u_2(d)$ is given by

- if $t\equiv 1\,(8)$ then $u_2(d)=\begin{cases}1 \text{ if } d\equiv 2\,(8)\\ 2 \text{ if } d\equiv -2\,(8);\end{cases}$

- if $t\equiv 5\,(8)$ then $u_2(d)=\begin{cases}2 \text{ if } d\equiv 2\,(8)\\ 1 \text{ if } d\equiv -2\,(8).\end{cases}$

6.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

t	$[u(E)]$	$[u(E)(d)]$	Prob
$v_{13}(t) > 0$	$(1 : 13)$	$(1 : 1)$	$(0, 1)$
$v_{13}(t) \leq 0$	$(1 : 1)$	$(1 : 1)$	$(1, 0)$

The contents of this table are the main ingredients to prove the following result:

Proposition 6. *Let* $E_1 \overset{13}{\text{———}} E_{13}$ *be a* $\mathbf{Q}$*-isogeny graph of type* $L_2(13)$ *corresponding to a given* t *in* $\mathbf{Q}^*$*. For every square-free integer* d*, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph* $E_1^d \overset{13}{\text{———}} E_{13}^d$ *is given by:*

$L_2(13)$	*twisted isogeny graph*	*Prob*
$v_{13}(t) > 0$	$E_1^d \longleftarrow E_{13}^d$	1
$v_{13}(t) \leq 0$	$E_1^d \longrightarrow E_{13}^d$	1

7 Type $L_2(17)$

7.1 Settings

Graph

The isogeny graphs of type $L_2(17)$ are given by two 17-isogenous elliptic curves:

$$E_1 \overset{17}{\text{———}} E_{17}\,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(17)$ parametrize isogeny graphs of type $L_2(17)$. The modular curve $X_0(17)$ is an elliptic curve of rank 0 over the rationals. Its rational points are: two rational cusps and two non-cuspidal non-CM points $\tau, \tau' \in \mathbb{H}$.

j-invariants

The corresponding j-invariants of τ and τ' are:

$$j(\tau) = \frac{-17 \cdot 373^3}{2^{17}}, \qquad j(\tau') = \frac{-17^2 \cdot 101^3}{2}.$$

We have $j(17\tau) = j(\tau')$.

Signatures

We can (and do) choose Weierstrass equations in such a way that the isogeny graphs are normalized:

E	Minimal Weierstrass model	$j(E)$	label
E_1	$y^2 + xy = x^3 + x^2 - 660x - 7600$	$\frac{-17 \cdot 373^3}{2^{17}}$	14450n1
E_{17}	$y^2 + xy = x^3 + x^2 - 878710x + 316677750$	$\frac{-17^2 \cdot 101^3}{2}$	14450n2

Their signatures are:

E	E_1	E_{17}
$c_4(E)$	$5 \cdot 17 \cdot 373$	$5 \cdot 17^4 \cdot 101$
$c_6(E)$	$5^2 \cdot 17 \cdot 14891$	$-5^2 \cdot 17^5 \cdot 7717$
$\Delta(E)$	$-2^{17} \cdot 5^3 \cdot 17^2$	$-2 \cdot 5^3 \cdot 17^{10}$

We have that the Faltings curve (circled) in the graph is

$$\boxed{E_1 \longrightarrow E_{17}}$$

Note that any $\mathbb{Q}$-isogeny class of elliptic curves of type $L_2(17)$ is obtained by quadratic twist from

$$E_1 \overset{17}{\text{———}} E_{17}\,.$$

7.2 Kodaira symbols, minimal models, and Pal values

There are three bad reduction primes $p = 2$, 5, and 17.

$p = 2$					
E	$\mathrm{sig}_2(E)$	$\mathrm{K}_2(E)$	$u_2(d)$		
E_1	$(0, 0, 17)$	I_{17}	1	1/2	1/2
E_{17}	$(0, 0, 1)$	I_1	1	1/2	1/2
			$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
			$d \pmod 4$		

$p = 5$			
E	$\mathrm{sig}_5(E)$	$\mathrm{K}_5(E)$	$u_5(d)$
E_1	$(1, 2, 3)$	III	1
E_{17}	$(1, 2, 3)$	III	1

$p = 17$				
E	$\mathrm{sig}_{17}(E)$	$\mathrm{K}_{17}(E)$	$u_{17}(d)$	
E_1	$(1, 1, 2)$	II	1	1
E_{17}	$(4, 5, 10)$	II^*	17	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{17}$	

7.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

$[u(E)(d)]$	d	Prob
$(1 : 1)$	$d \not\equiv 0\,(17)$	$\left(\frac{17}{18}, \frac{1}{18}\right)$
$(1 : 17)$	$d \equiv 0\,(17)$	

This table is the main ingredient to prove the following result:

Proposition 7. *For every square-free integer d, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph* $E_1^d \overset{17}{\text{———}} E_{17}^d$ *is given by:*

twisted isogeny graph	*condition*	*Prob*
$E_1^d \longrightarrow E_{17}^d$	$d \not\equiv 0\,(17)$	17/18
$E_1^d \longrightarrow E_{17}^d$	$d \equiv 0\,(17)$	1/18

8 Type $L_2(19)$

8.1 Settings

Graph

The isogeny graphs of type $L_2(19)$ are given by two 19-isogenous elliptic curves:

$$E_1 \xrightarrow{\ 19\ } E_{19}\,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(19)$ parametrize isogeny graphs of type $L_2(19)$. The modular curve $X_0(19)$ is an elliptic curve of rank 0 over the rationals. Its rational points are: two rational cusps and one rational CM point $\tau = \frac{1}{2} + \frac{\sqrt{-19}}{2\cdot 19} \in \mathbb{H}$.

j-invariants

The corresponding j-invariant of τ is:

$$j(\tau) = -2^{15}\cdot 3^3.$$

We have $j(19\tau) = j(\tau)$.

Signatures

We can (and do) choose Weierstrass equations in such a way that the isogeny graphs are normalized:

E	Minimal Weierstrass model	$j(E)$	label
E_1	$y^2+y = x^3-38x+90$	$-2^{15}\cdot 3^3$	`361a1`
E_{19}	$y^2+y = x^3-13718x-619025$	$-2^{15}\cdot 3^3$	`361a2`

Their signatures are:

E	E_1	E_{19}
$c_4(E)$	$2^5\cdot 3\cdot 19$	$2^5\cdot 3\cdot 19^3$
$c_6(E)$	$-2^3\cdot 3^3\cdot 19^2$	$2^3\cdot 3^3\cdot 19^5$
$\Delta(E)$	-19^3	-19^9

We have that the Faltings curve (circled) in the graph is

$$\boxed{E_1 \longrightarrow E_{19}}$$

Note that any $\mathbb{Q}$-isogeny class of elliptic curves of type $L_2(19)$ can be obtained by quadratic twist from

$$E_1 \xrightarrow{\ 19\ } E_{19}\,.$$

Note that E_1 and E_{19} have complex multiplication by the ring of integers of $\mathbf{Q}(\sqrt{-19})$ and $E_{19} = E_1^{-19}$.

8.2 Kodaira symbols, minimal models, and Pal values

There are only one bad reduction prime $p = 19$.

$p = 19$				
E	$\mathrm{sig}_{19}(E)$	$\mathrm{K}_{19}(E)$	$u_{19}(d)$	
E_1	$(1, 2, 3)$	III	1	1
E_{19}	$(3, 5, 9)$	III*	19	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{19}$	

8.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

$[u(E)(d)]$	d	Prob
$(1 : 1)$	$d \not\equiv 0\,(19)$	$\left(\frac{19}{20}, \frac{1}{20}\right)$
$(1 : 19)$	$d \equiv 0\,(19)$	

This table is the main ingredient to prove the following result:

Proposition 8. *For every square-free integer d, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph* $E_1^d \overset{19}{\text{---}} E_{19}^d$ *is given by:*

twisted isogeny graph	*condition*	*Prob*
$E_1^d \longrightarrow E_{19}^d$	$d \not\equiv 0\,(19)$	19/20
$E_1^d \longrightarrow E_{19}^d$	$d \equiv 0\,(19)$	1/20

9 Type $L_2(37)$

9.1 Settings

Graph

The isogeny graphs of type $L_2(37)$ are given by two 37-isogenous elliptic curves:

$$E_1 \stackrel{37}{\text{---}} E_{37}\,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(37)$ parametrize isogeny graphs of type $L_2(37)$. The modular curve $X_0(37)$ has genus 2. Its rational points are: two cusps and two non-CM points corresponding to a pair $\tau, \tau' \in \mathbb{H}$.

j-invariants

The corresponding j-invariants of τ and τ' are:

$$j(\tau) = -7 \cdot 11^3, \qquad j(\tau') = -7 \cdot 137^3 \cdot 2083^3,$$

and $j(37\tau) = j(\tau')$.

Signatures

We can (and do) choose Weierstrass equations in such a way that the isogeny graphs are normalized:

E	Minimal Weierstrass model	$j(E)$	label
E_1	$y^2 + xy + y = x^3 + x^2 - 8x + 6$	$-7 \cdot 11^3$	`1225h1`
E_{37}	$y^2 + xy + y = x^3 + x^2 - 208083x - 36621194$	$-7 \cdot 137^3 \cdot 2083^3$	`1225h2`

Their signatures are:

E	E_1	E_{37}
$c_4(E)$	$5 \cdot 7 \cdot 11$	$5 \cdot 7 \cdot 137 \cdot 2083$
$c_6(E)$	$-5^2 \cdot 7 \cdot 47$	$5^2 \cdot 7 \cdot 11 \cdot 1433 \cdot 11443$
$\Delta(E)$	$-5^3 \cdot 7^2$	$-5^3 \cdot 7^2$

We have that the Faltings curve (circled) in the graph is

$$E_1 \longrightarrow E_{37}$$

Note that any $\mathbb{Q}$-isogeny class of elliptic curves of type $L_2(37)$ can be obtained by quadratic twist from

$$E_1 \stackrel{37}{\text{---}} E_{37}\,.$$

9.2 Kodaira symbols, minimal models, and Pal values

There are two primes of bad reduction $p = 5$ and 7.

$p = 5$			
E	$\mathrm{sig}_5(E)$	$\mathrm{K}_5(E)$	$u_5(d)$
E_1	$(1, 2, 3)$	III	1
E_{37}	$(1, 2, 3)$	III	1

$p = 7$			
E	$\mathrm{sig}_7(E)$	$\mathrm{K}_7(E)$	$u_7(d)$
E_1	$(1, 1, 2)$	II	1
E_{37}	$(1, 1, 2)$	II	1

9.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

$[u(E)(d)]$	Prob
$(1 : 1)$	$(1, 0)$

This table is the main ingredient to prove the following result:

Proposition 9. *For every square-free integer d, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph $E_1^d \xrightarrow{37} E_{37}^d$ is given by:*

twisted isogeny graph	*Prob*
$E_1^d \longrightarrow E_{37}^d$	1

10 Type $L_2(43)$

10.1 Settings

Graph

The isogeny graphs of type $L_2(43)$ are given by two 43-isogenous elliptic curves:

$$E_1 \overset{43}{\text{------}} E_{43}\,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(43)$ parametrize isogeny graphs of type $L_2(43)$. The modular curve $X_0(43)$ has genus 3. Its rational points are: two rational cusps and one rational CM point given by $\tau = \frac{1}{2} + \frac{\sqrt{-43}}{2\cdot 43} \in \mathbb{H}$.

j-invariants

The corresponding j-invariant of τ is:

$$j(\tau) = -2^{18} \cdot 3^3 \cdot 5^3.$$

We have $j(43\tau) = j(\tau)$.

Signatures

We can (and do) choose Weierstrass equations in such a way that the isogeny graphs are normalized:

E	Minimal Weierstrass model	$j(E)$	label
E_1	$y^2 + y = x^3 - 860x + 9707$	$-2^{18} \cdot 3^3 \cdot 5^3$	`1849a1`
E_{43}	$y^2 + y = x^3 - 1590140x - 771794326$	$-2^{18} \cdot 3^3 \cdot 5^3$	`1849a2`

Their signatures are:

E	E_1	E_{43}
$c_4(E)$	$2^6 \cdot 3 \cdot 5 \cdot 43$	$2^6 \cdot 3 \cdot 5 \cdot 43^3$
$c_6(E)$	$-2^3 \cdot 3^4 \cdot 7 \cdot 43^2$	$2^3 \cdot 3^4 \cdot 7 \cdot 43^5$
$\Delta(E)$	-43^3	-43^9

We have that the Faltings curve (circled) in the graph is

$$\boxed{E_1 \longrightarrow E_{43}}$$

Note that any $\mathbb{Q}$-isogeny class of type $L_2(43)$ is obtained by quadratic twist from

$$E_1 \overset{43}{\text{------}} E_{43}\,.$$

Note that E_1 and E_{43} have complex multiplication by the ring of integers of $\mathbf{Q}(\sqrt{-43})$ and $E_{43} = E_1^{-43}$.

10.2 Kodaira symbols, minimal models, and Pal values

There is only one bad reduction prime $p = 43$.

$p = 43$				
E	$\mathrm{sig}_{43}(E)$	$\mathrm{K}_{43}(E)$	$u_{43}(d)$	
E_1	$(1, 2, 3)$	III	1	1
E_{43}	$(3, 5, 9)$	III*	43	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{43}$	

10.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

$[u(E)(d)]$	d	Prob
$(1 : 1)$	$d \not\equiv 0\,(43)$	$\left(\frac{43}{44}, \frac{1}{44}\right)$
$(1 : 43)$	$d \equiv 0\,(43)$	

This table is the main ingredient to prove the following result:

Proposition 10. *For every square-free integer d, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph* $E_1^d \overset{43}{\text{——}} E_{43}^d$ *is given by:*

twisted isogeny graph	*condition*	*Prob*
$E_1^d \longrightarrow E_{43}^d$	$d \not\equiv 0\,(43)$	$43/44$
$E_1^d \longrightarrow E_{43}^d$	$d \equiv 0\,(43)$	$1/44$

11 Type $L_2(67)$

11.1 Settings

Graph

The isogeny graphs of type $L_2(67)$ are given by two 67-isogenous elliptic curves:

$$E_1 \overset{67}{\text{———}} E_{67}\,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(67)$ parametrize isogeny graphs of type $L_2(67)$. The modular curve $X_0(67)$ has genus 5. Its rational points are: two cusps and one CM point associated with $\tau = \frac{1}{2} + \frac{\sqrt{-67}}{2\cdot 67} \in \mathbb{H}$.

j-invariant

The corresponding j-invariant of τ is:

$$j(\tau) = -2^{15} \cdot 3^3 \cdot 5^3 \cdot 11^3,$$

with $j(67\tau) = j(\tau)$.

Signatures

We can (and do) choose Weierstrass equations in such a way that the isogeny graphs are normalized:

E	Minimal Weierstrass model	$j(E)$	label
E_1	$y^2 + y = x^3 - 7370x + 243528$	$-2^{15} \cdot 3^3 \cdot 5^3 \cdot 11^3$	`4489a1`
E_{67}	$y^2 + y = x^3 - 33083930x - 73244287055$	$-2^{15} \cdot 3^3 \cdot 5^3 \cdot 11^3$	`4489a2`

Their signatures are:

E	E_1	E_{67}
$c_4(E)$	$2^5 \cdot 3 \cdot 5 \cdot 11 \cdot 67$	$2^5 \cdot 3 \cdot 5 \cdot 11 \cdot 67^3$
$c_6(E)$	$-2^3 \cdot 3^3 \cdot 7 \cdot 31 \cdot 67^2$	$2^3 \cdot 3^3 \cdot 7 \cdot 31 \cdot 67^5$
$\Delta(E)$	-67^3	-67^9

We have that the Faltings curve (circled) in the graph is

$$\boxed{E_1 \longrightarrow E_{67}}$$

Note that any $\mathbb{Q}$-isogeny class of type $L_2(67)$ can be obtained by quadratic twist from

$$E_1 \overset{67}{\text{———}} E_{67}\,.$$

Note that E_1 and E_{67} have complex multiplication by the ring of integers of $\mathbf{Q}(\sqrt{-67})$ and $E_{67} = E_1^{-67}$.

11.2 Kodaira symbols, minimal models, and Pal values

There is only one bad reduction prime $p = 67$.

$p = 67$				
E	$\mathrm{sig}_{67}(E)$	$\mathrm{K}_{67}(E)$	$u_{67}(d)$	
E_1	$(1, 2, 3)$	III	1	1
E_{67}	$(3, 5, 9)$	III^*	67	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod{67}$	

11.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

$[u(E)(d)]$	d	Prob
$(1 : 1)$	$d \not\equiv 0\,(67)$	$\left(\frac{67}{68}, \frac{1}{68}\right)$
$(1 : 67)$	$d \equiv 0\,(67)$	

This table is the main ingredient to prove the following result:

Proposition 11. *For every square-free integer d, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph* $E_1^d \overset{67}{\text{———}} E_{67}^d$ *is given by:*

twisted isogeny graph	*condition*	*Prob*
$E_1^d \longrightarrow E_{67}^d$	$d \not\equiv 0\,(67)$	$67/68$
$E_1^d \longrightarrow E_{67}^d$	$d \equiv 0\,(67)$	$1/68$

12 Type $L_2(163)$

12.1 Settings

Graph

The isogeny graphs of type $L_2(163)$ are given by two 163-isogenous elliptic curves:

$$E_1 \xrightarrow{163} E_{163}\,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(163)$ parametrize isogeny graphs of type $L_2(163)$. The modular curve $X_0(163)$ has genus 13. Its rational points are: two rational cusps and one rational CM point corresponding to $\tau = \frac{1}{2} + \frac{\sqrt{-163}}{2\cdot 163} \in \mathbb{H}$.

j-invariants

The corresponding j-invariant of τ is:

$$j(\tau) = -2^{18}\cdot 3^3\cdot 5^3\cdot 23^3\cdot 29^3,$$

with $j(163\tau) = j(\tau)$.

Signatures

We can (and do) choose Weierstrass equations in such a way that the isogeny graphs are normalized:

E	Minimal Weierstrass model	$j(E)$	label
E_1	$y^2 + y = x^3 - 2174420x + 1234136692$	$-2^{18}\cdot 3^3\cdot 5^3\cdot 23^3\cdot 29^3$	26569a1
E_{163}	$y^2 + y = x^3 - 57772164980x - 5344733777551611$	$-2^{18}\cdot 3^3\cdot 5^3\cdot 23^3\cdot 29^3$	26569a2

Their signatures are:

E	E_1	E_{163}
$c_4(E)$	$2^6\cdot 3\cdot 5\cdot 23\cdot 29\cdot 163$	$2^6\cdot 3\cdot 5\cdot 23\cdot 29\cdot 163^3$
$c_6(E)$	$-2^3\cdot 3^3\cdot 7\cdot 11\cdot 19\cdot 127\cdot 163^2$	$2^3\cdot 3^3\cdot 7\cdot 11\cdot 19\cdot 127\cdot 163^5$
$\Delta(E)$	-163^3	-163^9

We have that the Faltings curve (circled) in the graph is

$$\boxed{E_1 \longrightarrow E_{163}}$$

Note any $\mathbb{Q}$-isogeny class of type $L_2(163)$ can be obtained by quadratic twist from

$$E_1 \xrightarrow{163} E_{163}\,.$$

Note that E_1 and E_{163} have complex multiplication by the ring of integers of $\mathbf{Q}(\sqrt{-163})$ and $E_{163} = E_1^{-163}$.

12.2 Kodaira symbols, minimal models, and Pal values

There is only one bad reduction prime $p = 163$.

<table>
<tr><td colspan="5">$p = 163$</td></tr>
<tr><td>E</td><td>$\mathrm{sig}_{163}(E)$</td><td>$\mathrm{K}_{163}(E)$</td><td colspan="2">$u_{163}(d)$</td></tr>
<tr><td>E_1</td><td>$(1, 2, 3)$</td><td>III</td><td>1</td><td>1</td></tr>
<tr><td>E_{163}</td><td>$(3, 5, 9)$</td><td>III*</td><td>163</td><td>1</td></tr>
<tr><td></td><td></td><td></td><td>$d \equiv 0$</td><td>$d \not\equiv 0$</td></tr>
<tr><td></td><td></td><td></td><td colspan="2">$d \pmod{163}$</td></tr>
</table>

12.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

<table>
<tr><td>$[u(E)(d)]$</td><td>d</td><td>Prob</td></tr>
<tr><td>$(1 : 1)$</td><td>$d \not\equiv 0\,(163)$</td><td rowspan="2">$\left(\frac{163}{164}, \frac{1}{164}\right)$</td></tr>
<tr><td>$(1 : 163)$</td><td>$d \equiv 0\,(163)$</td></tr>
</table>

This table is the main ingredient to prove the following result:

Proposition 12. *For every square-free integer d, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph* $E_1^d \overset{163}{\text{---}} E_{163}^d$ *is given by:*

twisted isogeny graph	*condition*	*Prob*
$E_1^d \longrightarrow E_{163}^d$	$d \not\equiv 0\,(163)$	$163/164$
$E_1^d \longrightarrow E_{163}^d$	$d \equiv 0\,(163)$	$1/164$

13 Type $L_3(9)$

13.1 Setting

Graph

The isogeny graphs of type $L_3(9)$ are given by three isogenous elliptic curves:

$$E_1 \overset{3}{\longrightarrow} E_3 \overset{3}{\longrightarrow} E_9 \,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(9)$ parametrize isogeny graphs of type $L_3(9)$. The curve $X_0(9)$ has genus 0 and a hauptmodul for this curve is:

$$t(\tau) = 3^3 \left(\frac{\eta(9\tau)}{\eta(\tau)}\right)^3 .$$

j-invariants

Letting $t = t(\tau)$, one can write

$$\begin{aligned}
j(E_1) = j(\tau) &= \frac{(t+3)^3(t^3+9\,t^2+27\,t+3)^3}{t(t^2+9\,t+27)}\,,\\
j(E_3) = j(3\tau) &= \frac{(t+3)^3(t+9)^3}{t^3(t^2+9\,t+27)^3}\,,\\
j(E_9) = j(9\tau) &= \frac{(t+9)^3(t^3+243\,t^2+2187\,t+6561)^3}{t^9(t^2+9\,t+27)}\,.
\end{aligned}$$

Signatures We can (and do) choose Weierstrass equations for (E_1, E_3, E_9) in such a way that the isogeny graph is normalized. Their signatures are:

	$L_3(9)$
$c_4(E_1)$	$(t+3)(t^3+9\,t^2+27\,t+3)$
$c_6(E_1)$	$t^6+18\,t^5+135\,t^4+504\,t^3+891\,t^2+486\,t-27$
$\Delta(E_1)$	$t(t^2+9\,t+27)$
$c_4(E_3)$	$(t+3)(t+9)(t^2+27)$
$c_6(E_3)$	$(t^2-27)(t^4+18\,t^3+162\,t^2+486\,t+729)$
$\Delta(E_3)$	$t^3(t^2+9\,t+27)^3$
$c_4(E_9)$	$(t+9)(t^3+243\,t^2+2187\,t+6561)$
$c_6(E_9)$	$t^6-486\,t^5-24057\,t^4-367416\,t^3-2657205\,t^2-9565938\,t-14348907$
$\Delta(E_9)$	$t^9(t^2+9\,t+27)$

Automorphisms

The subgroup of $\operatorname{Aut} X_0(9)$ that fixes the set of vertices of the graph is generated by the Fricke involution of $X_0(9)$, given by $W_9(t) = 3^3/t$. The involution W_9 acts on the isogeny graphs of type $L_3(9)$ as:

$$W_9(E_1 \xrightarrow{\quad 3 \quad} E_3 \xrightarrow{\quad 3 \quad} E_9) = E_9^{-3} \xrightarrow{\quad 3 \quad} E_3^{-3} \xrightarrow{\quad 3 \quad} E_1^{-3}.$$

13.2 Kodaira symbols, minimal models, and Pal values

Table 18: $L_3(9)$ data for $p \neq 2$, 3

$L_3(9)$	$p \neq 2, 3$				
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$
$v_p(t) = m > 0$	E_1	$(0, 0, m)$	1	I_m	1
	E_3	$(0, 0, 3m)$	1	I_{3m}	1
	E_9	$(0, 0, 9m)$	1	I_{9m}	1
$v_p(t) = 0$, $m = v_p(t^2 + 9\,t + 27) \geq 0$	E_1	$(0, 0, m)$	1	I_m	1
	E_3	$(0, 0, 3m)$	1	I_{3m}	1
	E_9	$(0, 0, m)$	1	I_m	1
$v_p(t) = -m < 0$	E_1	$(0, 0, 9m)$	p^{-m}	I_{9m}	1
	E_3	$(0, 0, 3m)$	p^{-m}	I_{3m}	1
	E_9	$(0, 0, m)$	p^{-m}	I_m	1

