(29) Verifiqueu que les següents equacions són exactes i resoleu-les. (Indicació: busqueu $U(x,y)$ resolent el sistema d'equacions $\frac{\partial U}{\partial x} = P$ i $\frac{\partial U}{\partial x} = Q$ per integració directa respecte x i y.)

Recordem: $P(x,y) + Q(x,y)\,y' = 0$ és exacta sii $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$; llavors existeix $U(x,y)$ t.q. $\frac{\partial U}{\partial x} = P$ i $\frac{\partial U}{\partial x} = Q$. Aleshores, $U(x,y) = C$, $C \in \mathbb{R}$, dóna una familia de solucions de l'edo exacta

(a) $2x + \frac{1}{y} + \left(\frac{1}{y} - \frac{x}{y^2}\right) y' = 0$

- temim $P = 2x + \frac{1}{y}$, $Q = \frac{1}{y} - \frac{x}{y^2}$

$$\frac{\partial P}{\partial y} = -\frac{1}{y^2} = \frac{\partial Q}{\partial x} \Rightarrow \text{edo exacta.}$$

Busquem $U(x,y)$ t.q. $\frac{\partial U}{\partial x} = P$, $\frac{\partial U}{\partial y} = Q$.

- $\frac{\partial U}{\partial x} = P = 2x + \frac{1}{y} \underset{\substack{\uparrow \\ \text{integrem resp. } x}}{\Longrightarrow} U(x,y) = x + \frac{x}{y} + \underbrace{\varphi(y)}_{\substack{\text{constant} \\ \text{d'integració} = \\ \text{[depèn de } y]}}$

- $\frac{\partial U}{\partial y} = -\frac{x}{y^2} + \varphi'(y) = Q = \frac{1}{y} - \frac{x}{y^2}$

Per tant: $\varphi'(y) = \frac{1}{y} \Rightarrow \varphi(y) = \ln|y| + \text{const.}$

Així: $U(x,y) = x^2 + \frac{x}{y} + \ln|y|$. Solució: $U(x,y) = C$.

(b) $\underbrace{x(2x^2+y^2)}_{\substack{\shortparallel \\ P(x,y)}} + \underbrace{y(x^2+2y^2)}_{\substack{\shortparallel \\ Q(x,y)}} y' = 0$

$$\frac{\partial P}{\partial y} = 2xy = \frac{\partial Q}{\partial x} \Rightarrow \text{edo exacta}$$

Busquem $U(x,y)$ t.q. $\frac{\partial U}{\partial x} = P$ i $\frac{\partial U}{\partial y} = Q$.

- $\frac{\partial U}{\partial x} = P = x(2x^2+y^2) \underset{\substack{\uparrow \\ \text{integrem resp. } x}}{\Longrightarrow} U(x,y) = \frac{x^4}{2} + \frac{x^2y^2}{2} + \varphi(y).$

- $\frac{\partial U}{\partial y} = x^2y + \varphi'(y) = Q = y(x^2+2y^2) \Rightarrow \varphi'(y) = 2y^3 \Rightarrow$

$\Rightarrow \varphi(y) = y^4/2$

Així: $U(x,y) = \frac{x^4}{2} + \frac{x^2y^2}{2} + \frac{y^4}{2}$. Solució: $U(x,y) = C$

(e) $\underbrace{\sin y + y\sin x + \frac{1}{x}}_{\substack{\shortparallel \\ P(x,y)}} + \underbrace{\left(x\cos y - \cos x + \frac{1}{y}\right)}_{\substack{\shortparallel \\ Q(x,y)}} y' = 0$

$$\frac{\partial P}{\partial y} = \cos y + \sin x = \frac{\partial Q}{\partial x}$$

Busquem $U(x,y)$ t.q. $\frac{\partial U}{\partial x} = P$ i $\frac{\partial U}{\partial y} = Q$.

- $\frac{\partial U}{\partial y} = Q = x