Table 19: $L_3(9)$ data for p=3

$L_3(9)$	$p = 3$					
t	E	$\mathrm{sig}_3(E)$	u_3	$\mathrm{K}_3(E)$	$u_3(d)$	
$v_3(t) = m \geq 3$	E_1	$(2, 3, m + 3)$	1	I^*_{m-3}	3	1
	E_3	$(2, 3, 3m - 3)$	3	$\mathrm{I}^*_{3(m-3)}$	3	1
	E_9	$(2, 3, 9m - 21)$	3^2	$\mathrm{I}^*_{9(m-3)}$	3	1
$v_3(t) = 2$	E_1	$(2, 3, 5)$	1	IV	1	1
	E_3	$(\geq 2, 3, 3)$	3	II	1	1
	E_9	$(\geq 4, 6, 9)$	3	IV*	3	1
$v_3(t) = 1$	E_1	$(\geq 2, 3, 3)$	1	II	1	1
	E_3	$(\geq 4, 6, 9)$	1	IV*	3	1
	E_9	$(4, 6, 11)$	1	II*	3	1
$v_3(t) = -m \leq 0$	E_1	$(0, 0, 9m)$	3^{-m}	I_{9m}	1	1
	E_3	$(0, 0, 3m)$	3^{-m}	I_{3m}	1	1
	E_9	$(0, 0, m)$	3^{-m}	I_m	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod 3$	

Table 20: $L_3(9)$ data for p=2

$L_3(9)$	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=m>0$	E_1	$(4,6,m+12)$	2^{-1}	I^*_{m+4}	1	1	2
	E_3	$(4,6,3m+12)$	2^{-1}	I^*_{3m+4}	1	1	2
	E_9	$(4,6,9m+12)$	2^{-1}	I^*_{9m+4}	1	1	2
$v_2(t)=0$	E_1	$(\geq 8,9,12)$	2^{-1}	II^*	1	2	2
	E_3	$(\geq 8,9,12)$	2^{-1}	II^*	1	2	2
	E_9	$(\geq 8,9,12)$	2^{-1}	II^*	1	2	2
$v_2(t)=-m<0$	E_1	$(4,6,9m+12)$	2^{-m-1}	I^*_{9m+4}	1	1	2
	E_3	$(4,6,3m+12)$	2^{-m-1}	I^*_{3m+4}	1	1	2
	E_9	$(4,6,m+12)$	2^{-m-1}	I^*_{m+4}	1	1	2
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					$d \pmod 4$		

13.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

t	$[u(E)]$	$[u(E)(d)]$	d	Prob
$v_3(t) \leq 0$	$(1:1:1)$	$(1:1:1)$		$(1,0,0)$
$v_3(t) = 1$	$(1:1:1)$	$(1:1:1)$	$d \not\equiv 0\,(3)$	$\left(\frac{3}{4}, \frac{1}{4}, 0\right)$
		$(1:3:3)$	$d \equiv 0\,(3)$	
$v_3(t) = 2$	$(1:3:3)$	$(1:1:1)$	$d \not\equiv 0\,(3)$	$\left(0, \frac{3}{4}, \frac{1}{4}\right)$
		$(1:1:3)$	$d \equiv 0\,(3)$	
$v_3(t) \geq 3$	$(1:3:3^2)$	$(1:1:1)$		$(0,0,1)$

The contents of the above table are the main ingredients to prove the following result:

Proposition 13. *Let* $E_1 \overset{3}{\text{——}} E_3 \overset{3}{\text{——}} E_9$ *be a* $\mathbf{Q}$*-isogeny graph of type* $L_3(9)$ *corresponding to a given* t *in* $\mathbf{Q}^*$*. For every square-free integer* d*, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph* $E_1^d \overset{3}{\text{——}} E_3^d \overset{3}{\text{——}} E_9^d$ *is given by:*

$L_3(9)$	*twisted isogeny graph*	d	*Prob*
$v_3(t) \leq 0$	$E_1^d \longrightarrow E_3^d \longrightarrow E_9^d$		1
$v_3(t) = 1$	$E_1^d \longrightarrow E_3^d \longrightarrow E_9^d$	$d \not\equiv 0\,(3)$	3/4
	$E_1^d \longleftarrow E_3^d \longrightarrow E_9^d$	$d \equiv 0\,(3)$	1/4
$v_3(t) = 2$	$E_1^d \longleftarrow E_3^d \longrightarrow E_9^d$	$d \not\equiv 0\,(3)$	3/4
	$E_1^d \longleftarrow E_3^d \longleftarrow E_9^d$	$d \equiv 0\,(3)$	1/4
$v_3(t) \geq 3$	$E_1^d \longleftarrow E_3^d \longleftarrow E_9^d$		1

14 Type $L_3(25)$

14.1 Settings

Graph

The isogeny graphs of type $L_3(25)$ are given by three isogenous elliptic curves:

$$E_1 \overset{5}{\text{——}} E_5 \overset{5}{\text{——}} E_{25}\,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(25)$ parametrize isogeny graphs of type $L_3(25)$. The curve $X_0(25)$ has genus 0 and a hauptmodul for this curve is:

$$t(\tau) = 5\left(\frac{\eta(25\tau)}{\eta(\tau)}\right)\,.$$

j-invariants

Letting $t = t(\tau)$, one can write

$$\begin{aligned}
j(E_1) = j(\tau) &= \frac{\left(t^{10}+10t^9+55t^8+200t^7+525t^6+1010t^5+1425t^4+1400t^3+875t^2+250t+5\right)^3}{t\left(t^4+5t^3+15t^2+25t+25\right)}\,,\\
j(E_5) = j(5\tau) &= \frac{\left(t^2+5t+5\right)^3\left(t^4+5t^2+25\right)^3\left(t^4+5t^3+20t^2+25t+25\right)^3}{t^5\left(t^4+5t^3+15t^2+25t+25\right)^5}\,,\\
j(E_{25}) = j(25\tau) &= \frac{\left(t^{10}+250t^9+4375t^8+35000t^7+178125t^6+631250t^5+1640625t^4+3125000t^3+4296875t^2+3906250t+1953125\right)^3}{t^{25}\left(t^4+5t^3+15t^2+25t+25\right)}\,.
\end{aligned}$$

Signatures

We can (and do) choose Weierstrass equations for (E_1, E_5, E_{25}) in such a way that the isogeny graph is normalized. Their signatures are:

	$\mathrm{L}_3(25)$
$\mathrm{c}_4(E_1)$	$\left(t^2+2t+5\right)\left(t^{10}+10t^9+55t^8+200t^7+525t^6+1010t^5+1425t^4+1400t^3+875t^2+250t+5\right)$
$\mathrm{c}_6(E_1)$	$\left(t^2+2t+5\right)^2\left(t^4+4t^3+9t^2+10t+5\right)\left(t^{10}+10t^9+55t^8+200t^7+525t^6+1004t^5+1395t^4+1310t^3+725t^2+100t-1\right)$
$\Delta(E_1)$	$\mathrm{t}\,\left(t^2+2t+5\right)^3\left(t^4+5t^3+15t^2+25t+25\right)$
$\mathrm{c}_4(E_5)$	$\left(t^2+2t+5\right)\left(t^2+5t+5\right)\left(t^4+5t^2+25\right)\left(t^4+5t^3+20t^2+25t+25\right)$
$\mathrm{c}_6(E_5)$	$\left(t^2-5\right)\left(t^2+2t+5\right)^2\left(t^4+15t^2+25\right)\left(t^4+4t^3+9t^2+10t+5\right)\left(t^4+10t^3+45t^2+100t+125\right)$
$\Delta(E_5)$	$\mathrm{t}^5\left(t^2+2t+5\right)^3\left(t^4+5t^3+15t^2+25t+25\right)^5$
$\mathrm{c}_4(E_{25})$	$\left(t^2+2t+5\right)\left(t^{10}+250t^9+4375t^8+35000t^7+178125t^6+631250t^5+1640625t^4+3125000t^3+4296875t^2+3906250t+1953125\right)$
$\mathrm{c}_6(E_{25})$	$\left(t^2+2t+5\right)^2\left(t^4+10t^3+45t^2+100t+125\right)\left(t^{10}-500t^9-18125t^8-163750t^7-871875t^6-3137500t^5-8203125t^4-15625000t^3-21484375t^2-19531250t-9765625\right)$
$\Delta(E_{25})$	$\mathrm{t}^{25}\left(t^2+2t+5\right)^3\left(t^4+5t^3+15t^2+25t+25\right)$

Automorphisms

The subgroup of $\operatorname{Aut} X_0(25)$ that fixes the set of vertices of the graph is generated by the Fricke involution of $X_0(25)$, given by $W_{25}(t) = 5/t$. The involution W_{25} acts on the isogeny graphs of type $L_3(25)$ as:

$$W_{25}(E_1 \xrightarrow{\;5\;} E_5 \xrightarrow{\;5\;} E_{25}) = E_{25}^{-5} \xrightarrow{\;5\;} E_5^{-5} \xrightarrow{\;5\;} E_1^{-5}.$$

14.2 Kodaira symbols, minimal models, and Pal values

Table 21: $L_3(25)$ data for $p \neq 2, 5$

$L_3(25)$	$p \neq 2,3,5$					
t	E	$\operatorname{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = m > 0$	E_1	$(0,0,m)$	1	I_m	1	1
	E_5	$(0,0,5m)$	1	I_{5m}	1	1
	E_{25}	$(0,0,25m)$	1	I_{25m}	1	1
$v_p(t) = 0$ $v_p(t^2+2t+5) = 4k$	E_1	$(0,2k,0)$	p^k	I_0	1	1
	E_5	$(0,2k,0)$	p^k	I_0	1	1
	E_{25}	$(0,2k,0)$	p^k	I_0	1	1
$v_p(t) = 0$ $v_p(t^2+2t+5) = 4k+1$	E_1	$(1,2+2k,3)$	p^k	III	1	1
	E_5	$(1,2+2k,3)$	p^k	III	1	1
	E_{25}	$(1,2+2k,3)$	p^k	III	1	1
$v_p(t) = 0$ $v_p(t^2+2t+5) = 4k+2$	E_1	$(2,4+2k,6)$	p^k	I_0^*	p	1
	E_5	$(2,4+2k,6)$	p^k	I_0^*	p	1
	E_{25}	$(2,4+2k,6)$	p^k	I_0^*	p	1
$v_p(t) = 0$ $v_p(t^2+2t+5) = 4k+3$	E_1	$(3,6+2k,9)$	p^k	III^*	p	1
	E_5	$(3,6+2k,9)$	p^k	III^*	p	1
	E_{25}	$(3,6+2k,9)$	p^k	III^*	p	1
$v_p(t) = 0$ $m = v_p(t^4+5t^3+15t^2+25t+25) \geq 0$	E_1	$(0,0,m)$	1	I_m	1	1
	E_5	$(0,0,5m)$	1	I_{5m}	1	1
	E_{25}	$(0,0,m)$	1	I_m	1	1
$v_p(t) = -m < 0$	E_1	$(0,0,25m)$	p^{-3m}	I_{25m}	1	1
	E_5	$(0,0,5m)$	p^{-3m}	I_{5m}	1	1
	E_{25}	$(0,0,m)$	p^{-3m}	I_m	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod p$	

Table 22: $L_3(25)$ data for $p = 5$

$L_3(25)$	$p = 5$					
t	E	$\mathrm{sig}_5(E)$	u_5	$\mathrm{K}_5(E)$	$u_5(d)$	
$v_5(t) = m > 1$	E_1	$(2, 3, m + 5)$	1	I^*_{m-1}	5	1
	E_1	$(2, 3, 5m + 1)$	5	$\mathrm{I}^*_{5(m-1)}$	5	1
	E_1	$(2, 3, 25m - 19)$	5^2	$\mathrm{I}^*_{25(m-1)}$	5	1
$v_5(t) = 1$ $v_5(t^2 + 2t + 5) = 4k$	E_1	$(1, 1 + 2k, 3)$	5^k	III	1	1
	E_5	$(1, \geq 1 + 2k, 3)$	5^{k+1}	III	1	1
	E_{25}	$(1, \geq 1 + 2k, 3)$	5^{k+2}	III	1	1
$v_5(t) = 1$ $v_5(t^2 + 2t + 5) = 4k + 1$	E_1	$(2, 3 + 2k, 6)$	5^k	I^*_0	5	1
	E_5	$(2, \geq 3 + 2k, 6)$	5^{k+1}	I^*_0	5	1
	E_{25}	$(2, \geq 3 + 2k, 6)$	5^{k+2}	I^*_0	5	1
$v_5(t) = 1$ $v_5(t^2 + 2t + 5) = 4k + 2$	E_1	$(3, 5 + 2k, 9)$	5^k	III^*	5	1
	E_5	$(3, \geq 5 + 2k, 9)$	5^{k+1}	III^*	5	1
	E_{25}	$(3, \geq 5 + 2k, 9)$	5^{k+2}	III^*	5	1
$v_5(t) = 1$ $m = v_5(t^2 + 2t + 5) = 4k + 3$	E_1	$(0, 1 + 2k, 0)$	5^{k+1}	I_0	1	1
	E_5	$(0, \geq 1 + 2k, 0)$	5^{k+2}	I_0	1	1
	E_{25}	$(0, \geq 1 + 2k, 0)$	5^{k+3}	I_0	1	1
$v_5(t) = 0$ $v_5(t^2 + 2t + 5) = 4k$	E_1	$(0, \geq 1, 0)$	5^k	I_0	1	1
	E_5	$(0, \geq 1, 0)$	5^k	I_0	1	1
	E_{25}	$(0, \geq 1, 0)$	5^k	I_0	1	1
$v_5(t) = 0$ $v_5(t^2 + 2t + 5) = 4k + 1$	E_1	$(1, \geq 2, 3)$	5^k	III	1	1
	E_5	$(1, \geq 2, 3)$	5^k	III	1	1
	E_{25}	$(1, \geq 2, 3)$	5^k	III	1	1
$v_5(t) = 0$ $v_5(t^2 + 2t + 5) = 4k + 2$	E_1	$(2, \geq 4, 6)$	5^k	I^*_0	5	1
	E_5	$(2, \geq 4, 6)$	5^k	I^*_0	5	1
	E_{25}	$(2, \geq 4, 6)$	5^k	I^*_0	5	1
$v_5(t) = 0$ $v_5(t^2 + 2t + 5) = 4k + 3$	E_1	$(3, \geq 6, 9)$	5^k	III^*	5	1
	E_5	$(3, \geq 6, 9)$	5^k	III^*	5	1
	E_{25}	$(3, \geq 6, 9)$	5^k	III^*	5	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod 5$	

Table 22: $L_3(25)$ data for $p = 5$ (Continued)

$L_3(25)$	$p = 5$					
t	E	$\mathrm{sig}_5(E)$	u_5	$\mathrm{K}_5(E)$	$u_5(d)$	
$v_5(t) = -m < 0$	E_1	$(0, 0, 25m)$	5^{-3m}	I_{25m}	1	1
	E_5	$(0, 0, 5m)$	5^{-3m}	I_{5m}	1	1
	E_{25}	$(0, 0, m)$	5^{-3m}	I_{m}	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod 5$	

Table 23: $L_3(25)$ data for p=2

$L_3(25)$	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=m>0$	E_1	$(0,0,m)$	1	I_m	1	2^{-1}	2^{-1}
	E_5	$(0,0,5m)$	1	I_{5m}	1	2^{-1}	2^{-1}
	E_{25}	$(0,0,25m)$	1	I_{25m}	1	2^{-1}	2^{-1}
$v_2(t)=0$ $t\equiv 1\,(4)$	E_1	$(5,8,9)$	1	III	1	1	1
	E_5	$(5,8,9)$	1	III	1	1	1
	E_{25}	$(5,8,9)$	1	III	1	1	1
$v_2(t)=0$ $t\equiv 3\,(4)$	E_1	$(6,6,6)$	1	II	1	1^* or 2^*	1
	E_5	$(6,6,6)$	1	II	1	1^* or 2^*	1
	E_{25}	$(6,6,6)$	1	II	1	1^* or 2^*	1
$v_2(t)=-m<0$	E_1	$(4,6,25m+12)$	2^{-3m-1}	I^*_{25m+4}	1	1	2
	E_5	$(4,6,5m+12)$	2^{-3m-1}	I^*_{5m+4}	1	1	2
	E_{25}	$(4,6,m+12)$	2^{-3m-1}	I^*_{m+4}	1	1	2
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					d (mod 4)		

Remark: If $t\equiv 3\,(4)$ and $d\equiv 2\,(4)$ then, for E_1, E_5, E_{25}, $u_2(d)$ is given by

- $t\equiv 3\,(8)$: $u_2(d)=\begin{cases} 1 \text{ if } d\equiv -2\,(8) \\ 2 \text{ if } d\equiv 2\,(8); \end{cases}$
- $t\equiv 7\,(8)$: $u_2(d)=\begin{cases} 2 \text{ if } d\equiv -2\,(8) \\ 1 \text{ if } d\equiv 2\,(8). \end{cases}$

14.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

t	$[u(E)]$	$[u(E)(d)]$	Prob
$v_5(t) \geq 1$	$(1:5:5^2)$	$(1:1:1)$	$(0,0,1)$
$v_5(t) \leq 0$	$(1:1:1)$	$(1:1:1)$	$(1,0,0)$

The contents of the above table are the main ingredients to prove the following result:

Proposition 14. *Let* $E_1 \overset{5}{\text{------}} E_5 \overset{5}{\text{------}} E_{25}$ *be a* $\mathbf{Q}$*-isogeny graph of type* $L_3(25)$ *corresponding to a given* t *in* $\mathbf{Q}^*$*. For every square-free integer* d*, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph* $E_1^d \overset{5}{\text{------}} E_5^d \overset{5}{\text{------}} E_{25}^d$ *is given by:*

$L_3(25)$	*twisted isogeny graph*	*Prob*
$v_5(t) \geq 1$	$E_1^d \longleftarrow E_5^d \longleftarrow E_{25}^d$	1
$v_5(t) \leq 0$	$E_1^d \longrightarrow E_5^d \longrightarrow E_{25}^d$	1

15 Type L_4

15.1 Settings

Graph

The isogeny graphs of type L_4 are given by four isogenous elliptic curves:

$$E_1 \overset{3}{\longrightarrow} E_3 \overset{3}{\longrightarrow} E_9 \overset{3}{\longrightarrow} E_{27}\,.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(27)$ parametrize isogeny graphs of type L_4. The modular curve $X_0(27)$ is elliptic of rank 0 over the rationals. Its rational points are: two rational cusps and one rational CM point $\tau \in \mathbb{H}$.

j-invariants

The corresponding j-invariant of τ is:

$$j(\tau) = -2^{15} \cdot 3 \cdot 5^3.$$

Signatures

We can (and do) choose Weierstrass equations in such a way that the isogeny graphs are normalized:

E	Minimal Weierstrass equation	$j(E)$	label
E_1	$y^2 + y = x^3 - 30x + 63$	$-2^{15} \cdot 3 \cdot 5^3$	27a4
E_3	$y^2 + y = x^3$	0	27a3
E_9	$y^2 + y = x^3 - 7$	0	27a1
E_{27}	$y^2 + y = x^3 - 270x - 1708$	$-2^{15} \cdot 3 \cdot 5^3$	27a2

Their signatures are:

E	E_1	E_3	E_9	E_{27}
$c_4(E)$	$2^5 \cdot 3^2 \cdot 5$	0	0	$2^5 \cdot 3^4 \cdot 5$
$c_6(E)$	$-2^3 \cdot 3^3 \cdot 11 \cdot 23$	$-2^3 \cdot 3^3$	$2^3 \cdot 3^6$	$2^3 \cdot 3^6 \cdot 11 \cdot 23$
$\Delta(E)$	-3^5	-3^3	-3^9	-3^{11}

We have that the Faltings curve (circled) in the graph is

$$E_1 \longleftarrow E_3 \longrightarrow E_9 \longrightarrow E_{27}\,.$$

Note that any $\mathbb{Q}$-isogeny class of type L_4 can be obtained by quadratic twist from

$$E_1 \overset{3}{\longrightarrow} E_3 \overset{3}{\longrightarrow} E_9 \overset{3}{\longrightarrow} E_{27}\,.$$

Note that E_1, E_3, E_9 and E_{27} have complex multiplication by an order of the ring of integers of $\mathbb{Q}(\sqrt{-3})$, and $E_9 = E_3^{-3}$ and $E_{27} = E_1^{-3}$.

15.2 Kodaira symbols, minimal models, and Pal values

There is only one bad reduction prime $p = 3$.

$p = 3$				
E	$\mathrm{sig}_3(E)$	$\mathrm{K}_3(E)$	$u_3(d)$	
E_1	$(2, 3, 5)$	IV	1	1
E_3	$(0, 3, 3)$	II	1	1
E_9	$(0, 6, 9)$	IV*	3	1
E_{27}	$(4, 6, 11)$	II*	3	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod 3$	

15.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

$[u(E)(d)]$	d	Prob
$(1 : 1 : 1 : 1)$	$d \not\equiv 0\,(3)$	$\left(0, \frac{3}{4}, \frac{1}{4}, 0\right)$
$(1 : 1 : 3 : 3)$	$d \equiv 0\,(3)$	

This table is the main ingredient to prove the following result:

Proposition 15. *For every square-free integer d, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph*

$$E_1^d \overset{3}{\text{------}} E_3^d \overset{3}{\text{------}} E_9^d \overset{3}{\text{------}} E_{27}^d$$

is given by:

twisted isogeny graph	*condition*	*prob*
$E_1 \longleftarrow E_3 \longrightarrow E_9 \longrightarrow E_{27}$	$d \not\equiv 0\,(3)$	$3/4$
$E_1 \longleftarrow E_3 \longleftarrow E_9 \longrightarrow E_{27}$	$d \equiv 0\,(3)$	$1/4$

16 Type $R_4(6)$

16.1 Settings

Graph

The isogeny graphs of type $R_4(6)$ are given by four isogenous elliptic curves:

$$\begin{array}{ccc} E_1 & \overset{3}{\text{——}} & E_3 \\ \Big| {\scriptstyle 2} & & \Big| {\scriptstyle 2} \\ E_2 & \overset{3}{\text{——}} & E_6\,. \end{array}$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(6)$ parametrize isogeny graphs of type $R_4(6)$. The curve $X_0(6)$ has genus 0 and a hauptmodul for this curve is:

$$t = 2^3 3^2 \frac{\eta(2\tau)\eta(6\tau)^5}{\eta(\tau)^5\eta(3\tau)}\,.$$

j-invariants

Letting $t = t(\tau)$, one has

$$\begin{aligned}
j(E_1) = j(\tau) \ &= \frac{(t+6)^3(t^3+18t^2+84t+24)^3}{t(t+8)^3(t+9)^2}\,,\\
j(E_2) = j(2\tau) &= \frac{(t+12)^3\left(t^3+12t^2+48t+192\right)^3}{t^2(t+8)^6(t+9)}\,,\\
j(E_3) = j(3\tau) &= \frac{(t+6)^3\left(t^3+18t^2+324t+1944\right)^3}{t^3(t+8)(t+9)^6}\,,\\
j(E_6) = j(6\tau) &= \frac{(t+12)^3(t^3+252t^2+3888t+15552)^3}{t^6(t+8)^2(t+9)^3}\,.
\end{aligned}$$

Signatures

We can (and do) choose Weierstrass equations for (E_1, E_2, E_3, E_6) in such a way that the isogeny graph is normalized. Their signatures are:

$R_4(6)$	
$c_4(E_1)$	$(t+6)(t^3+18\,t^2+84\,t+24)$
$c_6(E_1)$	$(t^2+12\,t+24)(t^4+24\,t^3+192\,t^2+504\,t-72)$
$\Delta(E_1)$	$t(t+8)^3(t+9)^2$
$c_4(E_2)$	$(t+12)(t^3+12\,t^2+48\,t+192)$
$c_6(E_2)$	$(t^2+12\,t+24)(t^4+24\,t^3+192\,t^2-4608)$
$\Delta(E_2)$	$t^2(t+8)^6(t+9)$
$c_4(E_3)$	$(t+6)(t^3+18\,t^2+324\,t+1944)$
$c_6(E_3)$	$(t^2+36\,t+216)(t^4-216\,t^2-1944\,t-5832)$
$\Delta(E_3)$	$t^3(t+8)(t+9)^6$
$c_4(E_6)$	$(t+12)(t^3+252\,t^2+3888\,t+15552)$
$c_6(E_6)$	$(t^2+36\,t+216)(t^4-504\,t^3-13824\,t^2-124416\,t-373248)$
$\Delta(E_6)$	$t^6(t+8)^2(t+9)^3$

Automorphisms

The subgroup of $\operatorname{Aut} X_0(6)$ that fixes the set of vertices of the graph is generated by the Fricke involutions of $X_0(6)$, given by

$$W_2(t) = -8(t+9)/(t+8)\,, \qquad W_3(t) = -9(t+8)/(t+9)\,, \qquad W_6(t) = 72/t\,.$$

With regard to the action of the Fricke involutions on the isogeny graph, it can be displayed as follows:

$$\begin{tikzcd} E_1 \arrow[r, dash, "3"] \arrow[d, dash, "2"] & E_3 \arrow[d, dash, "2"] \\ E_2 \arrow[r, dash, "3"] & E_6 \end{tikzcd} : \mathrm{Id} \qquad \begin{tikzcd} E_2 \arrow[r, dash, "3"] \arrow[d, dash, "2"] & E_6 \arrow[d, dash, "2"] \\ E_1 \arrow[r, dash, "3"] & E_3 \end{tikzcd} : W_2$$

$$\begin{tikzcd} E_3^{-3} \arrow[r, dash, "3"] \arrow[d, dash, "2"] & E_1^{-3} \arrow[d, dash, "2"] \\ E_6^{-3} \arrow[r, dash, "3"] & E_2^{-3} \end{tikzcd} : W_3 \qquad \begin{tikzcd} E_6^{-3} \arrow[r, dash, "3"] \arrow[d, dash, "2"] & E_2^{-3} \arrow[d, dash, "2"] \\ E_3^{-3} \arrow[r, dash, "3"] & E_1^{-3} \end{tikzcd} : W_6$$

16.2 Kodaira symbols, minimal models, and Pal values

Table 24: $R_4(6)$ data for $p \neq 2$, 3

$R_4(6)$	$p \neq 2,3$				
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$
$v_p(t) = m > 0$	E_1	$(0,0,m)$	1	I_m	1
	E_2	$(0,0,2m)$	1	I_{2m}	1
	E_3	$(0,0,3m)$	1	I_{3m}	1
	E_6	$(0,0,6m)$	1	I_{6m}	1
$v_p(t) = 0$ $m = v_p(t+9)$	E_1	$(0,0,2m)$	1	I_{2m}	1
	E_2	$(0,0,m)$	1	I_m	1
	E_3	$(0,0,6m)$	1	I_{6m}	1
	E_6	$(0,0,3m)$	1	I_{3m}	1
$v_p(t) = 0$ $m = v_p(t+8)$	E_1	$(0,0,3m)$	1	I_{3m}	1
	E_2	$(0,0,6m)$	1	I_{6m}	1
	E_3	$(0,0,m)$	1	I_m	1
	E_6	$(0,0,2m)$	1	I_{2m}	1
$v_p(t) = -m < 0$	E_1	$(0,0,6m)$	p^{-m}	I_{6m}	1
	E_2	$(0,0,3m)$	p^{-m}	I_{3m}	1
	E_3	$(0,0,2m)$	p^{-m}	I_{2m}	1
	E_6	$(0,0,m)$	p^{-m}	I_m	1

Table 25: $R_4(6)$ data for p=3

$R_4(6)$	$p=3$					
t	E	$\mathrm{sig}_3(E)$	u_3	$\mathrm{K}_3(E)$	$u_3(d)$	
$v_3(t)=m>2$	E_1	$(2,3,m+4)$	1	I^*_{m-2}	3	1
	E_2	$(2,3,2m+2)$	1	$\mathrm{I}^*_{2(m-2)}$	3	1
	E_3	$(2,3,3m)$	3	$\mathrm{I}^*_{3(m-2)}$	3	1
	E_6	$(2,3,6m-6)$	3	$\mathrm{I}^*_{6(m-2)}$	3	1
$v_3(t)=2$ $m=v_3(t+9)$	E_1	$(2,3,2m+2)$	1	$\mathrm{I}^*_{2(m-2)}$	3	1
	E_2	$(2,3,m+4)$	1	I^*_{m-2}	3	1
	E_3	$(2,3,6m-6)$	3	$\mathrm{I}^*_{6(m-2)}$	3	1
	E_6	$(2,3,3m)$	3	$\mathrm{I}^*_{3(m-2)}$	3	1
$v_3(t)=1$	E_1	$(\geq 2,3,3)$	1	III	1	1
	E_2	$(\geq 2,3,3)$	1	III	1	1
	E_3	$(\geq 4,6,9)$	1	III*	3	1
	E_6	$(\geq 4,6,9)$	1	III*	3	1
$v_3(t)=0$ $m=v_3(t+8)$	E_1	$(0,0,3m)$	1	I_{3m}	1	1
	E_2	$(0,0,6m)$	1	I_{6m}	1	1
	E_3	$(0,0,m)$	1	I_{m}	1	1
	E_6	$(0,0,2m)$	1	I_{2m}	1	1
$v_3(t)=-m<0$	E_1	$(0,0,6m)$	3^{-m}	I_{6m}	1	1
	E_2	$(0,0,3m)$	3^{-m}	I_{3m}	1	1
	E_3	$(0,0,2m)$	3^{-m}	I_{2m}	1	1
	E_6	$(0,0,m)$	3^{-m}	I_{m}	1	1
					$d\equiv 0$	$d\not\equiv 0$
					$d \pmod 3$	

Table 26: $R_4(6)$ data for p=2

$R_4(6)$	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=m>3$	E_1	$(4,6,m+9)$	1	I^*_{m+1}	1	1	2
	E_2	$(4,6,2m+6)$	2	I^*_{2m-2}	1	1	2
	E_3	$(4,6,3m+3)$	1	I^*_{3m-5}	1	1	2
	E_6	$(4,6,6m-6)$	2	I^*_{6m-14}	1	1	2
$v_2(t)=3$ $m=v_2(t+8)$	E_1	$(4,6,3+3m)$	1	I^*_{3m-5}	1	1	2
	E_2	$(4,6,6m-6)$	2	I^*_{6m-14}	1	1	2
	E_3	$(4,6,m+9)$	1	I^*_{m+1}	1	1	2
	E_6	$(4,6,2m+6)$	2	I^*_{2m-2}	1	1	2
$v_2(t)=2$	E_1	$(4,6,8)$	1	I^*_0	1	1	1
	E_2	$(\geq 4,5,4)$	2	II	1	1	1
	E_3	$(4,6,8)$	1	I^*_0	1	1	1
	E_6	$(\geq 4,5,4)$	2	II	1	1	1
$v_2(t)=1$	E_1	$(\geq 4,5,4)$	1	II	1	1	1
	E_2	$(4,6,8)$	1	I^*_0	1	1	1
	E_3	$(\geq 4,5,4)$	1	II	1	1	1
	E_6	$(4,6,8)$	1	I^*_0	1	1	1
$v_2(t)=0$ $m=v_2(t+9)$	E_1	$(4,6,2m+12)$	2^{-1}	I^*_{2m+4}	1	1	2
	E_2	$(4,6,m+12)$	2^{-1}	I^*_{m+4}	1	1	2
	E_3	$(4,6,6m+12)$	2^{-1}	I^*_{6m+4}	1	1	2
	E_6	$(4,6,3m+12)$	2^{-1}	I^*_{3m+4}	1	1	2
$v_2(t)=-m<0$	E_1	$(4,6,6m+12)$	2^{-m-1}	I^*_{6m+4}	1	1	2
	E_2	$(4,6,3m+12)$	2^{-m-1}	I^*_{3m+4}	1	1	2
	E_3	$(4,6,2m+12)$	2^{-m-1}	I^*_{2m+4}	1	1	2
	E_6	$(4,6,m+12)$	2^{-m-1}	I^*_{m+4}	1	1	2
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					$d \pmod 4$		

16.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u}_p = [u_p(E)]$ and $\mathbf{u}_p(d) = [u_p(E)(d)]$:

t	$[u_2(E)]$	$[u_2(E)(d)]$
$v_2(t) \geq 2$	$(1:2:1:2)$	$(1:1:1:1)$
$v_2(t) \leq 1$	$(1:1:1:1)$	$(1:1:1:1)$

t	$[u_3(E)]$	$[u_3(E)(d)]$	d
$v_3(t) \geq 2$	$(1:1:3:3)$	$(1:1:1:1)$	
$v_3(t) = 1$	$(1:1:1:1)$	$(1:1:1:1)$	$d \not\equiv 0\,(3)$
		$(1:1:3:3)$	$d \equiv 0\,(3)$
$v_3(t) \leq 0$	$(1:1:1:1)$	$(1:1:1:1)$	

The contents of these tables are the main ingredients to prove the following result:

Proposition 16. *Let*

$$\begin{tikzcd} E_1 \arrow[r, dash, "3"] \arrow[d, dash, "2"] & E_3 \arrow[d, dash, "2"] \\ E_2 \arrow[r, dash, "3"] & E_6 \end{tikzcd}$$

be a $\mathbf{Q}$*-isogeny graph of type* $R_4(6)$ *corresponding to a given* t *in* $\mathbf{Q} \setminus \{0, -8, -9\}$ *as above. For every square-free integer* d*, the probability of a vertex to be the Faltings curve (circled) in the twisted graph*

$$\begin{tikzcd} E_1^d \arrow[r, dash, "3"] \arrow[d, dash, "2"] & E_3^d \arrow[d, dash, "2"] \\ E_2^d \arrow[r, dash, "3"] & E_6^d \end{tikzcd}$$

is given by:

Table 27: Faltings curves in $R_4(6)$

$R_4(6)$		*twisted isogeny graph*	d	*prob*
$v_2(t) \leq 1$	$v_3(t) \leq 0$	$\begin{tikzcd} E_1^d \arrow[r] \arrow[d] & E_3^d \arrow[d] \\ E_2^d \arrow[r] & E_6^d \end{tikzcd}$		1
$v_2(t) \leq 1$	$v_3(t) = 1$	$\begin{tikzcd} E_1^d \arrow[r] \arrow[d] & E_3^d \arrow[d] \\ E_2^d \arrow[r] & E_6^d \end{tikzcd}$	$d \equiv 0\,(3)$	1/4

Continued on next page

Table 27: Faltings curves in $R_4(6)$ (Continued)

		$\begin{array}{ccc} E_1^d & \longleftarrow & E_3^d \\ \downarrow & & \downarrow \\ E_2^d & \longleftarrow & E_6^d \end{array}$	$d \not\equiv 0\,(3)$	3/4
$v_2(t) \leq 1$	$v_3(t) \geq 2$	$\begin{array}{ccc} E_1^d & \longleftarrow & E_3^d \\ \downarrow & & \downarrow \\ E_2^d & \longleftarrow & E_6^d \end{array}$		1
$v_2(t) \geq 2$	$v_3(t) \leq 0$	$\begin{array}{ccc} E_1^d & \longrightarrow & E_3^d \\ \uparrow & & \uparrow \\ E_2^d & \longrightarrow & E_6^d \end{array}$		1
$v_2(t) \geq 2$	$v_3(t) = 1$	$\begin{array}{ccc} E_1^d & \longrightarrow & E_3^d \\ \uparrow & & \uparrow \\ E_2^d & \longrightarrow & E_6^d \end{array}$	$d \equiv 0\,(3)$	1/4
		$\begin{array}{ccc} E_1^d & \longleftarrow & E_3^d \\ \uparrow & & \uparrow \\ E_2^d & \longleftarrow & E_6^d \end{array}$	$d \not\equiv 0\,(3)$	3/4
$v_2(t) \geq 2$	$v_3(t) \geq 2$	$\begin{array}{ccc} E_1^d & \longleftarrow & E_3^d \\ \uparrow & & \uparrow \\ E_2^d & \longleftarrow & E_6^d \end{array}$		1

17 Type $R_4(10)$

17.1 Settings

Graph

The isogeny graphs of type $R_4(10)$ are given by four isogenous elliptic curves:

$$\begin{array}{ccc} E_1 & \overset{5}{\text{———}} & E_5 \\ {\scriptstyle 2}\Big| & & \Big|{\scriptstyle 2} \\ E_2 & \underset{5}{\text{———}} & E_{10}\,. \end{array}$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(10)$ parametrize isogeny graphs of type $R_4(10)$. The curve $X_0(10)$ has genus 0 and a hauptmodul for this curve is:

$$t = 4 + 2^2 5 \frac{\eta(2\tau)\eta(10\tau)^3}{\eta(\tau)^3\eta(5\tau)}\,.$$

j-invariants

Letting $t = t(\tau)$, one has

$$\begin{aligned}
j(E_1) = j(\tau) \quad &= \frac{\left(t^6 - 4t^5 + 16t + 16\right)^3}{(t-4)t^5(t+1)^2} \\
j(E_2) = j(2\tau) \quad &= \frac{\left(t^6 - 4t^5 + 256t + 256\right)^3}{(t-4)^2t^{10}(t+1)} \\
j(E_5) = j(5\tau) \quad &= \frac{\left(t^6 - 4t^5 + 240t^4 - 480t^3 + 1440t^2 - 944t + 16\right)^3}{(t-4)^5t(t+1)^{10}} \\
j(E_{10}) = j(10\tau) &= \frac{\left(t^6 + 236t^5 + 1440t^4 + 1920t^3 + 3840t^2 + 256t + 256\right)^3}{(t-4)^{10}t^2(t+1)^5}\,.
\end{aligned}$$

Signatures

We can (and do) choose Weierstrass equations for (E_1, E_2, E_5, E_{10}) in such a way that the isogeny graph is normalized. Their signatures are:

	$R_4(10)$
$c_4(E_1)$	$(t^2+4)(t^6-4t^5+16t+16)$
$c_6(E_1)$	$(t^2-2t-4)(t^2-2t+2)(t^2+4)^2(t^4-2t^3-6t^2-8t-4)$
$\Delta(E_1)$	$(t-4)t^5(t+1)^2(t^2+4)^3$
$c_4(E_2)$	$(t^2+4)(t^6-4t^5+256t+256)$
$c_6(E_2)$	$(t^2-2t-4)(t^2+4)^2(t^2+4t+8)(t^4-8t^3+24t^2-32t-64)$
$\Delta(E_2)$	$(t-4)^2t^{10}(t+1)(t^2+4)^3$
$c_4(E_5)$	$(t^2+4)(t^6-4t^5+240t^4-480t^3+1440t^2-944t+16)$
$c_6(E_5)$	$(t^2-2t+2)(t^2+4)^2(t^2+22t-4)(t^4-26t^3+66t^2-536t-4)$
$\Delta(E_5)$	$(t-4)^5t(t+1)^{10}(t^2+4)^3$
$c_4(E_{10})$	$(t^2+4)(t^6+236t^5+1440t^4+1920t^3+3840t^2+256t+256)$
$c_6(E_{10})$	$(t^2+4)^2(t^2+4t+8)(t^2+22t-4)(t^4-536t^3-264t^2-416t-64)$
$\Delta(E_{10})$	$(t-4)^{10}t^2(t+1)^5(t^2+4)^3$

Automorphisms

The subgroup of $\operatorname{Aut} X_0(10)$ that fixes the set of vertices of the graph is generated by the Fricke involutions of $X_0(10)$, given by

$$W_{10}(t) = 4(t+1)/(t-4)\,, \qquad W_5(t) = (-t+4)/(t+1)\,, \qquad W_2(t) = -4/t\,.$$

With regard to the action of the Fricke involutions on the isogeny graph, it can be displayed as follows:

$$\begin{tikzcd} E_1 \arrow[r, dash, "5"] \arrow[d, dash, "2"] & E_5 \arrow[d, dash, "2"] \\ E_2 \arrow[r, dash, "5"] & E_{10} \end{tikzcd} : \mathrm{Id} \qquad \begin{tikzcd} E_2^2 \arrow[r, dash, "5"] \arrow[d, dash, "2"] & E_{10}^2 \arrow[d, dash, "2"] \\ E_1^2 \arrow[r, dash, "5"] & E_5^2 \end{tikzcd} : W_2$$

$$\begin{tikzcd} E_5^{-5} \arrow[r, dash, "5"] \arrow[d, dash, "2"] & E_1^{-5} \arrow[d, dash, "2"] \\ E_{10}^{-5} \arrow[r, dash, "5"] & E_2^{-5} \end{tikzcd} : W_5 \qquad \begin{tikzcd} E_{10}^{-10} \arrow[r, dash, "5"] \arrow[d, dash, "2"] & E_2^{-10} \arrow[d, dash, "2"] \\ E_5^{-10} \arrow[r, dash, "5"] & E_1^{-10} \end{tikzcd} : W_{10}$$

where the arrows correspond to the dual isogenies.

17.2 Kodaira symbols, minimal models, and Pal values

Table 28: $R_4(10)$ data for $p \neq 2, 5$

$R_4(10)$	$p \neq 2,5$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = m > 0$	E_1	$(0,0,5m)$	1	I_{5m}	1	1
	E_2	$(0,0,10m)$	1	I_{10m}	1	1
	E_5	$(0,0,m)$	1	I_{m}	1	1
	E_{10}	$(0,0,2m)$	1	I_{2m}	1	1
$v_p(t) = 0$ $m = v_p(t+1) > 0$	E_1	$(0,0,2m)$	1	I_{2m}	1	1
	E_2	$(0,0,m)$	1	I_{m}	1	1
	E_5	$(0,0,10m)$	1	I_{10m}	1	1
	E_{10}	$(0,0,5m)$	1	I_{5m}	1	1
$v_p(t) = 0$ $m = v_p(t-4) > 0$	E_1	$(0,0,m)$	1	I_{m}	1	1
	E_2	$(0,0,2m)$	1	I_{2m}	1	1
	E_5	$(0,0,5m)$	1	I_{5m}	1	1
	E_{10}	$(0,0,10m)$	1	I_{10m}	1	1
$v_p(t) = 0$ $v_p(t^2+4) = 4k$	E_1	$(0,2k,0)$	p^k	I_0	1	1
	E_2	$(0,2k,0)$	p^k	I_0	1	1
	E_5	$(0,2k,0)$	p^k	I_0	1	1
	E_{10}	$(0,2k,0)$	p^k	I_0	1	1
$v_p(t) = 0$ $v_p(t^2+4) = 4k+1$	E_1	$(1,2+2k,3)$	p^k	III	1	1
	E_2	$(1,2+2k,3)$	p^k	III	1	1
	E_5	$(1,2+2k,3)$	p^k	III	1	1
	E_{10}	$(1,2+2k,3)$	p^k	III	1	1
$v_p(t) = 0$ $v_p(t^2+4) = 4k+2$	E_1	$(2,4+2k,6)$	p^k	I_0^*	p	1
	E_2	$(2,4+2k,6)$	p^k	I_0^*	p	1
	E_5	$(2,4+2k,6)$	p^k	I_0^*	p	1
	E_{10}	$(2,4+2k,6)$	p^k	I_0^*	p	1
$v_p(t) = 0$ $v_p(t^2+4) = 4k+3$	E_1	$(3,6+2k,9)$	p^k	III^*	p	1
	E_2	$(3,6+2k,9)$	p^k	III^*	p	1
	E_5	$(3,6+2k,9)$	p^k	III^*	p	1
	E_{10}	$(3,6+2k,9)$	p^k	III^*	p	1

Table 28: $R_4(10)$ data for $p \neq 2, 5$ (Continued)

$v_p(t) = -m < 0$	E_1	$(0, 0, 10m)$	p^{-2m}	I_{10m}	1	1
	E_2	$(0, 0, 5m)$	p^{-2m}	I_{5m}	1	1
	E_5	$(0, 0, 2m)$	p^{-2m}	I_{2m}	1	1
	E_{10}	$(0, 0, m)$	p^{-2m}	I_{m}	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod p$	

The polynomial t^2+4 factors over $\mathbb{Q}_5[t]$ as $(t-\alpha_1)(t-\alpha_2)$ with

$$\alpha_1 = 1+2\cdot 5+2\cdot 5^3+3\cdot 5^4+4\cdot 5^6+2\cdot 5^7+3\cdot 5^8+3\cdot 5^9+4\cdot 5^{10}+4\cdot 5^{11}+3\cdot 5^{12}+O(5^{13}),$$
$$\alpha_2 = 4+2\cdot 5+4\cdot 5^2+2\cdot 5^3+5^4+4\cdot 5^5+2\cdot 5^7+5^8+5^9+5^{12}+3\cdot 5^{13}+3\cdot 5^{14}+O(5^{15}).$$

Table 29: $R_4(10)$ data for $p=5$

$R_4(10)$	$p=5$					
t	E	$\mathrm{sig}_5(E)$	u_5	$\mathrm{K}_5(E)$	$u_5(d)$	
$v_5(t)=m>0$	E_1	$(0,0,5m)$	1	I_{5m}	1	1
	E_2	$(0,0,10m)$	1	I_{10m}	1	1
	E_5	$(0,0,m)$	1	I_{m}	1	1
	E_{10}	$(0,0,2m)$	1	I_{2m}	1	1
$v_5(t)=0$ $t\equiv 1\,(5)$ $v_5(t^2+4)=4k$	E_1	$(0,1+2k,0)$	5^k	I_0	1	1
	E_2	$(0,1+2k,0)$	5^k	I_0	1	1
	E_5	$(0,2k,0)$	5^k	I_0	1	1
	E_{10}	$(0,2k,0)$	5^k	I_0	1	1
$v_5(t)=0$ $t\equiv 1\,(5)$ $v_5(t^2+4)=4k+1$	E_1	$(1,3+2k,3)$	5^k	III	1	1
	E_2	$(1,3+2k,3)$	5^k	III	1	1
	E_5	$(1,2+2k,3)$	5^k	III	1	1
	E_{10}	$(1,2+2k,3)$	5^k	III	1	1
$v_5(t)=0$ $t\equiv 1\,(5)$ $v_5(t^2+4)=4k+2$	E_1	$(2,5+2k,6)$	5^k	I_0^*	5	1
	E_2	$(2,5+2k,6)$	5^k	I_0^*	5	1
	E_5	$(2,4+2k,6)$	5^k	I_0^*	5	1
	E_{10}	$(2,4+2k,6)$	5^k	I_0^*	5	1
$v_5(t)=0$ $t\equiv 1\,(5)$ $v_5(t^2+4)=4k+3$	E_1	$(3,7+2k,9)$	5^k	III^*	5	1
	E_2	$(3,7+2k,9)$	5^k	III^*	5	1
	E_5	$(3,6+2k,9)$	5^k	III^*	5	1
	E_{10}	$(3,6+2k,9)$	5^k	III^*	5	1
$v_5(t)=0$ $t\equiv 14\,(25)$ $v_5(t^2+4)=4k$	E_1	$(1,\geq 2,3)$	5^k	III	1	1
	E_2	$(1,\geq 3,3)$	5^k	III	1	1
	E_5	$(1,\geq 2,3)$	5^{k+1}	III	1	1
	E_{10}	$(1,\geq 2,3)$	5^{k+1}	III	1	1
					$d\equiv 0$	$d\not\equiv 0$
					$d \pmod 5$	

Table 29: $R_4(10)$ data for $p = 5$ (Continued)

$R_4(10)$	$p=5$					
t	E	$\mathrm{sig}_5(E)$	u_5	$\mathrm{K}_5(E)$	$u_5(d)$	
$v_5(t)=0$ $t \equiv 14\,(25)$ $v_5(t^2+4)=4k+1$	E_1	$(2, \geq 3, 6)$	5^k	I_0^*	5	1
	E_2	$(2, \geq 3, 6)$	5^k	I_0^*	5	1
	E_5	$(2, \geq 3, 6)$	5^{k+1}	I_0^*	5	1
	E_{10}	$(2, \geq 3, 6)$	5^{k+1}	I_0^*	5	1
$v_5(t)=0$ $t \equiv 14\,(25)$ $v_5(t^2+4)=4k+2$	E_1	$(3, \geq 5, 9)$	5^k	III^*	5	1
	E_2	$(3, \geq 5, 9)$	5^k	III^*	5	1
	E_5	$(3, \geq 5, 9)$	5^{k+1}	III^*	5	1
	E_{10}	$(3, \geq 5, 9)$	5^{k+1}	III^*	5	1
$v_5(t)=0$ $t \equiv 14\,(25)$ $v_5(t^2+4)=4k+3$	E_1	$(0, \geq 1, 0)$	5^{k+1}	I_0	1	1
	E_2	$(0, \geq 1, 0)$	5^{k+1}	I_0	1	1
	E_5	$(0, \geq 1, 0)$	5^{k+2}	I_0	1	1
	E_{10}	$(0, \geq 1, 0)$	5^{k+2}	I_0	1	1
$v_5(t)=0$ $t \equiv 4\,(25)$ $v_5(t-4)=m$	E_1	$(2, 3, m+5)$	1	I_{m-1}^*	5	1
	E_2	$(2, 3, 2m+4)$	1	I_{2m-2}^*	5	1
	E_5	$(2, 3, 5m+1)$	5	I_{5m-5}^*	5	1
	E_{10}	$(2, 3, 10m-4)$	5	I_{10m-10}^*	5	1
$v_5(t)=0$ $t \equiv 9, 19\,(25)$	E_1	$(2, \geq 3, 6)$	1	I_0^*	5	1
	E_2	$(2, \geq 3, 6)$	1	I_0^*	5	1
	E_5	$(2, \geq 3, 6)$	5	I_0^*	5	1
	E_{10}	$(2, \geq 3, 6)$	5	I_0^*	5	1
$v_5(t)=0$ $t \equiv 24\,(25)$ $v_5(t+1)=m$	E_1	$(2, 3, 2m+4)$	1	I_{2m-2}^*	5	1
	E_2	$(2, 3, m+5)$	1	I_{m-1}^*	5	1
	E_5	$(2, 3, 10m-4)$	5	I_{10m-10}^*	5	1
	E_{10}	$(2, 3, 5m+1)$	5	I_{5m-5}^*	5	1
$v_5(t)=0$ $t \equiv 2, 3\,(5)$	E_1	$(0, \geq 0, 0)$	1	I_0	1	1
	E_2	$(0, \geq 0, 0)$	1	I_0	1	1
	E_5	$(0, \geq 0, 0)$	1	I_0	1	1
	E_{10}	$(0, \geq 0, 0)$	1	I_0	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod 5$	

Table 29: $R_4(10)$ data for $p = 5$ (Continued)

$R_4(10)$	$p = 5$					
t	E	$\mathrm{sig}_5(E)$	u_5	$\mathrm{K}_5(E)$	$u_5(d)$	
$v_5(t) = -m < 0$	E_1	$(0, 0, 10m)$	5^{-2m}	I_{10m}	1	1
	E_2	$(0, 0, 5m)$	5^{-2m}	I_{5m}	1	1
	E_5	$(0, 0, 2m)$	5^{-2m}	I_{2m}	1	1
	E_{10}	$(0, 0, m)$	5^{-2m}	I_m	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod 5$	

Table 30: $R_4(10)$ data for p=2

$R_4(10)$	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=m>2$	E_1	$(6,9,5m+8)$	1	I^*_{5m-2}	1	2^* or 4^*	1
	E_2	$(6,9,10m-2)$	2	I^*_{10m-12}	1	2^* or 4^*	1
	E_5	$(6,9,m+16)$	1	I^*_{m+6}	1	2^* or 4^*	1
	E_{10}	$(6,9,2m+14)$	2	I^*_{2m+4}	1	2^* or 4^*	1
$v_2(t)=2$ $m=v_2(t-4)$	E_1	$(6,9,m+16)$	1	I^*_{m+4}	1	4^* or 2^*	1
	E_2	$(6,9,2m+14)$	2	I^*_{2m+4}	1	4^* or 2^*	1
	E_5	$(6,9,5m+8)$	1	I^*_{5m-2}	1	4^* or 2^*	1
	E_{10}	$(6,9,10m-2)$	2	I^*_{10m-12}	1	4^* or 2^*	1
$v_2(t)=1$	E_1	$(7,11,15)$	1	III^*	1	2	1
	E_2	$(5,8,9)$	2	III	1	1	1
	E_5	$(7,11,15)$	1	III	1	2	1
	E_{10}	$(5,8,9)$	2	III	1	1	1
$v_2(t)=0$ $m=v_2(t+1)$	E_1	$(0,0,2m)$	1	I_{2m}	1	2^{-1}	2^{-1}
	E_2	$(0,0,m)$	1	I_m	1	2^{-1}	2^{-1}
	E_5	$(0,0,10m)$	1	I_{10m}	1	2^{-1}	2^{-1}
	E_{10}	$(0,0,5m)$	1	I_{5m}	1	2^{-1}	2^{-1}
$v_2(t)=-m<0$	E_1	$(4,6,10m+12)$	2^{-2m-1}	I^*_{10m+4}	1	1	2
	E_2	$(4,6,5m+12)$	2^{-2m-1}	I^*_{5m+4}	1	1	2
	E_5	$(4,6,2m+12)$	2^{-2m-1}	I^*_{2m+4}	1	1	2
	E_{10}	$(4,6,m+12)$	2^{-2m-1}	I^*_{m+4}	1	1	2
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					$d \pmod 4$		

Remark (2^* or 4^*): If $v_2(t)>2$ and $d\equiv 2\,(4)$ then, for E_1,E_2,E_5,E_{10}, the value $u_2(d)$ is given by

$$u_2(d)=\begin{cases} 2 \text{ if } d\equiv 2\,(8) \\ 4 \text{ if } d\equiv -2\,(8). \end{cases}$$

Remark (4^* or 2^*): If $v_2(t)=2$ and $d\equiv 2\,(4)$ then, for E_1,E_2,E_5,E_{10}, the value $u_2(d)$ is given by

$$u_2(d)=\begin{cases} 4 \text{ if } d\equiv 2\,(8) \\ 2 \text{ if } d\equiv -2\,(8). \end{cases}$$

17.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u}_p = [u_p(E)]$ and $\mathbf{u}_p(d) = [u_p(E)(d)]$:

<table>
<tr><th>t</th><th>$[u_2(E)]$</th><th>$[u_2(E)(d)]$</th><th>d</th></tr>
<tr><td>$v_2(t) > 1$</td><td>$(1:2:1:2)$</td><td>$(1:1:1:1)$</td><td></td></tr>
<tr><td rowspan="2">$v_2(t) = 1$</td><td rowspan="2">$(1:2:1:2)$</td><td>$(1:1:1:1)$</td><td>$d \not\equiv 0\,(2)$</td></tr>
<tr><td>$(2:1:2:1)$</td><td>$d \equiv 0\,(2)$</td></tr>
<tr><td>$v_2(t) \le 0$</td><td>$(1:1:1:1)$</td><td>$(1:1:1:1)$</td><td></td></tr>
</table>

<table>
<tr><th>t</th><th>$[u_5(E)]$</th><th>$[u_5(E)(d)]$</th></tr>
<tr><td>$v_5(t) \neq 0$</td><td rowspan="2">$(1:1:1:1)$</td><td rowspan="2">$(1:1:1:1)$</td></tr>
<tr><td>$v_5(t) = 0$
$t \not\equiv 4\,(5)$</td></tr>
<tr><td>$v_5(t) = 0$
$t \equiv 4\,(5)$</td><td>$(1:1:5:5)$</td><td>$(1:1:1:1)$</td></tr>
</table>

The contents of these tables are the main ingredients to prove the following result:

Proposition 17. *Let*

$$\begin{tikzcd} E_1 \arrow[r, dash, "5"] \arrow[d, dash, "2"] & E_5 \arrow[d, dash, "2"] \\ E_2 \arrow[r, dash, "5"] & E_{10} \end{tikzcd}$$

be a $\mathbf{Q}$*-isogeny graph of type* $R_4(10)$ *corresponding to a given* t *in* $\mathbf{Q} \setminus \{-1, \pm 4\}$ *as above. For every square-free integer* d*, the probability of a vertex to be the Faltings curve (circled) in the twisted graph*

$$\begin{tikzcd} E_1^d \arrow[r, dash, "5"] \arrow[d, dash, "2"] & E_5^d \arrow[d, dash, "2"] \\ E_2^d \arrow[r, dash, "5"] & E_{10}^d \end{tikzcd}$$

is given by:

Table 31: Faltings curves in $R_4(10)$

<table>
<tr><th colspan="2">$R_4(10)$</th><th>twisted isogeny graph</th><th>d</th><th>prob</th></tr>
<tr><td rowspan="2">$v_2(t) \le 0$</td><td>$v_5(t) \neq 0$</td><td rowspan="2">$\begin{tikzcd} E_1^d \arrow[r] \arrow[d] & E_5^d \arrow[d] \\ E_2^d \arrow[r] & E_{10}^d \end{tikzcd}$</td><td rowspan="2"></td><td rowspan="2">1</td></tr>
<tr><td>$v_5(t) = 0$
$t \not\equiv 4\,(5)$</td></tr>
</table>

Continued on next page

Table 31: Faltings curves in $R_4(10)$ (Continued)

<table>
<tr><td rowspan="2">$v_2(t) = 1$</td><td>$v_5(t) \neq 0$</td><td>$\begin{array}{ccc} E_1^d & \longrightarrow & E_5^d \\ \downarrow & & \downarrow \\ E_2^d & \longrightarrow & E_{10}^d \end{array}$</td><td>$d \equiv 0\,(2)$</td><td>1/3</td></tr>
<tr><td>$v_5(t) = 0$
$t \not\equiv 4\,(5)$</td><td>$\begin{array}{ccc} E_1^d & \longleftarrow & E_5^d \\ \downarrow & & \downarrow \\ E_2^d & \longleftarrow & E_{10}^d \end{array}$</td><td>$d \not\equiv 0\,(2)$</td><td>2/3</td></tr>
<tr><td rowspan="2">$v_2(t) > 1$</td><td>$v_5(t) \neq 0$</td><td rowspan="2">$\begin{array}{ccc} E_1^d & \longleftarrow & E_5^d \\ \downarrow & & \downarrow \\ E_2^d & \longleftarrow & E_{10}^d \end{array}$</td><td rowspan="2"></td><td rowspan="2">1</td></tr>
<tr><td>$v_5(t) = 0$
$t \not\equiv 4\,(5)$</td></tr>
<tr><td>$v_2(t) > 1$</td><td>$v_5(t) = 0$
$t \equiv 4\,(5)$</td><td>$\begin{array}{ccc} E_1^d & \longrightarrow & E_5^d \\ \uparrow & & \uparrow \\ E_2^d & \longrightarrow & E_{10}^d \end{array}$</td><td></td><td>1</td></tr>
<tr><td rowspan="2">$v_2(t) = 1$</td><td rowspan="2">$v_5(t) = 0$
$t \equiv 4\,(5)$</td><td>$\begin{array}{ccc} E_1^d & \longrightarrow & E_5^d \\ \uparrow & & \uparrow \\ E_2^d & \longrightarrow & E_{10}^d \end{array}$</td><td>$d \equiv 0\,(2)$</td><td>1/3</td></tr>
<tr><td>$\begin{array}{ccc} E_1^d & \longleftarrow & E_5^d \\ \uparrow & & \uparrow \\ E_2^d & \longleftarrow & E_{10}^d \end{array}$</td><td>$d \not\equiv 0\,(2)$</td><td>2/3</td></tr>
<tr><td>$v_2(t) \leq 0$</td><td>$v_5(t) = 0$
$t \equiv 4\,(5)$</td><td>$\begin{array}{ccc} E_1^d & \longleftarrow & E_5^d \\ \uparrow & & \uparrow \\ E_2^d & \longleftarrow & E_{10}^d \end{array}$</td><td></td><td>1</td></tr>
</table>

18 Type $R_4(14)$

18.1 Settings

Graph

The isogeny graphs of type $R_4(14)$ are given by four isogenous elliptic curves:

$$\begin{tikzcd} E_1 \arrow[r, dash, "7"] \arrow[d, dash, "2"] & E_7 \arrow[d, dash, "2"] \\ E_2 \arrow[r, dash, "7"] & E_{14}\,. \end{tikzcd}$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(14)$ parametrize isogeny graphs of type $R_4(14)$. The modular curve $X_0(14)$ is elliptic of rank 0 over the rationals. Its rational points are: four cusps and two CM points given by $\tau,\tau' \in \mathbb{H}$.

j-invariants

The corresponding j-invariants of τ and τ' are:

$$j(\tau) = -3^5 \cdot 5^3, \qquad j(\tau') = 3^5 \cdot 5^3 \cdot 17^3.$$

We have $j(14\tau) = j(\tau')$.

Signatures

We can (and do) choose Weierstrass equations in such a way that the isogeny graphs are normalized:

E	Minimal Weierstrass model	$j(E)$	label
E_1	$y^2 + xy = x^3 - x^2 - 2x - 1$	$-3^5 \cdot 5^3$	`49a1`
E_2	$y^2 + xy = x^3 - x^2 - 37x - 78$	$3^5 \cdot 5^3 \cdot 17^3$	`49a2`
E_7	$y^2 + xy = x^3 - x^2 - 107x + 552$	$-3^5 \cdot 5^3$	`49a3`
E_{14}	$y^2 + xy = x^3 - x^2 - 1822x + 30393$	$3^5 \cdot 5^3 \cdot 17^3$	`49a4`

Their signatures are:

E	E_1	E_2	E_7	E_{14}
$c_4(E)$	$3 \cdot 5 \cdot 7$	$3 \cdot 5 \cdot 7 \cdot 17$	$3 \cdot 5 \cdot 7^3$	$3 \cdot 5 \cdot 7^3 \cdot 17$
$c_6(E)$	$3^3 \cdot 7^2$	$3^4 \cdot 7^2 \cdot 19$	$-3^3 \cdot 7^5$	$-3^4 \cdot 7^5 \cdot 19$
$\Delta(E)$	-7^3	7^3	-7^9	7^3

We have that the Faltings curve (circled) in the graph is

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] & E_7 \arrow[d] \\ E_2 \arrow[r] & E_{14} \end{tikzcd}$$

Note that any $\mathbb{Q}$-isogeny class of type $R_4(14)$ can be obtained by quadratic twist from

$$\begin{tikzcd} E_1 \arrow[r, dash, "7"] \arrow[d, dash, "2"] & E_7 \arrow[d, dash, "2"] \\ E_2 \arrow[r, dash, "7"] & E_{14}. \end{tikzcd}$$

Note that E_1, E_2, E_7 and E_{14} have complex multiplication by an order in the ring of integers of $\mathbf{Q}(\sqrt{-7})$ and $E_7 = E_1^{-7}$, $E_{14} = E_2^{-7}$.

18.2 Kodaira symbols, minimal models, and Pal values

There is only one bad reduction prime $p = 7$.

$p=7$				
E	$\mathrm{sig}_7(E)$	$\mathrm{K}_7(E)$	$u_7(d)$	
E_1	$(1,2,3)$	III	1	1
E_2	$(1,2,3)$	III	1	1
E_7	$(3,5,9)$	III*	7	1
E_{14}	$(3,5,9)$	III*	7	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod 7$	

18.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

$[u(E)(d)]$	d	Prob
$(1:1:1:1)$	$d \not\equiv 0\,(7)$	$\left(\frac{7}{8}, 0, \frac{1}{8}, 0\right)$
$(1:1:7:7)$	$d \equiv 0\,(7)$	

This table is the main ingredient to prove the following result:

Proposition 18. *For every square-free integer d, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph*

$$\begin{tikzcd} E_1^d \arrow[r, dash, "7"] \arrow[d, dash, "2"] & E_7^d \arrow[d, dash, "2"] \\ E_2^d \arrow[r, dash, "7"] & E_{14}^d. \end{tikzcd}$$

is given by:

Table 32: Faltings curves in $R_4(14)$

twisted isogeny graph	*condition*	*prob*
$\begin{array}{ccc} E_1^d & \longrightarrow & E_7^d \\ \downarrow & & \downarrow \\ E_2^d & \longrightarrow & E_{14}^d \end{array}$	$d \not\equiv 0\,(7)$	7/8
$\begin{array}{ccc} E_1^d & \longleftarrow & E_7^d \\ \downarrow & & \downarrow \\ E_2^d & \longleftarrow & E_{14}^d \end{array}$	$d \equiv 0\,(7)$	1/8

19 Type $R_4(15)$

19.1 Settings

Graph

The isogeny graphs of type $R_4(15)$ are given by four isogenous elliptic curves:

$$\begin{tikzcd} E_1 \arrow[r, dash, "5"] \arrow[d, dash, "3"] & E_5 \arrow[d, dash, "3"] \\ E_3 \arrow[r, dash, "5"] & E_{15} \end{tikzcd}.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(15)$ parametrize isogeny graphs of type $R_4(15)$. The modular curve $X_0(15)$ is elliptic of rank 0 over the rationals. Its rational points are: four cusps and four CM points given by $\tau, \tau', \tau'', \tau''' \in \mathbb{H}$.

j-invariants

The corresponding j-invariants of τ, τ', τ'' and τ''' are:

$$j(\tau) = \frac{-5^2}{2}, \qquad j(\tau') = \frac{-5^2 \cdot 241^3}{2^3}, \quad j(\tau'') = \frac{-5 \cdot 29^3}{2^5}, \quad j(\tau''') = \frac{5 \cdot 211^3}{2^{15}}.$$

We have $j(15\tau) = j(\tau')$.

Signatures

We can (and do) choose Weierstrass equations in such a way that the isogeny graphs are normalized:

E	Minimal Weierstrass model	$j(E)$	label
E_1	$y^2 = x^3 - x^2 - 8x + 112$	$\frac{-5^2}{2}$	400c1
E_3	$y^2 = x^3 - x^2 - 2008x + 35312$	$\frac{-5^2 \cdot 241^3}{2^3}$	400c2
E_5	$y^2 = x^3 - x^2 - 1208x - 19088$	$\frac{-5 \cdot 29^3}{2^5}$	400c3
E_{15}	$y^2 = x^3 - x^2 + 8792x + 140912$	$\frac{5 \cdot 211^3}{2^{15}}$	400c4

Their signatures are:

E	E_1	E_3	E_7	E_{15}
$c_4(E)$	$2^4 \cdot 5^2$	$2^4 \cdot 5^2 \cdot 241$	$2^4 \cdot 5^3 \cdot 29$	$-2^4 \cdot 5^3 \cdot 211$
$c_6(E)$	$-2^6 \cdot 5^2 \cdot 59$	$-2^6 \cdot 5^2 \cdot 13 \cdot 1439$	$2^6 \cdot 5^4 \cdot 421$	$-2^6 \cdot 5^4 \cdot 13 \cdot 239$
$\Delta(E)$	$-2^{13} \cdot 5^4$	$-2^{15} \cdot 5^4$	$-2^{17} \cdot 5^8$	$-2^{27} \cdot 5^8$

We have that the Faltings curve (circled) in the graph is

$$\begin{array}{ccc} E_1 & \longrightarrow & E_5 \\ \downarrow & & \downarrow \\ E_3 & \longrightarrow & E_{15} \end{array}$$

Note that any other $\mathbb{Q}$-isogeny class of type $R_4(15)$ can be obtained by quadratic twist from

$$\begin{array}{ccc} E_1 & \text{---} & E_5 \\ | & & | \\ E_3 & \text{---} & E_{15}\,. \end{array}$$

19.2 Kodaira symbols, minimal models, and Pal values

There are two primes of bad reduction $p = 2$ and 5.

$p = 2$					
E	$\mathrm{sig}_2(E)$	$\mathrm{K}_2(E)$	$u_2(d)$		
E_1	$(4, 6, 13)$	I_5^*	1	1	2
E_3	$(4, 6, 15)$	I_7^*	1	1	2
E_5	$(4, 6, 17)$	I_9^*	1	1	2
E_{15}	$(4, 6, 27)$	I_{19}^*	1	1	2
			$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
			$d \pmod 4$		

$p = 5$				
E	$\mathrm{sig}_3(E)$	$\mathrm{K}_3(E)$	$u_3(d)$	
E_1	$(2, 2, 4)$	IV	1	1
E_3	$(2, 2, 4)$	IV	1	1
E_5	$(3, 4, 8)$	IV*	5	1
E_{15}	$(3, 4, 8)$	IV*	5	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod 5$	

19.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

$[u(E)(d)]$	d	Prob
$(1:1:1:1)$	$d \not\equiv 0\,(5)$	$\left(\frac{5}{6}, 0, \frac{1}{6}, 0\right)$
$(1:1:5:5)$	$d \equiv 0\,(5)$	

This table is the main ingredient to prove the following result:

Proposition 19. *For every square-free integer d, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph*

$$\begin{tikzcd} E_1^d \arrow[r, dash, "5"] \arrow[d, dash, "3"] & E_5^d \arrow[d, dash, "3"] \\ E_3^d \arrow[r, dash, "5"] & E_{15}^d . \end{tikzcd}$$

is given by:

Table 33: Faltings curves in $R_4(15)$

twisted isogeny graph	*condition*	*prob*
$\begin{tikzcd} E_1^d \arrow[r] \arrow[d] & E_5^d \arrow[d] \\ E_3^d \arrow[r] & E_{15}^d \end{tikzcd}$	$d \not\equiv 0\,(5)$	5/6
$\begin{tikzcd} E_1^d \arrow[d] & E_5^d \arrow[l] \arrow[d] \\ E_3^d & E_{15}^d \arrow[l] \end{tikzcd}$	$d \equiv 0\,(5)$	1/6

20 Type $R_4(21)$

20.1 Settings

The isogeny graphs of type $R_4(21)$ are given by four isogenous elliptic curves:

$$\begin{tikzcd} E_1 \arrow[r, dash, "7"] \arrow[d, dash, "3"] & E_5 \arrow[d, dash, "3"] \\ E_3 \arrow[r, dash, "7"] & E_{21} \end{tikzcd}.$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(21)$ parametrize isogeny graphs of type $R_4(21)$. The modular curve $X_0(21)$ is elliptic of rank 0 over the rationals. Its rational points are: four cusps and four non-cuspodial non-CM points corresponding to $\tau, \tau', \tau'', \tau''' \in \mathbb{H}$.

j-invariants

The corresponding j-invariants of τ, τ', τ'' and τ''' are:

$$j(\tau) = \frac{3^3 \cdot 5^3}{2}, \qquad j(\tau') = \frac{-3^2 \cdot 5^6}{2^3}, \quad j(\tau'') = \frac{-3^3 \cdot 5^3 \cdot 383^3}{2^7}, \quad j(\tau''') = \frac{-3^2 \cdot 5^3 \cdot 101^3}{2^{21}}.$$

Signatures

We can (and do) choose Weierstrass equations in such a way that the isogeny graphs are normalized:

E	Minimal Weierstrass model	$j(E)$	label
E_1	$y^2 = x^3 + 45x + 18$	$\frac{+3^3 \cdot 5^3}{2}$	1296k1
E_3	$y^2 = x^3 - 675x + 7074$	$\frac{-3^2 \cdot 5^6}{2^3}$	1296k2
E_7	$y^2 = x^3 - 17235x - 870894$	$\frac{-3^3 \cdot 5^3 \cdot 383^3}{2^7}$	1296k3
E_{21}	$y^2 = x^3 - 13635x - 1244862$	$\frac{-3^2 \cdot 5^3 \cdot 101^3}{2^{21}}$	1296k4

Their signatures are:

E	E_1	E_3	E_7	E_{21}
$c_4(E)$	$-2^4 \cdot 3^3 \cdot 5$	$2^4 \cdot 3^4 \cdot 5^2$	$2^4 \cdot 3^3 \cdot 5 \cdot 383$	$2^4 \cdot 3^4 \cdot 5 \cdot 101$
$c_6(E)$	$-2^6 \cdot 3^5$	$-2^6 \cdot 3^6 \cdot 131$	$2^6 \cdot 3^5 \cdot 48383$	$2^6 \cdot 3^6 \cdot 23053$
$\Delta(E)$	$-2^{13} \cdot 3^6$	$-2^{15} \cdot 3^{10}$	$-2^{19} \cdot 3^6$	$-2^{33} \cdot 3^{10}$

We have that the Faltings curve (circled) in the graph is

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] & E_7 \arrow[d] \\ E_3 \arrow[r] & E_{21} \end{tikzcd}$$

Note that any $\mathbb{Q}$-isogeny class of type $R_4(21)$ can be obtained by quadratic twist from

$$\begin{array}{ccc} E_1 & \text{———} & E_7 \\ | & & | \\ E_3 & \text{———} & E_{21}\,. \end{array}$$

20.2 Kodaira symbols, minimal models, and Pal values

There are two bad reduction rational primes $p = 2$ and 3.

$p = 2$					
E	$\mathrm{sig}_2(E)$	$\mathrm{K}_2(E)$	$u_2(d)$		
E_1	$(4, 6, 13)$	I_5^*	1	1	2
E_3	$(4, 6, 15)$	I_7^*	1	1	2
E_7	$(4, 6, 19)$	I_{11}^*	1	1	2
E_{21}	$(4, 6, 33)$	I_{25}^*	1	1	2
			$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
			$d \pmod 4$		

$p = 3$				
E	$\mathrm{sig}_3(E)$	$\mathrm{K}_3(E)$	$u_3(d)$	
E_1	$(3, 5, 6)$	IV	1	1
E_3	$(4, 6, 10)$	IV^*	3	1
E_7	$(3, 5, 6)$	IV	1	1
E_{21}	$(4, 6, 10)$	IV^*	3	1
			$d \equiv 0$	$d \not\equiv 0$
			$d \pmod 3$	

20.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

$[u(E)(d)]$	d	Prob
$(1:1:1:1)$	$d \not\equiv 0\,(3)$	$\left(\frac{3}{4}, 0, \frac{1}{4}, 0\right)$
$(1:3:1:3)$	$d \equiv 0\,(3)$	

This table is the main ingredient to prove the following result:

Proposition 20. *For every square-free integer d, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph*

$$\begin{tikzcd} E_1^d \arrow[r, dash, "7"] \arrow[d, dash, "3"] & E_7^d \arrow[d, dash, "3"] \\ E_3^d \arrow[r, dash, "7"] & E_{21}^d . \end{tikzcd}$$

is given by:

Table 34: Faltings curves in $R_4(21)$

twisted isogeny graph	*condition*	*prob*
$\begin{tikzcd} E_1^d \arrow[r] \arrow[d] & E_7^d \arrow[d] \\ E_3^d \arrow[r] & E_{21}^d \end{tikzcd}$	$d \not\equiv 0\,(3)$	$3/4$
$\begin{tikzcd} E_1^d \arrow[d] & E_7^d \arrow[l] \arrow[d] \\ E_3^d & E_{21}^d \arrow[l] \end{tikzcd}$	$d \equiv 0\,(3)$	$1/4$

21 Type R_6

21.1 Settings

Graph

The isogeny graphs of type R_6 are given by six isogenous elliptic curves:

$$\begin{tikzcd}
E_1 \arrow[r, dash, "3"] \arrow[d, dash, "2"'] & E_3 \arrow[r, dash, "3"] \arrow[d, dash, "2"] & E_9 \arrow[d, dash, "2"] \\
E_2 \arrow[r, dash, "3"'] & E_6 \arrow[r, dash, "3"'] & E_{18}
\end{tikzcd}$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(18)$ parametrize isogeny graphs of type R_6. The curve $X_0(18)$ has genus 0 and a hauptmodul for this curve is:

$$t = 2 + 2 \cdot 3 \cdot \frac{\eta(2\tau)\eta(3\tau)\eta(18\tau)^2}{\eta(\tau)^2\eta(6\tau)\eta(9\tau)}.$$

j-invariants

Letting $t = t(\tau)$, one can write

$$j(E_1) = j(\tau) = \frac{\left(t^3-2\right)^3\left(t^9-6t^6-12t^3-8\right)^3}{(t-2)t^9(t+1)^2\left(t^2-t+1\right)^2\left(t^2+2t+4\right)}$$

$$j(E_2) = j(2\tau) = \frac{\left(t^3+4\right)^3\left(t^9-12t^6+48t^3+64\right)^3}{(t-2)^2t^{18}(t+1)\left(t^2-t+1\right)\left(t^2+2t+4\right)^2}$$

$$j(E_3) = j(3\tau) = \frac{\left(t^3-2\right)^3\left(t^3+6t-2\right)^3\left(t^6-6t^4-4t^3+36t^2+12t+4\right)^3}{(t-2)^3t^3(t+1)^6\left(t^2-t+1\right)^6\left(t^2+2t+4\right)^3}$$

$$j(E_6) = j(6\tau) = \frac{\left(t^3+4\right)^3\left(t^3+6t^2+4\right)^3\left(t^6-6t^5+36t^4+8t^3-24t^2+16\right)^3}{(t-2)^6t^6(t+1)^3\left(t^2-t+1\right)^3\left(t^2+2t+4\right)^6}$$

$$j(E_9) = j(9\tau) = \frac{\left(t^3+6t-2\right)^3\left(t^9+234t^7-6t^6+756t^5-936t^4+2172t^3-1512t^2+936t-8\right)^3}{(t-2)^9t(t+1)^{18}\left(t^2-t+1\right)^2\left(t^2+2t+4\right)}$$

$$j(E_{18}) = j(18\tau) = \frac{\left(t^3+6t^2+4\right)^3\left(t^9+234t^8+756t^7+2172t^6+1872t^5+3024t^4+48t^3+3744t^2+64\right)^3}{(t-2)^{18}t^2(t+1)^9\left(t^2-t+1\right)\left(t^2+2t+4\right)^2}.$$

Signatures

We can (and do) choose Weierstrass equations for $(E_1, E_2, E_3, E_6, E_9, E_{18})$ in such a way that the isogeny graph is normalized. Their signatures are:

R$_6$ signatures	
$c_4(E_1)$	$(t^3-2)\cdot(t^9-6t^6-12t^3-8)$
$c_6(E_1)$	$(t^6-4t^3-8)\cdot(t^{12}-8t^9-8t^3-8)$
$\Delta(E_1)$	$(t-2)\cdot(t+1)^2\cdot t^9\cdot(t^2+2t+4)\cdot(t^2-t+1)^2$
$c_4(E_2)$	$(t^3+4)\cdot(t^9-12t^6+48t^3+64)$
$c_6(E_2)$	$(t^6-4t^3-8)\cdot(t^{12}-8t^9-512t^3-512)$
$\Delta(E_2)$	$(t+1)\cdot(t-2)^2\cdot t^{18}\cdot(t^2-t+1)\cdot(t^2+2t+4)^2$
$c_4(E_3)$	$(t^3-2)\cdot(t^3+6t-2)\cdot(t^6-6t^4-4t^3+36t^2+12t+4)$
$c_6(E_3)$	$(t^2+2t-2)\cdot(t^4-2t^3-8t-2)\cdot(t^4-2t^3+6t^2+4t+4)\cdot(t^8+2t^7+4t^6-16t^5-14t^4+8t^3+64t^2-16t+4)$
$\Delta(E_3)$	$(t-2)^3\cdot t^3\cdot(t+1)^6\cdot(t^2+2t+4)^3\cdot(t^2-t+1)^6$
$c_4(E_6)$	$(t^3+4)\cdot(t^3+6t^2+4)\cdot(t^6-6t^5+36t^4+8t^3-24t^2+16)$
$c_6(E_6)$	$(t^2+2t-2)\cdot(t^4-8t^3-8t-8)\cdot(t^4-2t^3+6t^2+4t+4)\cdot(t^8+8t^7+64t^6-16t^5-56t^4+128t^3+64t^2-64t+64)$
$\Delta(E_6)$	$(t+1)^3\cdot(t-2)^6\cdot t^6\cdot(t^2-t+1)^3\cdot(t^2+2t+4)^6$
$c_4(E_9)$	$(t^3+6t-2)\cdot(t^9+234t^7-6t^6+756t^5-936t^4+2172t^3-1512t^2+936t-8)$
$c_6(E_9)$	$(t^6+24t^5+24t^4+92t^3-48t^2+96t-8)\cdot(t^{12}-24t^{11}+48t^{10}-680t^9+792t^8-3312t^7+4704t^6-10656t^5+13968t^4-14792t^3+7968t^2-2112t-8)$
$\Delta(E_9)$	$t\cdot(t-2)^9\cdot(t+1)^{18}\cdot(t^2+2t+4)\cdot(t^2-t+1)^2$
$c_4(E_{18})$	$(t^3+6t^2+4)\cdot(t^9+234t^8+756t^7+2172t^6+1872t^5+3024t^4+48t^3+3744t^2+64)$
$c_6(E_{18})$	$(t^6+24t^5+24t^4+92t^3-48t^2+96t-8)\cdot(t^{12}-528t^{11}-3984t^{10}-14792t^9-27936t^8-42624t^7-37632t^6-52992t^5-25344t^4-43520t^3-6144t^2-6144t-512)$
$\Delta(E_{18})$	$t^2\cdot(t+1)^9\cdot(t-2)^{18}\cdot(t^2-t+1)\cdot(t^2+2t+4)^2$

Automorphisms

The subgroup of $\operatorname{Aut} X_0(18)$ that fixes the set of vertices of the graph is isomorphic to the Klein group of order 4.

automorphism	permutation	order
$\mathrm{id}(t) = t$	$()$	1
$\sigma(t) = 2(t+1)/(t-2)$	$(j_1, j_{18})(j_2, j_9)(j_3, j_6)$	2
$\tau(t) = -2/t$	$(j_1, j_2)(j_3, j_6)(j_9, j_{18})$	2
$\sigma\tau(t) = -(t-2)/(t+1)$	$(j_1, j_9)(j_2, j_{18})(j_3)(j_6)$	2

Automorphism action on the graph	
id	$()$
σ	$(E_1, E_{18})^{\otimes -3}(E_2, E_9)^{\otimes -3}(E_3, E_6)^{\otimes -3}$
τ	$(E_1, E_2)(E_3, E_6)(E_9, E_{18})$
$\sigma\tau$	$(E_1, E_9)^{\otimes -3}(E_2, E_{18})^{\otimes -3}(E_3)^{\otimes -3}(E_6)^{\otimes -3}$

21.2 Kodaira symbols, minimal models, and Pal values

Table 35: R_6 data for $p \neq 2,3$

R_6	$p \neq 2,3$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = m > 0$	E_1	$(0,0,9m)$	1	I_{9m}	1	1
	E_2	$(0,0,18m)$	1	I_{18m}	1	1
	E_3	$(0,0,3m)$	1	I_{3m}	1	1
	E_6	$(0,0,6m)$	1	I_{6m}	1	1
	E_9	$(0,0,m)$	1	I_{m}	1	1
	E_{18}	$(0,0,2m)$	1	I_{2m}	1	1
$v_p(t) = 0$ $v_p(t-2) = m > 0$	E_1	$(0,0,m)$	1	I_{m}	1	1
	E_2	$(0,0,2m)$	1	I_{2m}	1	1
	E_3	$(0,0,3m)$	1	I_{3m}	1	1
	E_6	$(0,0,6m)$	1	I_{6m}	1	1
	E_9	$(0,0,9m)$	1	I_{9m}	1	1
	E_{18}	$(0,0,18m)$	1	I_{18m}	1	1
$v_p(t) = 0$ $v_p(t+1) = m > 0$	E_1	$(0,0,2m)$	1	I_{2m}	1	1
	E_2	$(0,0,m)$	1	I_{m}	1	1
	E_3	$(0,0,6m)$	1	I_{6m}	1	1
	E_6	$(0,0,3m)$	1	I_{3m}	1	1
	E_9	$(0,0,18m)$	1	I_{18m}	1	1
	E_{18}	$(0,0,9m)$	1	I_{9m}	1	1
$v_p(t) = 0$ $v_p(t^2+2t+4) = m > 0$	E_1	$(0,0,m)$	1	I_{m}	1	1
	E_2	$(0,0,2m)$	1	I_{2m}	1	1
	E_3	$(0,0,3m)$	1	I_{3m}	1	1
	E_6	$(0,0,6m)$	1	I_{6m}	1	1
	E_9	$(0,0,m)$	1	I_{m}	1	1
	E_{18}	$(0,0,2m)$	1	I_{2m}	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod p$	

Table 35: R_6 data for $p \neq 2,3$ (Continued)

R_6	$p \neq 2,3$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = 0$ $v_p(t^2 - t + 1) = m > 0$	E_1	$(0,0,2m)$	1	I_{2m}	1	1
	E_2	$(0,0,m)$	1	I_{m}	1	1
	E_3	$(0,0,6m)$	1	I_{6m}	1	1
	E_6	$(0,0,3m)$	1	I_{3m}	1	1
	E_9	$(0,0,2m)$	1	I_{2m}	1	1
	E_{18}	$(0,0,m)$	1	I_{m}	1	1
$v_p(t) = -m < 0$	E_1	$(0,0,18m)$	p^{-3m}	I_{18m}	1	1
	E_2	$(0,0,9m)$	p^{-3m}	I_{9m}	1	1
	E_3	$(0,0,6m)$	p^{-3m}	I_{6m}	1	1
	E_6	$(0,0,3m)$	p^{-3m}	I_{3m}	1	1
	E_9	$(0,0,2m)$	p^{-3m}	I_{2m}	1	1
	E_{18}	$(0,0,m)$	p^{-3m}	I_{m}	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod{p}$	

Table 36: R_6 data for $p = 3$

R_6	$p = 3$					
t	E	$\mathrm{sig}_3(E)$	u_3	$\mathrm{K}_3(E)$	$u_3(d)$	
$v_3(t) = m > 0$	E_1	$(0, 0, 9m)$	1	I_{9m}	1	1
	E_2	$(0, 0, 18m)$	1	I_{18m}	1	1
	E_3	$(0, 0, 3m)$	1	I_{3m}	1	1
	E_6	$(0, 0, 6m)$	1	I_{6m}	1	1
	E_9	$(0, 0, m)$	1	I_{m}	1	1
	E_{18}	$(0, 0, 2m)$	1	I_{2m}	1	1
$v_3(t) = 0$ $t \equiv 2, 5\,(8)$ $v_3(t-2) = m$	E_1	$(2, 3, m+5)$	1	I^*_{m-1}	3	1
	E_2	$(2, 3, 2m+4)$	1	I^*_{2m-2}	3	1
	E_3	$(2, 3, 3m+3)$	3	I^*_{3m-3}	3	1
	E_6	$(2, 3, 6m)$	3	I^*_{6m-6}	3	1
	E_9	$(2, 3, 9m-3)$	3^2	I^*_{9m-9}	3	1
	E_{18}	$(2, 3, 18m-12)$	3^2	I^*_{18m-18}	3	1
$v_3(t) = 0$ $t \equiv 8\,(9)$ $v_3(t+1) = m$	E_1	$(2, 3, 2m+4)$	1	I^*_{2m-2}	3	1
	E_2	$(2, 3, m+5)$	1	I^*_{m-1}	3	1
	E_3	$(2, 3, 6m)$	3	I^*_{6m-6}	3	1
	E_6	$(2, 3, 3m+3)$	3	I^*_{3m-3}	3	1
	E_9	$(2, 3, 18m-12)$	3^2	I^*_{18m-18}	3	1
	E_{18}	$(2, 3, 9m-3)$	3^2	I^*_{9m-9}	3	1
$v_3(t) = -m < 0$	E_1	$(0, 0, 18m)$	3^{-3m}	I_{18m}	1	1
	E_2	$(0, 0, 9m)$	3^{-3m}	I_{9m}	1	1
	E_3	$(0, 0, 6m)$	3^{-3m}	I_{6m}	1	1
	E_6	$(0, 0, 3m)$	3^{-3m}	I_{3m}	1	1
	E_9	$(0, 0, 2m)$	3^{-3m}	I_{2m}	1	1
	E_{18}	$(0, 0, m)$	3^{-3m}	I_{m}	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod 3$	

Table 37: R_6 data for p=2

R_6	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=m>1$	E_1	$(4,6,9m+3)$	1	I^*_{9m-5}	1	1	2
	E_2	$(4,6,18m-6)$	2	I^*_{18m-14}	1	1	2
	E_3	$(4,6,3m+9)$	1	I^*_{3m+1}	1	1	2
	E_6	$(4,6,6m+6)$	2	I^*_{6m-2}	1	1	2
	E_9	$(4,6,m+11)$	1	I^*_{m+3}	1	1	2
	E_{18}	$(4,6,2m+10)$	2	I^*_{2m+2}	1	1	2
$v_2(t)=1$ $v_2(t-2)=m$	E_1	$(4,6,m+11)$	1	I^*_{m+3}	1	1	2
	E_2	$(4,6,2m+10)$	2	I^*_{2m+2}	1	1	2
	E_3	$(4,6,3m+9)$	1	I^*_{3m+1}	1	1	2
	E_6	$(4,6,6m+6)$	2	I^*_{6m-2}	1	1	2
	E_9	$(4,6,9m+3)$	1	I^*_{9m-5}	1	1	2
	E_{18}	$(4,6,18m-6)$	2	I^*_{18m-14}	1	1	2
$v_2(t)=0$ $v_2(t+1)=m$	E_1	$(4,6,2m+12)$	2^{-1}	I^*_{2m+4}	1	1	2
	E_2	$(4,6,m+12)$	2^{-1}	I^*_{m+4}	1	1	2
	E_3	$(4,6,6m+12)$	2^{-1}	I^*_{6m+4}	1	1	2
	E_6	$(4,6,3m+12)$	2^{-1}	I^*_{3m+4}	1	1	2
	E_9	$(4,6,18m+12)$	2^{-1}	I^*_{18m+4}	1	1	2
	E_{18}	$(4,6,9m+12)$	2^{-1}	I^*_{9m+4}	1	1	2
$v_2(t)=-m<0$	E_1	$(4,6,18m+12)$	2^{-3m-1}	I^*_{18m+4}	1	1	2
	E_2	$(4,6,9m+12)$	2^{-3m-1}	I^*_{9m+4}	1	1	2
	E_3	$(4,6,6m+12)$	2^{-3m-1}	I^*_{6m+4}	1	1	2
	E_6	$(4,6,3m+12)$	2^{-3m-1}	I^*_{3m+4}	1	1	2
	E_9	$(4,6,2m+12)$	2^{-3m-1}	I^*_{2m+4}	1	1	2
	E_{18}	$(4,6,m+12)$	2^{-3m-1}	I^*_{m+4}	1	1	2
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					$d \pmod 4$		

21.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u}_p = [u_p(E)]$ and $\mathbf{u}_p(d) = [u_p(E)(d)]$:

t	$[u_2(E)]$	$[u_2(E)(d)]$
$v_2(t) > 0$	$(1:2:1:2:1:2)$	$(1:1:1:1:1:1)$
$v_2(t) \leq 0$	$(1:1:1:1:1:1)$	$(1:1:1:1:1:1)$

t	$[u_3(E)]$	$[u_3(E)(d)]$
$v_3(t) = 0$	$(1:1:3:3:3^2:3^2)$	$(1:1:1:1:1:1)$
$v_3(t) \neq 0$	$(1:1:1:1:1:1)$	$(1:1:1:1:1:1)$

The contents of these tables are the main ingredients to prove the following result:

Proposition 21. *Let*

$$\begin{array}{ccccc} E_1 & \overset{3}{\text{———}} & E_3 & \overset{3}{\text{———}} & E_9 \\ {\scriptstyle 2}\Big| & & \Big|{\scriptstyle 2} & & \Big|{\scriptstyle 2} \\ E_2 & \underset{3}{\text{———}} & E_6 & \underset{3}{\text{———}} & E_{18} \end{array}$$

be a $\mathbf{Q}$*-isogeny graph of type* R_6 *corresponding to a given* t *in* $\mathbf{Q} \setminus \{0, -1, 2\}$ *as above. For every square-free integer* d*, the probability of a vertex to be the Faltings curve (circled) in the twisted graph*

$$\begin{array}{ccccc} E_1^d & \overset{3}{\text{———}} & E_3^d & \overset{3}{\text{———}} & E_9^d \\ {\scriptstyle 2}\Big| & & \Big|{\scriptstyle 2} & & \Big|{\scriptstyle 2} \\ E_2^d & \underset{3}{\text{———}} & E_6^d & \underset{3}{\text{———}} & E_{18}^d \end{array}$$

is given by:

Table 38: Faltings curves in R_6

R_6		*twisted isogeny graph*	*prob*
$v_2(t) > 0$	$v_3(t) \neq 0$	$\begin{array}{ccccc} E_1^d & \overset{3}{\text{——}} & E_3^d & \longrightarrow & E_9^d \\ \downarrow & & \downarrow & & \downarrow \\ E_2^d & \longrightarrow & E_6^d & \longrightarrow & E_{18}^d \end{array}$	1
$v_2(t) > 0$	$v_3(t) = 0$	$\begin{array}{ccccc} E_1^d & \overset{3}{\text{——}} & E_3^d & \longrightarrow & E_9^d \\ \downarrow & & \downarrow & & \downarrow \\ E_2^d & \longrightarrow & E_6^d & \longrightarrow & E_{18}^d \end{array}$	1

Continued on next page

Table 38: Faltings curves in R_6 (Continued)

$v_2(t) \leq 0$	$v_3(t) \neq 0$	$\begin{tikzcd} E_1^d \arrow[r, dash, "3"] \arrow[d] & E_3^d \arrow[r] \arrow[d] & E_9^d \arrow[d] \\ E_2^d \arrow[r] & E_6^d \arrow[r] & E_{18}^d \end{tikzcd}$	1
$v_2(t) \leq 0$	$v_3(t) = 0$	$\begin{tikzcd} E_1^d \arrow[r, dash, "3"] \arrow[d] & E_3^d \arrow[r] \arrow[d] & E_9^d \arrow[d] \\ E_2^d \arrow[r] & E_6^d \arrow[r] & E_{18}^d \end{tikzcd}$	1

22 Type T_4

22.1 Settings

Graph

The isogeny graphs of type T_4 are given by four isogenous elliptic curves:

$$\begin{tikzcd} & E_1 \arrow[d, dash, "2"] & \\ & E_2 \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\ E_4 & & E_{12}\,. \end{tikzcd}$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(4)$ parametrize isogeny graphs of type T_4. The curve $X_0(4)$ has genus 0 and a hauptmodul for this curve is:

$$t = 2^8 \left(\frac{\eta(4\tau)}{\eta(\tau)} \right)^8 .$$

j-invariants

Letting $t = t(\tau)$, one can write

$$\begin{aligned}
j_1 = j(E_1) = j(\tau) &= \frac{(t^2+16t+16)^3}{t(t+16)} \\
j_2 = j(E_2) = j(2\tau) &= \frac{(t^2+16t+256)^3}{t^2(t+16)^2} \\
j_4 = j(E_4) = j(4\tau) &= \frac{(t^2+256t+4096)^3}{t^4(t+16)} \\
j_{12} = j(E_{12}) = j(\tau+1/2) &= -\frac{(t^2-224t+256)^3}{t(t+16)^4} .
\end{aligned}$$

Signatures

We can (and do) choose Weierstrass equations for (E_1, E_2, E_4, E_{12}) in such a way that the isogeny graph is normalized. Their signatures are:

$c_4(E_1)$	$\left(t^2 + 16\,t + 16\right)$
$c_6(E_1)$	$(t+8)\left(t^2 + 16\,t - 8\right)$
$\Delta(E_1)$	$t\,(t+16)$
$c_4(E_2)$	$\left(t^2 + 16\,t + 256\right)$
$c_6(E_2)$	$(t-16)(t+8)(t+32)$
$\Delta(E_2)$	$t^2(t+16)^2$
$c_4(E_4)$	$\left(t^2 + 256\,t + 4096\right)$
$c_6(E_4)$	$(t+32)\left(t^2 - 512\,t - 8192\right)$
$\Delta(E_4)$	$t^4(t+16)$
$c_4(E_{12})$	$\left(t^2 - 224\,t + 256\right)$
$c_6(E_{12})$	$(t-16)\left(t^2 + 544\,t + 256\right)$
$\Delta(E_{12})$	$-t\,(t+16)^4$

Automorphisms

The subgroup of $\operatorname{Aut} X_0(4)$ that fixes the set of vertices of the graph is isomorphic to the symmetric group $\mathcal{S}_3$ with elements:

		permutation	order
$\mathrm{id}(t)$	$= t$	(j_1, j_2, j_4, j_{12})	1
$\sigma(t)$	$= -256/(t+16)$	(j_{12}, j_2, j_1, j_4)	3
$\sigma^2(t)$	$= -16\,(t+16)/t$	(j_4, j_2, j_{12}, j_1)	3
$\tau(t)$	$= 256/t$	(j_4, j_2, j_4, j_{12})	2
$\sigma\tau(t)$	$= -(t+16)$	(j_1, j_2, j_4, j_{12})	2
$\sigma^2\tau(t)$	$= -16\,t/(t+16)$	(j_{12}, j_2, j_4, j_1)	2

22.2 Kodaira symbols, minimal models, and Pal values

Table 39: T_4 data for $p \neq 2$

T_4	$p \neq 2$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = m > 0$	E_1	$(0, 0, m)$	1	I_m	1	1
	E_2	$(0, 0, 2m)$	1	I_{2m}	1	1
	E_4	$(0, 0, 4m)$	1	I_{4m}	1	1
	E_{12}	$(0, 0, m)$	1	I_m	1	1
$v_p(t) = 0$ $v_p(t + 16) = m > 0$	E_1	$(0, 0, m)$	1	I_m	1	1
	E_2	$(0, 0, 2m)$	1	I_{2m}	1	1
	E_4	$(0, 0, m)$	1	I_m	1	1
	E_{12}	$(0, 0, 4m)$	1	I_{4m}	1	1
$v_p(t)? - m < 0$ m odd	E_1	$(2, 3, 4m + 6)$	$p^{-(m+1)/2}$	I^*_{4m}	p	1
	E_2	$(2, 3, 2m + 6)$	$p^{-(m+1)/2}$	I^*_{2m}	p	1
	E_4	$(2, 3, m + 6)$	$p^{-(m+1)/2}$	I^*_m	p	1
	E_{12}	$(2, 3, m + 6)$	$p^{-(m+1)/2}$	I^*_m	p	1
$v_p(t) = -m < 0$ m even	E_1	$(0, 0, 4m)$	$p^{-m/2}$	I_{4m}	1	1
	E_2	$(0, 0, 2m)$	$p^{-m/2}$	I_{2m}	1	1
	E_4	$(0, 0, m)$	$p^{-m/2}$	I_m	1	1
	E_{12}	$(0, 0, m)$	$p^{-m/2}$	I_m	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod p$	

Table 40: T_4 data for p=2

T_4	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=m>7$	E_1	$(0,0,m-8)$	2	I_{m-8}	1	2^{-1}	2^{-1}
	E_2	$(0,0,2(m-8))$	2^2	$\mathrm{I}_{2(m-8)}$	1	2^{-1}	2^{-1}
	E_4	$(0,0,4(m-8))$	2^3	$\mathrm{I}_{4(m-8)}$	1	2^{-1}	2^{-1}
	E_{12}	$(0,0,m-8)$	2^2	I_{m-8}	1	2^{-1}	2^{-1}
$v_2(t)=7$	E_1	$(4,6,11)$	1	II^*	1	1	1
	E_2	$(4,6,10)$	2	III^*	1	1	1
	E_4	$(4,6,8)$	2^2	I_1^*	1	1	1
	E_{12}	$(4,6,11)$	2	II^*	1	1	1
$v_2(t)=6$	E_1	$(4,6,10)$	1	III^*	1	1	1
	E_2	$(4,6,8)$	2	I_1^*	1	1	1
	E_4	$(5,5,4)$	2^2	III	1	1	1
	E_{12}	$(4,6,10)$	2	III^*	1	1	1
$v_2(t)=5$	E_1	$(4,6,9)$	1	I_0^*	1	1	1
	E_2	$(4,\geq 7,6)$	2	III	1	1	1
	E_4	$(6,\geq 10,12)$	2	I_3^*	1	2	1
	E_{12}	$(4,6,9)$	2	I_0^*	1	1	1
$v_2(t)=4$ $t/2^4\equiv 1\,(4)$	E_1	$(4,6,9)$	1	I_0^*	1	1	1
	E_2	$(4,\geq 7,6)$	2	III	1	1	1
	E_4	$(4,6,9)$	2	I_0^*	1	1	1
	E_{12}	$(6,\geq 10,12)$	2	I_3^*	1	2	1
$v_2(t)=4$ $t/2^4\equiv -1\,(16)$ $v_2(t+16)=m>7$	E_1	$(4,6,4+m)$	1	I_{m-4}^*	1	1	2
	E_2	$(4,6,2m-4)$	2	I_{2m-12}^*	1	1	2
	E_4	$(4,6,4+m)$	2	I_{m-4}^*	1	1	2
	E_{12}	$(4,6,4m-20)$	2^2	I_{4m-28}^*	1	1	2
$v_2(t)=4$ $t/2^4\equiv 7\,(16)$	E_1	$(4,6,11)$	1	I_3^*	1	1	1
	E_2	$(4,6,10)$	2	I_2^*	1	1	1
	E_4	$(4,6,11)$	2	I_3^*	1	1	1
	E_{12}	$(4,6,8)$	2^2	I_0^*	1	1	1
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					$d \pmod 4$		

Table 40: T_4 data for p=2 (Continued)

T_4	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=4$ $t/2^4 \equiv 3\,(8)$	E_1	$(4,6,10)$	1	I_2^*	1	1	1
	E_2	$(4,6,8)$	2	I_0^*	1	1	1
	E_4	$(4,6,10)$	2	I_2^*	1	1	1
	E_{12}	$(5,5,4)$	2^2	II	1	1	1
$v_2(t)=3$	E_1	$(4,\geq 7,6)$	1	II	1	1	1
	E_2	$(6,\geq 10,12)$	1	I_2^*	1	2	1
	E_4	$(6,9,15)$	1	I_5^*	1	2	1
	E_{12}	$(6,9,15)$	1	I_5^*	1	2	1
$v_2(t)=2$ $t/2^2 \equiv 1\,(4)$	E_1	$(5,5,4)$	1	II	1	1	1
	E_2	$(4,6,8)$	1	I_0^*	1	1	1
	E_4	$(4,6,10)$	1	I_2^*	1	1	1
	E_{12}	$(4,6,10)$	1	I_2^*	1	1	1
$v_2(t)=2$ $t/2^2 \equiv 3\,(4)$	E_1	$(5,5,4)$	1	III	1	1	1
	E_2	$(4,6,8)$	1	I_1^*	1	1	1
	E_4	$(4,6,10)$	1	III^*	1	1	1
	E_{12}	$(4,6,10)$	1	III^*	1	1	1
$v_2(t)=1$	E_1	$(6,9,14)$	2^{-1}	I_4^*	1	2	1
	E_2	$(6,9,16)$	2^{-1}	I_6^*	1	2	1
	E_4	$(6,9,17)$	2^{-1}	I_7^*	1	2	1
	E_{12}	$(6,9,17)$	2^{-1}	I_7^*	1	2	1
$v_p(t)=-2m \leq 0$ $2^{2m}t \equiv 1\,(4)$	E_1	$(4,6,12+8m)$	$2^{-(m+1)}$	I_{4+8m}^*	1	1	2
	E_2	$(4,6,12+4m)$	$2^{-(m+1)}$	I_{4+4m}^*	1	1	2
	E_4	$(4,6,12+2m)$	$2^{-(m+1)}$	I_{4+2m}^*	1	1	2
	E_{12}	$(4,6,12+2m)$	$2^{-(m+1)}$	I_{4+2m}^*	1	1	2
$v_p(t)=-2m \leq 0$ $2^{2m}t \equiv 3\,(4)$	E_1	$(0,0,8m)$	2^{-m}	I_{8m}	1	2^{-1}	2^{-1}
	E_2	$(0,0,4m)$	2^{-m}	I_{4m}	1	2^{-1}	2^{-1}
	E_4	$(0,0,2m)$	2^{-m}	I_{2m}	1	2^{-1}	2^{-1}
	E_{12}	$(0,0,2m)$	2^{-m}	I_{2m}	1	2^{-1}	2^{-1}
					$d \equiv 1$	$d \equiv 2$	$d \equiv 3$
					d (mod 4)		

Table 40: T_4 data for p=2 (Continued)

T_4	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_p(t)=-(2m+1)<0$	E_1	$(6,9,8m+22)$	$2^{-(m+2)}$	I^*_{8m+12}	1	2^2	1
	E_2	$(6,9,4m+20)$	$2^{-(m+2)}$	I^*_{4m+10}	1	2^2	1
	E_4	$(6,9,2m+19)$	$2^{-(m+2)}$	I^*_{2m+9}	1	2^2	1
	E_{12}	$(6,9,2m+19)$	$2^{-(m+2)}$	I^*_{2m+9}	1	2^2	1
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					$d \pmod 4$		

22.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

t	$[u(E)]$	$[u(E)(d)]$	d	Prob
$v_2(t) \geq 6$	$(1:2:2^2:1)$	$(1:1:1:1)$		$(0,0,1,0)$
$v_2(t) = 5$	$(1:2:2:2)$	$(1:1:1:1)$	$d \not\equiv 0\,(2)$	$\left(0, \frac{1}{3}, \frac{2}{3}, 0\right)$
		$(1:1:2:1)$	$d \equiv 0\,(2)$	
$v_2(t) = 4$ $t/2^4 \equiv 1\,(4)$	$(1:2:2:2)$	$(1:1:1:1)$	$d \not\equiv 0\,(2)$	$\left(0, \frac{2}{3}, 0, \frac{1}{3}\right)$
		$(1:1:1:2)$	$d \equiv 0\,(2)$	
$v_2(t) = 4$ $t/2^4 \equiv 3\,(4)$	$(1:2:2:2^2)$	$(1:1:1:1)$		$(0,0,0,1)$
$v_2(t) = 3$	$(1:1:1:1)$	$(1:2:2:2)$	$d \not\equiv 0\,(2)$	$\left(\frac{2}{3}, \frac{1}{3}, 0, 0\right)$
		$(1:1:1:1)$	$d \equiv 0\,(2)$	
$v_2(t) \leq 2$	$(1:1:1:1)$	$(1:1:1:1)$		$(1,0,0,0)$

The contents of this table are the main ingredients to prove the following result:

Proposition 22. *Let*

$$\begin{tikzcd} & E_1 \arrow[d, dash, "2"] & \\ & E_2 \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\ E_4 & & E_{12} \end{tikzcd}$$

be a $\mathbf{Q}$*-isogeny graph of type* T_4 *corresponding to a given* t *in* $\mathbf{Q}$*,* $t \neq 0, -16$*. For every square-free integer* d*, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph*

$$\begin{tikzcd} & E_1^d \arrow[d, dash, "2"] & \\ & E_2^d \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\ E_4^d & & E_{12}^d \end{tikzcd}$$

is given by:

Table 41: Faltings curves in T_4

T_4	*twisted isogeny graph*	d	*Prob*

Continued on next page

Table 41: Faltings curves in T_4 (Continued)

$v_2(t) \geq 6$	$\begin{tikzcd} & E_1^d & \\ & E_2^d \arrow[u] \arrow[dr] & \\ E_4^d \arrow[ur] & & E_{12}^d \end{tikzcd}$		1
$v_2(t) = 5$	$\begin{tikzcd} & E_1^d & \\ & E_2^d \arrow[u] \arrow[dr] & \\ E_4^d \arrow[ur] & & E_{12}^d \end{tikzcd}$	$d \equiv 0\,(2)$	1/3
	$\begin{tikzcd} & E_1^d & \\ & E_2^d \arrow[u] \arrow[dl] \arrow[dr] & \\ E_4^d & & E_{12}^d \end{tikzcd}$	$d \not\equiv 0\,(2)$	2/3
$v_2(t) = 4$ $t/2^4 \equiv 1\,(4)$	$\begin{tikzcd} & E_1^d & \\ & E_2^d \arrow[u] \arrow[dl] & \\ E_4^d & & E_{12}^d \arrow[ul] \end{tikzcd}$	$d \equiv 0\,(2)$	1/3
	$\begin{tikzcd} & E_1^d & \\ & E_2^d \arrow[u] \arrow[dl] \arrow[dr] & \\ E_4^d & & E_{12}^d \end{tikzcd}$	$d \not\equiv 0\,(2)$	2/3
$v_2(t) = 4$ $t/2^4 \equiv 3\,(4)$	$\begin{tikzcd} & E_1^d & \\ & E_2^d \arrow[u] \arrow[dl] & \\ E_4^d & & E_{12}^d \arrow[ul] \end{tikzcd}$		1
$v_2(t) = 3$	$\begin{tikzcd} & E_1^d & \\ & E_2^d \arrow[u] \arrow[dl] \arrow[dr] & \\ E_4^d & & E_{12}^d \end{tikzcd}$	$d \equiv 0\,(2)$	1/3
	$\begin{tikzcd} & E_1^d \arrow[d] & \\ & E_2^d \arrow[dl] \arrow[dr] & \\ E_4^d & & E_{12}^d \end{tikzcd}$	$d \not\equiv 0\,(2)$	2/3

Continued on next page

Table 41: Faltings curves in T_4 (Continued)

$v_2(t) \leq 2$	$\begin{array}{c} E_1^d \\ \downarrow \\ E_2^d \\ \swarrow \qquad \searrow \\ E_4^d \qquad\qquad E_{12}^d \end{array}$		1

$$
\begin{tikzcd}
& \omega\langle 1, \tau\rangle \arrow[d, dash] & \\
& \frac{1}{2}\omega\langle 1, 2\,\tau\rangle \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{1}{4}\,\omega\langle 1, 4\,\tau\rangle & & \frac{1}{2}\,\omega\langle 1, \tau + 1/2\rangle\,.
\end{tikzcd}
$$

$$
\begin{tikzcd}
& V \arrow[d, dash] & \\
& \frac{1}{2}\,V \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{1}{4}\,V & & \frac{1}{4}\,V\,.
\end{tikzcd}
$$

$$
\begin{tikzcd}
& V u_1^2 \arrow[d, dash] & \\
& \frac{1}{2}\,V u_2^2 \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{1}{4}\,V u_4^2 & & \frac{1}{4}\,V u_{12}^2\,.
\end{tikzcd}
$$

$$
\begin{tikzcd}
& V u_1^2 \dfrac{u_1(d)^2}{|d|} \arrow[d, dash] & \\
& \frac{1}{2}\,V u_2^2 \dfrac{u_2(d)^2}{|d|} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{1}{4}\,V u_4^2 \dfrac{u_4(d)^2}{|d|} & & \frac{1}{4}\,V u_{12}^2 \dfrac{u_{12}(d)^2}{|d|}\,.
\end{tikzcd}
$$

23 Type T_6

23.1 Setting

Graph

The isogeny graphs of type T_6 are given by six isogenous elliptic curves:

$$
\begin{tikzcd}
E_1 \arrow[dr, dash, "2"] & & & E_8 \\
& E_2 \arrow[r, dash, "2"] & E_4 \arrow[ur, dash, "2"] \arrow[dr, dash, "2"] & \\
E_{12} \arrow[ur, dash, "2"] & & & E_{22}\,.
\end{tikzcd}
$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(8)$ parametrize isogeny graphs of type T_6. The curve $X_0(8)$ has genus 0 and a hauptmodul for this curve is:

$$t = 4 + 2^5 \frac{\eta(2\tau)^2\eta(8t)^4}{\eta(\tau)^4\eta(4\tau)^2}\,.$$

j-invariants

Letting $t = t(\tau)$, one can write

$$
\begin{aligned}
j_1 = j(E_1) = j(\tau) \quad &= \frac{\left(t^4 - 16t^2 + 16\right)^3}{(t-4)t^2(t+4)}\,, \\
j_2 = j(E_2) = j(2\tau) \quad &= \frac{\left(t^4 - 16t^2 + 256\right)^3}{(t-4)^2t^4(t+4)^2}\,, \\
j_{12} = j(E_{12}) = j(\tau+1/2) \quad &= -\frac{\left(t^4 - 256t^2 + 4096\right)^3}{(t-4)t^8(t+4)}\,, \\
j_4 = j(E_4) = j(4\tau) \quad &= \frac{\left(t^4 + 224t^2 + 256\right)^3}{(t-4)^4t^2(t+4)^4}\,, \\
j_8 = j(E_8) = j(8\tau) \quad &= \frac{\left(t^4 + 240t^3 + 2144t^2 + 3840t + 256\right)^3}{(t-4)^8t(t+4)^2}\,, \\
j_{22} = j(E_{22}) = j(2\tau+1/2) &= -\frac{\left(t^4 - 240t^3 + 2144t^2 - 3840t + 256\right)^3}{(t-4)^2t(t+4)^8}\,.
\end{aligned}
$$

Signatures

We can (and do) choose Weierstrass equations for $(E_1, E_2, E_{12}, E_4, E_8, E_{22})$ in such a way that the isogeny graph is normalized. Their signatures are:

T_6 signatures	
$c_4(E_1)$	$(t^4 - 16\,t^2 + 16)$
$c_6(E_1)$	$(t^2 - 8)(t^4 - 16\,t^2 - 8)$
$\Delta(E_1)$	$t^2(t-4)(t+4)$
$c_4(E_2)$	$(t^4 - 16\,t^2 + 256)$
$c_6(E_2)$	$(t^2 - 32)(t^2 - 8)(t^2 + 16)$
$\Delta(E_2)$	$t^4(t-4)^2(t+4)^2$
$c_4(E_{12})$	$(t^4 - 256\,t^2 + 4096)$
$c_6(E_{12})$	$(t^2 - 32)(t^4 + 512\,t^2 - 8192)$
$\Delta(E_{12})$	$-t^8(t-4)(t+4)$
$c_4(E_4)$	$(t^4 + 224\,t^2 + 256)$
$c_6(E_4)$	$(t^2 - 24\,t + 16)(t^2 + 16)(t^2 + 24\,t + 16)$
$\Delta(E_4)$	$t^2(t-4)^4(t+4)^4$
$c_4(E_8)$	$(t^4 + 240\,t^3 + 2144\,t^2 + 3840\,t + 256)$
$c_6(E_8)$	$(t^2 + 24\,t + 16)(t^4 - 528\,t^3 - 4000\,t^2 - 8448\,t + 256)$
$\Delta(E_8)$	$t\,(t-4)^8(t+4)^2$
$c_4(E_{22})$	$(t^4 - 240\,t^3 + 2144\,t^2 - 3840\,t + 256)$
$c_6(E_{22})$	$(t^2 - 24\,t + 16)(t^4 + 528\,t^3 - 4000\,t^2 + 8448\,t + 256)$
$\Delta(E_{22})$	$-t\,(t-4)^2(t+4)^8$

Automorphisms

The subgroup of $\operatorname{Aut} X_0(8)$ that fixes the set of vertices of the graph is isomorphic to the dihedral group D_4 of eight elements:

```
[1,2,3,4,5,6] <---> [E1,E2,E12,E4,E8,E22]
===== T6 =====
w=-t --(ord(w)=2)--> d=1 --> [ 1, 2, 3, 4, 6, 5 ]
w=16/t --(ord(w)=2)--> d=1 --> [ 3, 2, 1, 4, 5, 6 ]
w=(4*t + 16)/(t - 4) --(ord(w)=2)--> d=-1 --> [ 5, 4, 6, 2, 1, 3 ]
w=(-4*t + 16)/(t + 4) --(ord(w)=2)--> d=-1 --> [ 6, 4, 5, 2, 3, 1 ]
w=(-4*t - 16)/(t - 4) --(ord(w)=4)--> d=-1 --> [ 5, 4, 6, 2, 3, 1 ]
w=(4*t - 16)/(t + 4) --(ord(w)=4)--> d=-1 --> [ 6, 4, 5, 2, 1, 3 ]
w=-16/t --(ord(w)=2)--> d=1 --> [ 3, 2, 1, 4, 6, 5 ]
```

23.2 Kodaira symbols, minimal models, and Pal values

Table 42: T_6 data for $p \neq 2$

T_6	$p \neq 2$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = m > 0$	E_1	$(0,0,2m)$	1	I_{2m}	1	1
	E_2	$(0,0,4m)$	1	I_{4m}	1	1
	E_{12}	$(0,0,8m)$	1	I_{8m}	1	1
	E_4	$(0,0,2m)$	1	I_{2m}	1	1
	E_8	$(0,0,m)$	1	I_{m}	1	1
	E_{22}	$(0,0,m)$	1	I_{m}	1	1
$v_p(t) = 0$ $v_p(t-4) = m > 0$	E_1	$(0,0,m)$	1	I_{m}	1	1
	E_2	$(0,0,2m)$	1	I_{2m}	1	1
	E_{12}	$(0,0,m)$	1	I_{m}	1	1
	E_4	$(0,0,4m)$	1	I_{4m}	1	1
	E_8	$(0,0,8m)$	1	I_{8m}	1	1
	E_{22}	$(0,0,2m)$	1	I_{2m}	1	1
$v_p(t) = 0$ $v_p(t+4) = m > 0$	E_1	$(0,0,m)$	1	I_{m}	1	1
	E_2	$(0,0,2m)$	1	I_{2m}	1	1
	E_{12}	$(0,0,m)$	1	I_{m}	1	1
	E_4	$(0,0,4m)$	1	I_{4m}	1	1
	E_8	$(0,0,2m)$	1	I_{2m}	1	1
	E_{22}	$(0,0,8m)$	1	I_{8m}	1	1
$v_p(t) = -m < 0$	E_1	$(0,0,8m)$	p^{-m}	I_{8m}	1	1
	E_2	$(0,0,4m)$	p^{-m}	I_{4m}	1	1
	E_{12}	$(0,0,2m)$	p^{-m}	I_{2m}	1	1
	E_4	$(0,0,2m)$	p^{-m}	I_{2m}	1	1
	E_8	$(0,0,m)$	p^{-m}	I_{m}	1	1
	E_{22}	$(0,0,m)$	p^{-m}	I_{m}	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod p$	

Table 43: T_6 data for p=2

T_6	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=m>3$	E_1	$(4,6,2m+4)$	1	I^*_{2m-4}	1	1	2
	E_2	$(4,6,4m-4)$	2	I^*_{4m-12}	1	1	2
	E_{12}	$(4,6,8m-20)$	2^2	I^*_{8m-28}	1	1	2
	E_4	$(4,6,2m+4)$	2	I^*_{2m-4}	1	1	2
	E_8	$(4,6,m+8)$	2	I^*_{m}	1	1	2
	E_{22}	$(4,6,m+8)$	2	I^*_{m}	1	1	2
$v_2(t)=3$	E_1	$(4,6,10)$	1	I^*_{2}	1	1	1
	E_2	$(4,6,8)$	2	I^*_{0}	1	1	1
	E_{12}	$(5,5,4)$	2^2	II	1	1	1
	E_4	$(4,6,10)$	2	I^*_{2}	1	1	1
	E_8	$(4,6,11)$	2	I^*_{3}	1	1	1
	E_{22}	$(4,6,11)$	2	I^*_{3}	1	1	1
$v_2(t)=2$ $t\equiv 4\,(32)$ $v_2(t-4)=m+5$	E_1	$(0,0,m)$	2	I_{m}	1	2^{-1}	2^{-1}
	E_2	$(0,0,2m)$	2^2	I_{2m}	1	2^{-1}	2^{-1}
	E_{12}	$(0,0,m)$	2^2	I_{m}	1	2^{-1}	2^{-1}
	E_4	$(0,0,4m)$	2^3	I_{4m}	1	2^{-1}	2^{-1}
	E_8	$(0,0,8m)$	2^4	I_{8m}	1	2^{-1}	2^{-1}
	E_{22}	$(0,0,2m)$	2^3	I_{2m}	1	2^{-1}	2^{-1}
$v_2(t)=2$ $t\equiv 28\,(32)$ $v_2(t+4)=m+5$	E_1	$(0,0,m)$	2	I_{m}	1	2^{-1}	2^{-1}
	E_2	$(0,0,2m)$	2^2	I_{2m}	1	2^{-1}	2^{-1}
	E_{12}	$(0,0,m)$	2^2	I_{m}	1	2^{-1}	2^{-1}
	E_4	$(0,0,4m)$	2^3	I_{4m}	1	2^{-1}	2^{-1}
	E_8	$(0,0,2m)$	2^3	I_{2m}	1	2^{-1}	2^{-1}
	E_{22}	$(0,0,8m)$	2^4	I_{8m}	1	2^{-1}	2^{-1}
$v_2(t)=2$ $t\equiv 12\,(32)$	E_1	$(4,6,11)$	1	II^*	1	1	1
	E_2	$(4,6,10)$	2	III^*	1	1	1
	E_{12}	$(4,6,11)$	2	II^*	1	1	1
	E_4	$(4,6,8)$	2^2	I^*_{1}	1	1	1
	E_8	$(4,6,10)$	2^2	III^*	1	1	1
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					$d \pmod 4$		

Continued on next page

Table 43: T_6 data for p=2 (Continued)

T_6	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
	E_{22}	$(5,5,4)$	2^3	III	1	1	1
$v_2(t)=2$ $t\equiv 20\,(32)$	E_1	$(4,6,11)$	1	II*	1	1	1
	E_2	$(4,6,10)$	2	III*	1	1	1
	E_{12}	$(4,6,11)$	2	II*	1	1	1
	E_4	$(4,6,8)$	2^2	I_1^*	1	1	1
	E_8	$(5,5,4)$	2^3	III	1	1	1
	E_{22}	$(4,6,10)$	2^2	III*	1	1	1
$v_2(t)=1$	E_1	$(5,5,4)$	1	II	1	1	1
	E_2	$(4,6,8)$	1	I_0^*	1	1	1
	E_{12}	$(4,6,10)$	1	I_2^*	1	1	1
	E_4	$(4,6,10)$	1	I_2^*	1	1	1
	E_8	$(4,6,11)$	1	I_3^*	1	1	1
	E_{22}	$(4,6,11)$	1	I_3^*	1	1	1
$v_2(t)=-m\leq 0$	E_1	$(4,6,8m+12)$	$2^{-(m+1)}$	I_{8m+4}^*	1	1	2
	E_2	$(4,6,4m+12)$	$2^{-(m+1)}$	I_{4m+4}^*	1	1	2
	E_{12}	$(4,6,2m+12)$	$2^{-(m+1)}$	I_{2m+4}^*	1	1	2
	E_4	$(4,6,2m+12)$	$2^{-(m+1)}$	I_{2m+4}^*	1	1	2
	E_8	$(4,6,m+12)$	$2^{-(m+1)}$	I_{m+4}^*	1	1	2
	E_{22}	$(4,6,m+12)$	$2^{-(m+1)}$	I_{m+4}^*	1	1	2
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					$d \pmod 4$		

23.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

t	$[u(E)]$	$[u(E)(d)]$	Prob
$v_2(t) \geq 3$	$(1:2:2^2:2:2:2)$	$(1:1:1:1:1:1)$	$(0,0,1,0,0,0)$
$v_2(t) = 2$ $t/2^2 \equiv 3\,(4)$	$(1:2:2:2^2:2^2:2^3)$	$(1:1:1:1:1:1)$	$(0,0,0,0,0,1)$
$v_2(t) = 2$ $t/2^2 \equiv 1\,(4)$	$(1:2:2:2^2:2^3:2^2)$	$(1:1:1:1:1:1)$	$(0,0,0,0,1,0)$
$v_2(t) \leq 1$	$(1:1:1:1:1:1)$	$(1:1:1:1:1:1)$	$(1,0,0,0,0,0)$

The contents of this table are the main ingredients to prove the following result:

Proposition 23. *Let*

$$\begin{tikzcd}
E_1 \arrow[dr, dash, "2"] & & & E_8 \\
& E_2 \arrow[r, dash, "2"] \arrow[dl, dash, "2"'] & E_4 \arrow[ur, dash, "2"'] \arrow[dr, dash, "2"] & \\
E_{12} & & & E_{22}
\end{tikzcd}$$

be a $\mathbf{Q}$*-isogeny graph of type* T_6 *corresponding to a given* t *in* $\mathbf{Q}$*,* $t \neq 0, \pm 4$*. For every square-free integer* d*, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph*

$$\begin{tikzcd}
E_1^d \arrow[dr, dash, "2"] & & & E_8^d \\
& E_2^d \arrow[r, dash, "2"] \arrow[dl, dash, "2"'] & E_4^d \arrow[ur, dash, "2"'] \arrow[dr, dash, "2"] & \\
E_{12}^d & & & E_{22}^d
\end{tikzcd}$$

is given by:

Table 44: Faltings curves in T_6

T_6	*twisted isogeny graph*	*Prob*
$v_2(t) \geq 3$	$\begin{tikzcd} E_1 & & & E_8 \\ & E_2 \arrow[ul] \arrow[r] & E_4 \arrow[ur] \arrow[dr] & \\ E_{12} \arrow[ur] & & & E_{22} \end{tikzcd}$	1
$v_2(t) = 2$ $t/2^2 \equiv 3\,(4)$	$\begin{tikzcd} E_1 & & & E_8 \arrow[dl] \\ & E_2 \arrow[ul] \arrow[dl] & E_4 \arrow[l] \arrow[dr] & \\ E_{12} & & & E_{22} \end{tikzcd}$	1
$v_2(t) = 2$ $t/2^2 \equiv 1\,(4)$	$\begin{tikzcd} E_1 & & & E_8 \\ & E_2 \arrow[ul] \arrow[dl] & E_4 \arrow[l] \arrow[ur] & \\ E_{12} & & & E_{22} \arrow[ul] \end{tikzcd}$	1
$v_2(t) \leq 1$	$\begin{tikzcd} E_1 \arrow[dr] & & & E_8 \\ & E_2 \arrow[dl] \arrow[r] & E_4 \arrow[ur] \arrow[dr] & \\ E_{12} & & & E_{22} \end{tikzcd}$	1

24 Type T_8

24.1 Settings

Graph

The isogeny graphs of type T_8 are given by eight isogenous elliptic curves:

$$
\begin{tikzcd}
& E_{21} \arrow[d, no head, "2"] & & E_{81} \arrow[d, no head, "2"] & \\
& E_2 \arrow[dl, no head, "2"'] \arrow[dr, no head, "2"] & & E_8 \arrow[dl, no head, "2"'] \arrow[dr, no head, "2"] & \\
E_1 & & E_4 \arrow[d, no head, "2"] & & E_{16} \\
& & E_{41} & & .
\end{tikzcd}
$$

Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(16)$ parametrize isogeny graphs of type T_8. The curve $X_0(16)$ has genus 0 and a hauptmodul for this curve is:

$$t = 2 + 2^3 \, \frac{\eta(2\tau)\eta(16t)^2}{\eta(\tau)^2\eta(8\tau)} \, .$$

j-invariants

Letting $t = t(\tau)$, one can write

$$j(E_1) = j(\tau) = \frac{\left(t^8 - 16t^4 + 16\right)^3}{(t-2)t^4(t+2)\left(t^2+4\right)} ,$$

$$j(E_2) = j(2\tau) = \frac{\left(t^8 - 16t^4 + 256\right)^3}{(t-2)^2t^8(t+2)^2\left(t^2+4\right)^2} ,$$

$$j(E_{21}) = j(\tau + 1/2) = -\frac{\left(t^8 - 256t^4 + 4096\right)^3}{(t-2)t^{16}(t+2)\left(t^2+4\right)} ,$$

$$j(E_4) = j(4\tau) = \frac{\left(t^4 - 4t^3 + 8t^2 + 16t + 16\right)^3\left(t^4 + 4t^3 + 8t^2 - 16t + 16\right)^3}{(t-2)^4t^4(t+2)^4\left(t^2+4\right)^4} ,$$

$$j(E_{41}) = j(2\tau + 1/2) = -\frac{\left(t^4 - 16t^3 + 8t^2 + 64t + 16\right)^3\left(t^4 + 16t^3 + 8t^2 - 64t + 16\right)^3}{(t-2)^2t^2(t+2)^2\left(t^2+4\right)^8} ,$$

$$j(E_8) = j(8\tau) = \frac{\left(t^8 + 240t^6 + 2144t^4 + 3840t^2 + 256\right)^3}{(t-2)^8t^2(t+2)^8\left(t^2+4\right)^2} ,$$

$$j(E_{81}) = j(4\tau + 1/2) = \\ -\frac{\left(t^8 - 240t^7 + 2160t^6 - 6720t^5 + 17504t^4 - 26880t^3 + 34560t^2 - 15360t + 256\right)^3}{(t-2)^4t(t+2)^{16}\left(t^2+4\right)} ,$$

$$j(E_{16}) = j(16\tau) = \frac{\left(t^8 + 240t^7 + 2160t^6 + 6720t^5 + 17504t^4 + 26880t^3 + 34560t^2 + 15360t + 256\right)^3}{(t-2)^{16}t(t+2)^4\left(t^2+4\right)} .$$

Signatures

We can (and do) choose Weierstrass equations for $(E_1, E_2, E_{21}, E_4, E_{41}, E_8, E_{81}, E_{16})$ in such a way that the isogeny graph is normalized. Their signatures are:

	T_8
$c_4(E_1)$	$(t^8 - 16t^4 + 16)$
$c_6(E_1)$	$(t^4 - 8) \cdot (t^8 - 16t^4 - 8)$
$\Delta(E_1)$	$(t - 2) \cdot (t+2) \cdot t^4 \cdot (t^2+4)$
$c_4(E_2)$	$(t^8 - 16t^4 + 256)$
$c_6(E_2)$	$(t^4 - 32) \cdot (t^4 - 8) \cdot (t^4 + 16)$
$\Delta(E_2)$	$(t - 2)^2 \cdot (t+2)^2 \cdot t^8 \cdot (t^2+4)^2$
$c_4(E_{21})$	$(t^8 - 256t^4 + 4096)$
$c_6(E_{21})$	$(t^4 - 32) \cdot (t^8 + 512t^4 - 8192)$
$\Delta(E_{21})$	$(-1) \cdot (t-2) \cdot (t+2) \cdot t^{16} \cdot (t^2+4)$
$c_4(E_4)$	$(t^4 - 4t^3 + 8t^2 + 16t + 16) \cdot (t^4 + 4t^3 + 8t^2 - 16t + 16)$
$c_6(E_4)$	$(t^2 - 4t - 4) \cdot (t^2 + 4t - 4) \cdot (t^4 + 16) \cdot (t^4 + 24t^2 + 16)$
$\Delta(E_4)$	$(t - 2)^4 \cdot t^4 \cdot (t+2)^4 \cdot (t^2+4)^4$
$c_4(E_{41})$	$(t^4 - 16t^3 + 8t^2 + 64t + 16) \cdot (t^4 + 16t^3 + 8t^2 - 64t + 16)$
$c_6(E_{41})$	$(t^2 - 4t - 4) \cdot (t^2 + 4t - 4) \cdot (t^8 + 528t^6 - 4000t^4 + 8448t^2 + 256)$
$\Delta(E_{41})$	$(-1) \cdot (t-2)^2 \cdot t^2 \cdot (t+2)^2 \cdot (t^2+4)^8$
$c_4(E_8)$	$(t^8 + 240t^6 + 2144t^4 + 3840t^2 + 256)$
$c_6(E_8)$	$(t^4 - 24t^3 + 24t^2 - 96t + 16) \cdot (t^4 + 24t^2 + 16) \cdot (t^4 + 24t^3 + 24t^2 + 96t + 16)$
$\Delta(E_8)$	$t^2 \cdot (t-2)^8 \cdot (t+2)^8 \cdot (t^2+4)^2$
$c_4(E_{81})$	$(t^8 - 240t^7 + 2160t^6 - 6720t^5 + 17504t^4 - 26880t^3 + 34560t^2 - 15360t + 256)$
$c_6(E_{81})$	$(t^4 - 24t^3 + 24t^2 - 96t + 16) \cdot (t^8 + 528t^7 - 3984t^6 + 14784t^5 - 31648t^4 + 59136t^3 - 63744t^2 + 33792t + 256)$
$\Delta(E_{81})$	$(-1) \cdot t \cdot (t-2)^4 \cdot (t+2)^{16} \cdot (t^2+4)$
$c_4(E_{16})$	$(t^8 + 240t^7 + 2160t^6 + 6720t^5 + 17504t^4 + 26880t^3 + 34560t^2 + 15360t + 256)$
$c_6(E_{16})$	$(t^4 + 24t^3 + 24t^2 + 96t + 16) \cdot (t^8 - 528t^7 - 3984t^6 - 14784t^5 - 31648t^4 - 59136t^3 - 63744t^2 - 33792t + 256)$
$\Delta(E_{16})$	$t \cdot (t+2)^4 \cdot (t-2)^{16} \cdot (t^2+4)$

Automorphisms

The subgroup of Aut $X_0(16)$ that fixes the set of vertices of the graph is isomorphic to the dihedral group of order 8:

automorphism	permutation	order
$\mathrm{id}(t) = t$	$()$	1
$\sigma(t) = 2(t-2)/(t+2)$	$(j_1, j_{81}, j_{21}, j_{16})(j_2, j_8)$	4
$\sigma^2(t) = -4/t$	$(j_1, j_{21})(j_{81}, j_{16})$	2
$\sigma^3(t) = -2(t+2)/(t-2)$	$(j_1, j_{16}, j_{21}, j_{81})(j_2, j_8)$	4
$\tau(t) = -t$	(j_{81}, j_{16})	2
$\sigma\tau(t) = 2(t+2)/(t-2)$	$(j_1, j_{16})(j_2, j_8)(j_{21}, j_{81})$	2
$\sigma^2\tau(t) = 4/t$	(j_1, j_{21})	2
$\sigma^3\tau(t) = -2(t-2)/(t+2)$	$(j_1, j_{81})(j_2, j_8)(j_{21}, j_{16})$	2

Automorphism action on the graph	
id	$()$
σ	$(E_1, E_{81}, E_{21}, E_{16})^{\otimes -1}(E_2, E_8)^{\otimes -1}(E_4)^{\otimes -1}(E_{41})^{\otimes -1}$
σ^2	$(E_1, E_{21})(E_{81}, E_{16})$
σ^3	$(E_1, E_{16}, E_{21}, E_{81})^{\otimes -1}(E_2, E_8)^{\otimes -1}$
τ	(E_{81}, E_{16})
$\sigma\tau$	$(E_1, E_{16})^{\otimes -1}(E_2, E_8)^{\otimes -1}(E_{21}, E_{81})^{\otimes -1}(E_4)^{\otimes -1}(E_{41})^{\otimes -1}$
$\sigma^2\tau$	(E_1, E_{21})
$\sigma^3\tau$	$(E_1, E_{81})^{\otimes -1}(E_2, E_8)^{\otimes -1}(E_{21}, E_{16})^{\otimes -1}(E_4)^{\otimes -1}(E_{41})^{\otimes -1}$

24.2 Kodaira symbols, minimal models, and Pal values

Table 45: T_8 data for $p \neq 2$

T_8	$p \neq 2$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = m > 0$	E_1	$(0,0,4m)$	1	I_{4m}	1	1
	E_2	$(0,0,8m)$	1	I_{8m}	1	1
	E_{21}	$(0,0,16m)$	1	I_{16m}	1	1
	E_4	$(0,0,4m)$	1	I_{4m}	1	1
	E_{41}	$(0,0,2m)$	1	I_{2m}	1	1
	E_8	$(0,0,2m)$	1	I_{2m}	1	1
	E_{81}	$(0,0,m)$	1	I_{m}	1	1
	E_{16}	$(0,0,m)$	1	I_{m}	1	1
$v_p(t) = 0$ $v_p(t-2) = m > 0$	E_1	$(0,0,m)$	1	I_{m}	1	1
	E_2	$(0,0,2m)$	1	I_{2m}	1	1
	E_{21}	$(0,0,m)$	1	I_{m}	1	1
	E_4	$(0,0,4m)$	1	I_{4m}	1	1
	E_{41}	$(0,0,2m)$	1	I_{2m}	1	1
	E_8	$(0,0,8m)$	1	I_{8m}	1	1
	E_{81}	$(0,0,4m)$	1	I_{4m}	1	1
	E_{16}	$(0,0,16m)$	1	I_{16m}	1	1
$v_p(t) = 0$ $v_p(t+2) = m > 0$	E_1	$(0,0,m)$	1	I_{m}	1	1
	E_2	$(0,0,2m)$	1	I_{2m}	1	1
	E_{21}	$(0,0,m)$	1	I_{m}	1	1
	E_4	$(0,0,4m)$	1	I_{4m}	1	1
	E_{41}	$(0,0,2m)$	1	I_{2m}	1	1
	E_8	$(0,0,8m)$	1	I_{8m}	1	1
	E_{81}	$(0,0,16m)$	1	I_{16m}	1	1
	E_{16}	$(0,0,4m)$	1	I_{4m}	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod p$	

Table 45: T_8 data for $p \neq 2$ (Continued)

T_8	$p \neq 2$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = 0$ $v_p(t^2+4) = m > 0$	E_1	$(0,0,m)$	1	I_m	1	1
	E_2	$(0,0,2m)$	1	I_{2m}	1	1
	E_{21}	$(0,0,m)$	1	I_m	1	1
	E_4	$(0,0,4m)$	1	I_{4m}	1	1
	E_{41}	$(0,0,8m)$	1	I_{8m}	1	1
	E_8	$(0,0,2m)$	1	I_{2m}	1	1
	E_{81}	$(0,0,m)$	1	I_m	1	1
	E_{16}	$(0,0,m)$	1	I_m	1	1
$v_p(t) = -m < 0$	E_1	$(0,0,16m)$	p^{-2m}	I_{16m}	1	1
	E_2	$(0,0,8m)$	p^{-2m}	I_{8m}	1	1
	E_{21}	$(0,0,4m)$	p^{-2m}	I_{4m}	1	1
	E_4	$(0,0,4m)$	p^{-2m}	I_{4m}	1	1
	E_{41}	$(0,0,2m)$	p^{-2m}	I_{2m}	1	1
	E_8	$(0,0,2m)$	p^{-2m}	I_{2m}	1	1
	E_{81}	$(0,0,m)$	p^{-2m}	I_m	1	1
	E_{16}	$(0,0,m)$	p^{-2m}	I_m	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod p$	

Table 46: T_8 data for p=2

T_8	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=m>1$	E_1	$(4,6,4m+4)$	1	I^*_{4m-4}	1	1	2
	E_2	$(4,6,8m-4)$	2	I^*_{8m-12}	1	1	2
	E_{21}	$(4,6,16m-20)$	2^2	I^*_{16m-28}	1	1	2
	E_4	$(4,6,4m+4)$	2	I^*_{4m-4}	1	1	2
	E_{41}	$(4,6,2m+8)$	2	I^*_{2m}	1	1	2
	E_8	$(4,6,2m+8)$	2	I^*_{2m}	1	1	2
	E_{81}	$(4,6,m+10)$	2	I^*_{m+2}	1	1	2
	E_{16}	$(4,6,m+10)$	2	I^*_{m+2}	1	1	2
$v_2(t)=1$ $t\equiv 2\,(8)$ $v_2(t-2)=m$	E_1	$(0,0,m-3)$	2	I_{m-3}	1	2^{-1}	2^{-1}
	E_2	$(0,0,2(m-3))$	2^2	$\mathrm{I}_{2(m-3)}$	1	2^{-1}	2^{-1}
	E_{21}	$(0,0,m-3)$	2^2	I_{m-3}	1	2^{-1}	2^{-1}
	E_4	$(0,0,4(m-3))$	2^3	$\mathrm{I}_{4(m-3)}$	1	2^{-1}	2^{-1}
	E_{41}	$(0,0,2(m-3))$	2^3	$\mathrm{I}_{2(m-3)}$	1	2^{-1}	2^{-1}
	E_8	$(0,0,8(m-3))$	2^4	$\mathrm{I}_{8(m-3)}$	1	2^{-1}	2^{-1}
	E_{81}	$(0,0,4(m-3))$	2^4	$\mathrm{I}_{4(m-3)}$	1	2^{-1}	2^{-1}
	E_{16}	$(0,0,16(m-3))$	2^5	$\mathrm{I}_{16(m-3)}$	1	2^{-1}	2^{-1}
$v_2(t)=1$ $t/2\equiv 3\,(4)$ $v_2(t^2-4)=m$ $v_2(t^2+4)=n$	E_1	$(0,0,m-3)$	2	I_{m-3}	1	2^{-1}	2^{-1}
	E_2	$(0,0,2(m-3))$	2^2	$\mathrm{I}_{2(m-3)}$	1	2^{-1}	2^{-1}
	E_{21}	$(0,0,m-3)$	2^2	I_{m-3}	1	2^{-1}	2^{-1}
	E_4	$(0,0,4(m-3))$	2^3	$\mathrm{I}_{4(m-3)}$	1	2^{-1}	2^{-1}
	E_{41}	$(0,0,2(m-3))$	2^3	$\mathrm{I}_{2(m-3)}$	1	2^{-1}	2^{-1}
	E_8	$(0,0,8(m-3))$	2^4	$\mathrm{I}_{8(m-3)}$	1	2^{-1}	2^{-1}
	E_{81}	$(0,0,16(m-3))$	2^5	$\mathrm{I}_{16(m-3)}$	1	2^{-1}	2^{-1}
	E_{16}	$(0,0,4(m-3))$	2^4	$\mathrm{I}_{4(m-3)}$	1	2^{-1}	2^{-1}
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					d (mod 4)		

Table 46: T_8 data for p=2 (Continued)

T_8	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=-m\leq 0$	E_1	$(4,6,12+16m)$	2^{-2m-1}	I^*_{4+16m}	1	1	2
	E_2	$(4,6,12+8m)$	2^{-2m-1}	I^*_{4+8m}	1	1	2
	E_{21}	$(4,6,12+4m)$	2^{-2m-1}	I^*_{4+4m}	1	1	2
	E_4	$(4,6,12+4m)$	2^{-2m-1}	I^*_{4+4m}	1	1	2
	E_{41}	$(4,6,12+2m)$	2^{-2m-1}	I^*_{4+2m}	1	1	2
	E_8	$(4,6,12+2m)$	2^{-2m-1}	I^*_{4+2m}	1	1	2
	E_{81}	$(4,6,12+m)$	2^{-2m-1}	I^*_{4+m}	1	1	2
	E_{16}	$(4,6,12+m)$	2^{-2m-1}	I^*_{4+m}	1	1	2
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					$d \pmod 4$		

24.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u} = [u(E)]$ and $\mathbf{u}(d) = [u(E)(d)]$:

t	$[u(E)]$	$[u(E)(d)]$	Prob
$v_2(t) \geq 2$	$(1:2:2^2:2:2:2:2:2)$	$(1:1:1:1:1:1:1:1)$	$(0,0,1,0,0,0,0,0)$
$v_2(t) = 1$ $t/2 \equiv 3\,(4)$	$(1:2:2:2^2:2^2:2^3:2^4:2^3)$	$(1:1:1:1:1:1:1:1)$	$(0,0,0,0,0,0,1,0)$
$v_2(t) = 1$ $t/2 \equiv 1\,(4)$	$(1:2:2:2^2:2^2:2^3:2^3:2^4)$	$(1:1:1:1:1:1:1:1)$	$(0,0,0,0,0,0,0,1)$
$v_2(t) \leq 0$	$(1:1:1:1:1:1:1:1)$	$(1:1:1:1:1:1:1:1)$	$(1,0,0,0,0,0,0,0)$

The contents of this table are the main ingredients to prove the following result:

Proposition 24. *Let*

$$\begin{tikzcd}
 & E_{21} \arrow[d, dash, "2"] & & E_{81} \arrow[d, dash, "2"] & \\
 & E_2 \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & & E_8 \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\
E_1 & & E_4 \arrow[d, dash, "2"] & & E_{16} \\
 & & E_{41} & &
\end{tikzcd}.$$

be a $\mathbf{Q}$*-isogeny graph of type* T_8 *corresponding to a given* t *in* $\mathbf{Q}$*,* $t \neq 0, \pm 2$*. For every square-free integer* d*, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph*

$$\begin{tikzcd}
 & E^d_{21} \arrow[d, dash, "2"] & & E^d_{81} \arrow[d, dash, "2"] & \\
 & E^d_2 \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & & E^d_8 \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\
E^d_1 & & E^d_4 \arrow[d, dash, "2"] & & E^d_{16} \\
 & & E^d_{41} & &
\end{tikzcd}.$$

is given by:

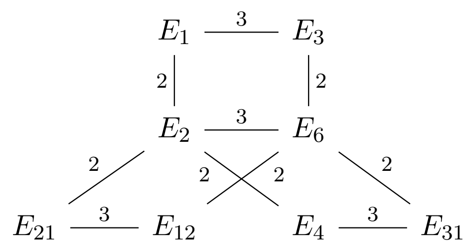
Table 47: Faltings curves in T_8

<table>
<tr><th>T_8</th><th>twisted isogeny graph</th><th>Prob</th></tr>
<tr><td>$v_2(t) \geq 2$</td><td>$$\begin{tikzcd} & E_{21} \arrow[d] & & E_{81} & \\ & E_2 \arrow[dl] \arrow[dr] & & E_8 \arrow[u] \arrow[dr] & \\ E_1 & & E_4 \arrow[ur] \arrow[d] & & E_{16} \\ & & E_{41} & & \end{tikzcd}$$</td><td>1</td></tr>
<tr><td>$v_2(t) = 1$
$t/2^2 \equiv 3\,(4)$</td><td>$$\begin{tikzcd} & E_{21} & & E_{81} \arrow[d] & \\ & E_2 \arrow[u] \arrow[dl] & & E_8 \arrow[dl] \arrow[dr] & \\ E_1 & & E_4 \arrow[ul] \arrow[d] & & E_{16} \\ & & E_{41} & & \end{tikzcd}$$</td><td>1</td></tr>
<tr><td>$v_2(t) = 1$
$t/2^2 \equiv 1\,(4)$</td><td>$$\begin{tikzcd} & E_{21} \arrow[d] & & E_{81} \arrow[d] & \\ & E_2 \arrow[dl] \arrow[dr] & & E_8 \arrow[dl] \arrow[dr] & \\ E_1 & & E_4 \arrow[d] & & E_{16} \\ & & E_{41} & & \end{tikzcd}$$</td><td>1</td></tr>
<tr><td>$v_2(t) \leq 0$</td><td>$$\begin{tikzcd} & E_{21} \arrow[d] & & E_{81} \arrow[d] & \\ & E_2 \arrow[dl] \arrow[dr] & & E_8 \arrow[dl] \arrow[dr] & \\ E_1 & & E_4 \arrow[d] & & E_{16} \\ & & E_{41} & & \end{tikzcd}$$</td><td>1</td></tr>
</table>

25 Type S

25.1 Settings

The isogeny graphs of type S are given by eight isogenous elliptic curves:



Modular curve

The $\mathbb{Q}$-rational points of the modular curve $X_0(12)$ parametrize isogeny graphs of type S. The curve $X_0(12)$ has genus 0 and a hauptmodul for this curve is:

$$t = 3 + 2^2 3 \frac{\eta(2\tau)^2 \eta(3\tau) \eta(12\tau)^3}{\eta(\tau)^3 \eta(4\tau) \eta(6\tau)^2}.$$

j-invariants.

Letting $t = t(\tau)$, one can write

$$\begin{aligned}
j(E_1) = j(\tau) &= \frac{\left(t^2-3\right)^3 \left(t^6-9t^4+3t^2-3\right)^3}{(t-3)(t-1)^3 t^4 (t+1)^3 (t+3)}, \\
j(E_3) = j(3\tau) &= \frac{\left(t^2-3\right)^3 \left(t^6-9t^4+243t^2-243\right)^3}{(t-3)^3 (t-1) t^{12} (t+1)(t+3)^3}, \\
j(E_2) = j(2\tau) &= \frac{\left(t^2+3\right)^3 \left(t^6-15t^4+75t^2+3\right)^3}{(t-3)^2 (t-1)^6 t^2 (t+1)^6 (t+3)^2}, \\
j(E_6) = j(6\tau) &= \frac{\left(t^2+3\right)^3 \left(t^6+225t^4-405t^2+243\right)^3}{(t-3)^6 (t-1)^2 t^6 (t+1)^2 (t+3)^6}, \\
j(E_{21}) = j(\tau+1/2) &= -\frac{\left(t^2-6t-3\right)^3 \left(t^6+6t^5+27t^4-60t^3-249t^2-234t-3\right)^3}{(t-3)(t-1)^{12} t (t+1)^3 (t+3)^4}, \\
j(E_{12}) = j(12\tau) &= \frac{\left(t^2+6t-3\right)^3 \left(t^6+234t^5+747t^4+540t^3-729t^2-486t-243\right)^3}{(t-3)^{12} (t-1) t^3 (t+1)^4 (t+3)^3}, \\
j(E_4) = j(4\tau) &= \frac{\left(t^2+6t-3\right)^3 \left(t^6-6t^5+27t^4+60t^3-249t^2+234t-3\right)^3}{(t-3)^4 (t-1)^3 t (t+1)^{12} (t+3)}, \\
j(E_{31}) = j(3\tau+1/2) &= -\frac{\left(t^2-6t-3\right)^3 \left(t^6-234t^5+747t^4-540t^3-729t^2+486t-243\right)^3}{(t-3)^3 (t-1)^4 t^3 (t+1)(t+3)^{12}}.
\end{aligned}$$

Signatures

We can (and do) choose Weierstrass equations for $(E_1, E_3, E_2, E_6, E_{21}, E_{12}, E_4, E_{31})$ in such a way that the isogeny graph is normalized. Their signatures are:

S signatures	
$c_4(E_1)$	$(t^2-3)\cdot(t^6-9t^4+3t^2-3)$
$c_6(E_1)$	$(t^4-6t^2-3)\cdot(t^8-12t^6+30t^4-36t^2+9)$
$\Delta(E_1)$	$(t-3)\cdot(t+3)\cdot(t-1)^3\cdot(t+1)^3\cdot t^4$
$c_4(E_3)$	$(t^2-3)\cdot(t^6-9t^4+243t^2-243)$
$c_6(E_3)$	$(t^4+18t^2-27)\cdot(t^8-36t^6+270t^4-972t^2+729)$
$\Delta(E_3)$	$(t-1)\cdot(t+1)\cdot(t-3)^3\cdot(t+3)^3\cdot t^{12}$
$c_4(E_2)$	$(t^2+3)\cdot(t^6-15t^4+75t^2+3)$
$c_6(E_2)$	$(t^4-6t^2-24t-3)\cdot(t^4-6t^2-3)\cdot(t^4-6t^2+24t-3)$
$\Delta(E_2)$	$(t-3)^2\cdot t^2\cdot(t+3)^2\cdot(t-1)^6\cdot(t+1)^6$
$c_4(E_6)$	$(t^2+3)\cdot(t^6+225t^4-405t^2+243)$
$c_6(E_6)$	$(t^4-24t^3+18t^2-27)\cdot(t^4+18t^2-27)\cdot(t^4+24t^3+18t^2-27)$
$\Delta(E_6)$	$(t-1)^2\cdot(t+1)^2\cdot(t-3)^6\cdot t^6\cdot(t+3)^6$
$c_4(E_{21})$	$(t^2-6t-3)\cdot(t^6+6t^5+27t^4-60t^3-249t^2-234t-3)$
$c_6(E_{21})$	$(t^4-6t^2-24t-3)\cdot(t^8-12t^6+528t^5+30t^4-3168t^3-3996t^2-1584t+9)$
$\Delta(E_{21})$	$(-1)\cdot(t-3)\cdot t\cdot(t+1)^3\cdot(t+3)^4\cdot(t-1)^{12}$
$c_4(E_{12})$	$(t^2+6t-3)\cdot(t^6+234t^5+747t^4+540t^3-729t^2-486t-243)$
$c_6(E_{12})$	$(t^4+24t^3+18t^2-27)\cdot(t^8-528t^7-3996t^6-9504t^5+270t^4+14256t^3-972t^2+729)$
$\Delta(E_{12})$	$(t-1)\cdot t^3\cdot(t+3)^3\cdot(t+1)^4\cdot(t-3)^{12}$
$c_4(E_4)$	$(t^2+6t-3)\cdot(t^6-6t^5+27t^4+60t^3-249t^2+234t-3)$
$c_6(E_4)$	$(t^4-6t^2+24t-3)\cdot(t^8-12t^6-528t^5+30t^4+3168t^3-3996t^2+1584t+9)$
$\Delta(E_4)$	$t\cdot(t+3)\cdot(t-1)^3\cdot(t-3)^4\cdot(t+1)^{12}$
$c_4(E_{31})$	$(t^2-6t-3)\cdot(t^6-234t^5+747t^4-540t^3-729t^2+486t-243)$
$c_6(E_{31})$	$(t^4-24t^3+18t^2-27)\cdot(t^8+528t^7-3996t^6+9504t^5+270t^4-14256t^3-972t^2+729)$
$\Delta(E_{31})$	$(-1)\cdot(t+1)\cdot(t-3)^3\cdot t^3\cdot(t-1)^4\cdot(t+3)^{12}$

Automorphisms

The subgroup of Aut $X_0(12)$ that fixes the set of vertices of the graph is isomorphic to the dihedral group of order 12 with elements:

automorphism	permutation	order
$\mathrm{id}(t) = t$	$()$	1
$\sigma(t) = 3(t-1)/(t+3)$	$(j_1, j_{31}, j_4, j_3, j_{21}, j_{12})(j_2, j_6)$	6
$\sigma^2(t) = (t-3)/(t+1)$	$(j_1, j_4, j_{21})(j_3, j_{12}, j_{31})$	3
$\sigma^3(t) = -3/t$	$(j_1, j_3)(j_2, j_6)(j_{21}, j_{31})(j_{12}, j_4)$	2
$\sigma^4(t) = -(t+3)/(t-1)$	$(j_1, j_{21}, j_4)(j_3, j_{31}, j_{12})$	3
$\sigma^5(t) = -3(t+1)/(t-3)$	$(j_1, j_{12}, j_{21}, j_3, j_4, j_{31})(j_2, j_6)$	6
$\tau(t) = -t$	$(j_{21}, j_4)(j_{12}, j_{31})$	2
$\sigma\tau(t) = 3(t+1)/(t-3)$	$(j_1, j_{31})(j_3, j_{21})(j_2, j_6)(j_{12}, j_4)$	2
$\sigma^2\tau(t) = (t+3)/(t-1)$	$(j_1, j_{21}, j_4)(j_3, j_{31})$	6
$\sigma^3\tau(t) = 3/t$	$(j_1, j_3)(j_2, j_6)(j_{21}, j_{12})(j_4, j_{31})$	2
$\sigma^4\tau(t) = -(t-3)/(t+1)$	$(j_1, j_{21})(j_3, j_{31})$	2
$\sigma^5\tau(t) = -3(t-1)/(t+3)$	$(j_1, j_{12})(j_3, j_4)(j_2, j_6)(j_{21}, j_{31})$	2

Automorphism action on the graph	
id	$()$
σ	$(E_1, E_{31}, E_4, E_3, E_{21}, E_{12})^{\otimes -3}(E_2, E_6)^{\otimes -3}(E_4)^{\otimes -3}(E_{41})^{\otimes -3}$
σ^2	$(E_1, E_4, E_{21})(E_3, E_{12}, E_{31})$
σ^3	$(E_1, E_3)^{\otimes -3}(E_2, E_6)^{\otimes -3}(E_{21}, E_{31})^{\otimes -3}(E_{12}, E_4)^{\otimes -3}$
σ^4	$(E_1, E_{21}, E_4)(E_3, E_{31}, E_{12})$
σ^5	$(E_1, E_{12}, E_{21}, E_3, E_4, E_{31})^{\otimes -3}(E_2, E_6)^{\otimes -3}$
τ	$(E_{21}, E_4)(E_{12}, E_{31})$
$\sigma\tau$	$(E_1, E_{31})^{\otimes -3}(E_3, E_{21})^{\otimes -3}(E_2, E_6)^{\otimes -3}(E_{12}, E_4)^{\otimes -3}$
$\sigma^2\tau$	$(E_1, E_{21}, E_4)(E_3, E_{31})$
$\sigma^3\tau$	$(E_1, E_3)^{\otimes -3}(E_2, E_6)^{\otimes -3}(E_{21}, E_{12})^{\otimes -3}(E_4, E_{31})^{\otimes -3}$
$\sigma^4\tau$	$(E_1, E_{21})(E_3, E_{31})$
$\sigma^5\tau$	$(E_1, E_{12})^{\otimes -3}(E_3, E_4)^{\otimes -3}(E_2, E_6)^{\otimes -3}(E_{21}, E_{31})^{\otimes -3}$

25.2 Kodaira symbols, minimal models, and Pal values

Table 48: S data for $p \neq 2,3$

S	$p \neq 2,3$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = m > 0$	E_1	$(0,0,4m)$	1	I_{4m}	1	1
	E_3	$(0,0,12m)$	1	I_{12m}	1	1
	E_2	$(0,0,2m)$	1	I_{2m}	1	1
	E_6	$(0,0,6m)$	1	I_{6m}	1	1
	E_{21}	$(0,0,m)$	1	I_{m}	1	1
	E_{12}	$(0,0,3m)$	1	I_{3m}	1	1
	E_4	$(0,0,m)$	1	I_{m}	1	1
	E_{31}	$(0,0,3m)$	1	I_{3m}	1	1
$v_p(t) = 0$ $v_p(t+1) = m$	E_1	$(0,0,3m)$	1	I_{3m}	1	1
	E_3	$(0,0,m)$	1	I_{m}	1	1
	E_2	$(0,0,6m)$	1	I_{6m}	1	1
	E_6	$(0,0,2m)$	1	I_{2m}	1	1
	E_{21}	$(0,0,3m)$	1	I_{3m}	1	1
	E_{12}	$(0,0,4m)$	1	I_{4m}	1	1
	E_4	$(0,0,12m)$	1	I_{12m}	1	1
	E_{31}	$(0,0,m)$	1	I_{m}	1	1
$v_p(t) = 0$ $v_p(t-1) = m$	E_1	$(0,0,3m)$	1	I_{3m}	1	1
	E_3	$(0,0,m)$	1	I_{m}	1	1
	E_2	$(0,0,6m)$	1	I_{6m}	1	1
	E_6	$(0,0,2m)$	1	I_{2m}	1	1
	E_{21}	$(0,0,12m)$	1	I_{12m}	1	1
	E_{12}	$(0,0,m)$	1	I_{m}	1	1
	E_4	$(0,0,3m)$	1	I_{3m}	1	1
	E_{31}	$(0,0,4m)$	1	I_{4m}	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod p$	

Table 48: S data for $p \neq 2, 3$ (Continued)

S	$p \neq 2,3$					
t	E	$\mathrm{sig}_p(E)$	u_p	$\mathrm{K}_p(E)$	$u_p(d)$	
$v_p(t) = 0$ $v_p(t+3) = m$	E_1	$(0, 0, m)$	1	I_m	1	1
	E_3	$(0, 0, 3m)$	1	I_{3m}	1	1
	E_2	$(0, 0, 2m)$	1	I_{2m}	1	1
	E_6	$(0, 0, 6m)$	1	I_{6m}	1	1
	E_{21}	$(0, 0, 4m)$	1	I_{4m}	1	1
	E_{12}	$(0, 0, 3m)$	1	I_{3m}	1	1
	E_4	$(0, 0, m)$	1	I_m	1	1
	E_{31}	$(0, 0, 12m)$	1	I_{12m}	1	1
$v_p(t) = 0$ $v_p(t-3) = m$	E_1	$(0, 0, m)$	1	I_m	1	1
	E_3	$(0, 0, 3m)$	1	I_{3m}	1	1
	E_2	$(0, 0, 2m)$	1	I_{2m}	1	1
	E_6	$(0, 0, 6m)$	1	I_{6m}	1	1
	E_{21}	$(0, 0, m)$	1	I_m	1	1
	E_{12}	$(0, 0, 12m)$	1	I_{12m}	1	1
	E_4	$(0, 0, 4m)$	1	I_{4m}	1	1
	E_{31}	$(0, 0, 3m)$	1	I_{3m}	1	1
$v_p(t) = -m < 0$	E_1	$(0, 0, 12m)$	p^{-2m}	I_{12m}	1	1
	E_3	$(0, 0, 4m)$	p^{-2m}	I_{4m}	1	1
	E_2	$(0, 0, 6m)$	p^{-2m}	I_{6m}	1	1
	E_6	$(0, 0, 2m)$	p^{-2m}	I_{2m}	1	1
	E_{21}	$(0, 0, 3m)$	p^{-2m}	I_{3m}	1	1
	E_{12}	$(0, 0, m)$	p^{-2m}	I_m	1	1
	E_4	$(0, 0, 3m)$	p^{-2m}	I_{3m}	1	1
	E_{31}	$(0, 0, m)$	p^{-2m}	I_m	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod p$	

Table 49: S data for $p = 3$

S	$p=3$					
t	E	$\mathrm{sig}_3(E)$	u_3	$\mathrm{K}_3(E)$	$u_3(d)$	
$v_3(t) = m > 1$	E_1	$(2,3,4m+2)$	1	I^*_{4m-4}	3	1
	E_3	$(2,3,12m-6)$	3	I^*_{12m-12}	3	1
	E_2	$(2,3,2m+4)$	1	I^*_{2m-2}	3	1
	E_6	$(2,3,6m)$	3	I^*_{6m-6}	3	1
	E_{21}	$(2,3,m+5)$	1	I^*_{m-1}	3	1
	E_{12}	$(2,3,3m+3)$	3	I^*_{3m-3}	3	1
	E_4	$(2,3,m+5)$	1	I^*_{m-1}	3	1
	E_{31}	$(2,3,3m+3)$	3	I^*_{3m-3}	3	1
$v_3(t) = 1$ $v_3(t-3) = m$ $v_3(t+3) = n$	E_1	$(2,3,m+n+4)$	1	I^*_{m+n-2}	3	1
	E_3	$(2,3,3m+3n)$	3	$\mathrm{I}^*_{3m+3n-6}$	3	1
	E_2	$(2,3,2m+2n+2)$	1	$\mathrm{I}^*_{2m+2n-4}$	3	1
	E_6	$(2,3,6m+6n-6)$	3	$\mathrm{I}^*_{6m+6n-12}$	3	1
	E_{21}	$(2,3,m+4n+1)$	1	I^*_{m+4n-5}	3	1
	E_{12}	$(2,3,12m+3n-9)$	3	$\mathrm{I}^*_{12m+3n-15}$	3	1
	E_4	$(2,3,4m+n+1)$	1	I^*_{4m+n-5}	3	1
	E_{31}	$(2,3,3m+12n-9)$	3	$\mathrm{I}^*_{3m+12n-15}$	3	1
$v_3(t) = 0$ $v_3(t-1) = m$ $v_3(t+1) = n$	E_1	$(0,0,3m+3n)$	1	I_{3m+3n}	1	1
	E_3	$(0,0,m+n)$	1	I_{m+n}	1	1
	E_2	$(0,0,6m+6n)$	1	I_{6m+6n}	1	1
	E_6	$(0,0,2m+2n)$	1	I_{2m+2n}	1	1
	E_{21}	$(0,0,12m+3n)$	1	I_{12m+3n}	1	1
	E_{12}	$(0,0,m+4n)$	1	I_{m+4n}	1	1
	E_4	$(0,0,3m+12n)$	1	I_{3m+12n}	1	1
	E_{31}	$(0,0,4m+n)$	1	I_{4m+n}	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod 3$	

Table 49: S data for $p = 3$ (Continued)

S	$p = 3$					
t	E	$\mathrm{sig}_3(E)$	u_3	$\mathrm{K}_3(E)$	$u_3(d)$	
$v_3(t) = -m < 0$	E_1	$(0, 0, 12m)$	3^{-2m}	I_{12m}	1	1
	E_3	$(0, 0, 4m)$	3^{-2m}	I_{4m}	1	1
	E_2	$(0, 0, 6m)$	3^{-2m}	I_{6m}	1	1
	E_6	$(0, 0, 2m)$	3^{-2m}	I_{2m}	1	1
	E_{21}	$(0, 0, 3m)$	3^{-2m}	I_{3m}	1	1
	E_{12}	$(0, 0, m)$	3^{-2m}	I_{m}	1	1
	E_4	$(0, 0, 3m)$	3^{-2m}	I_{3m}	1	1
	E_{31}	$(0, 0, m)$	3^{-2m}	I_{m}	1	1
					$d \equiv 0$	$d \not\equiv 0$
					$d \pmod 3$	

Table 50: S data for p=2

S	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=m>0$	E_1	$(4,6,4m+12)$	2^{-1}	I^*_{4m+4}	1	1	2
	E_3	$(4,6,12m+12)$	2^{-1}	I^*_{12m+4}	1	1	2
	E_2	$(4,6,2m+12)$	2^{-1}	I^*_{2m+4}	1	1	2
	E_6	$(4,6,6m+12)$	2^{-1}	I^*_{6m+4}	1	1	2
	E_{21}	$(4,6,m+12)$	2^{-1}	I^*_{m+4}	1	1	2
	E_{12}	$(4,6,3m+12)$	2^{-1}	I^*_{3m+4}	1	1	2
	E_4	$(4,6,m+12)$	2^{-1}	I^*_{m+4}	1	1	2
	E_{31}	$(4,6,3m+12)$	2^{-1}	I^*_{3m+4}	1	1	2
$v_2(t)=0$ $t\equiv 3\,(4)$ $v_2(t-3)=m$ $v_2(t+1)=n$	E_1	$(4,6,m+3n+4)$	1	I^*_{m+3n-4}	1	1	2
	E_3	$(4,6,3m+n+4)$	1	I^*_{3m+n-4}	1	1	2
	E_2	$(4,6,2m+6n-4)$	2	$\mathrm{I}^*_{2m+6n-12}$	1	1	2
	E_6	$(4,6,6m+2n-4)$	2	$\mathrm{I}^*_{6m+2n-12}$	1	1	2
	E_{21}	$(4,6,m+3n+4)$	2	I^*_{m+3n-4}	1	1	2
	E_{12}	$(4,6,12m+4n-20)$	4	$\mathrm{I}^*_{12m+4n-28}$	1	1	2
	E_4	$(4,6,4m+12n-20)$	4	$\mathrm{I}^*_{4m+12n-28}$	1	1	2
	E_{31}	$(4,6,3m+n+4)$	2	I^*_{3m+n-4}	1	1	2
$v_2(t)=0$ $t\equiv 1\,(4)$ $v_2(t-1)=m$ $v_2(t+3)=n$	E_1	$(4,6,3m+n+4)$	1	I^*_{3m+n-4}	1	1	2
	E_3	$(4,6,m+3n+4)$	1	I^*_{m+3n-4}	1	1	2
	E_2	$(4,6,6m+2n-4)$	2	$\mathrm{I}^*_{6m+2n-12}$	1	1	2
	E_6	$(4,6,2m+6n-4)$	2	$\mathrm{I}^*_{2m+6n-12}$	1	1	2
	E_{21}	$(4,6,12m+4n-20)$	4	$\mathrm{I}^*_{12m+4n-28}$	1	1	2
	E_{12}	$(4,6,m+3n+4)$	2	I^*_{m+3n-4}	1	1	2
	E_4	$(4,6,3m+n+4)$	2	I^*_{3m+n-4}	1	1	2
	E_{31}	$(4,6,4m+12n-20)$	4	$\mathrm{I}^*_{4m+12n-28}$	1	1	2
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					d (mod 4)		

Table 50: S data for p=2 (Continued)

S	$p=2$						
t	E	$\mathrm{sig}_2(E)$	u_2	$\mathrm{K}_2(E)$	$u_2(d)$		
$v_2(t)=-m<0$	E_1	$(4,6,12m+12)$	$2^{-(2m+1)}$	I^*_{12m+4}	1	1	2
	E_3	$(4,6,4m+12)$	$2^{-(2m+1)}$	I^*_{4m+4}	1	1	2
	E_2	$(4,6,6m+12)$	$2^{-(2m+1)}$	I^*_{6m+4}	1	1	2
	E_6	$(4,6,2m+12)$	$2^{-(2m+1)}$	I^*_{2m+4}	1	1	2
	E_{21}	$(4,6,3m+12)$	$2^{-(2m+1)}$	I^*_{3m+4}	1	1	2
	E_{12}	$(4,6,m+12)$	$2^{-(2m+1)}$	I^*_{m+4}	1	1	2
	E_4	$(4,6,3m+12)$	$2^{-(2m+1)}$	I^*_{3m+4}	1	1	2
	E_{31}	$(4,6,m+12)$	$2^{-(2m+1)}$	I^*_{m+4}	1	1	2
					$d\equiv 1$	$d\equiv 2$	$d\equiv 3$
					$d \pmod 4$		

25.3 Conclusion

From the above tables one gets the (projective) vectors $\mathbf{u}_p=[u_p(E)]$ and $\mathbf{u}_p(d)=[u_p(E)(d)]$:

t	$[u_2(E)]$	$[u_2(E)(d)]$
$v_2(t)\neq 0$	$(1:1:1:1:1:1:1:1)$	$(1:1:1:1:1:1:1:1)$
$v_2(t)=0$ $t/2\equiv 3\,(4)$	$(1:1:2:2:2:2^2:2^2:2)$	$(1:1:1:1:1:1:1:1)$
$v_2(t)=0$ $t/2\equiv 1\,(4)$	$(1:1:2:2:2^2:2:2:2^2)$	$(1:1:1:1:1:1:1:1)$

t	$[u_3(E)]$	$[u_3(E)(d)]$
$v_3(t)>0$	$(1:3:1:3:1:3:1:3)$	$(1:1:1:1:1:1:1:1)$
$v_3(t)\leq 0$	$(1:1:1:1:1:1:1:1)$	$(1:1:1:1:1:1:1:1)$

The contents of these tables are the main ingredients to prove the following result:

Proposition 25. *Let*

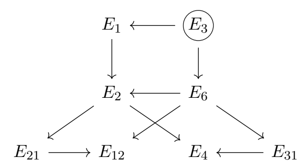
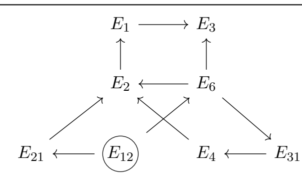
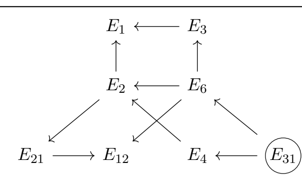
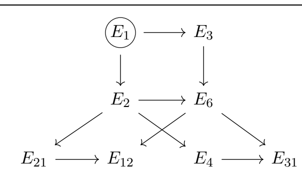
$$\begin{tikzcd}
 & E_1 \arrow[r, dash, "3"] \arrow[d, dash, "2"'] & E_3 \arrow[d, dash, "2"] & \\
 & E_2 \arrow[r, dash, "3"] \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & E_6 \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\
E_{21} \arrow[r, dash, "3"] & E_{12} & E_4 \arrow[r, dash, "3"] & E_{31}
\end{tikzcd}$$

be a $\mathbf{Q}$*-isogeny graph of type* S *corresponding to a given* t *in* $\mathbf{Q}$*,* $t \neq 1, \pm 3$ *as above. For every square-free integer* d*, the probability of a vertex to be the Faltings curve (circled) in the twisted isogeny graph*

$$\begin{tikzcd}[arrows=-] & E_1^d \arrow[r, "3"] \arrow[d, "2"'] & E_3^d \arrow[d, "2"] & \\ & E_2^d \arrow[r, "3"] \arrow[dl, "2"'] \arrow[dr, "2"] & E_6^d \arrow[dl, "2"'] \arrow[dr, "2"] & \\ E_{21}^d \arrow[r, "3"] & E_{12}^d & E_4^d \arrow[r, "3"] & E_{31}^d \end{tikzcd}$$

is given by:

Table 51: Faltings curves in S

S		*Twisted graph*	*Prob*
$v_3(t) > 1$	$v_2(t) \neq 0$		1
$v_3(t) > 1$	$v_2(t) = 0$ $t \equiv 3\,(4)$		1
$v_3(t) > 1$	$v_2(t) = 0$ $t \equiv 1\,(4)$		1
$v_3(t) \leq 0$	$v_2(t) \neq 0$		1

Continued on next page

Table 51: Faltings curves in S (Continued)

$v_3(t) \leq 0$	$v_2(t) = 0$ $t \equiv 3\,(4)$	$\begin{tikzcd} & E_1 & E_3 \arrow[l] & \\ & E_2 \arrow[u] \arrow[dl] & E_6 \arrow[l] \arrow[u] \arrow[dl] & \\ E_{21} & E_{12} \arrow[l] & E_4 \arrow[ul] \arrow[r] & E_{31} \arrow[ul] \end{tikzcd}$	1
$v_3(t) \leq 0$	$v_2(t) = 0$ $t \equiv 1\,(4)$	$\begin{tikzcd} & E_1 \arrow[r] & E_3 \arrow[d] & \\ & E_2 \arrow[u] \arrow[r] \arrow[dr] & E_6 \arrow[dr] & \\ E_{21} \arrow[ur] \arrow[r] & E_{12} \arrow[ur] & E_4 \arrow[r] & E_{31} \end{tikzcd}$	1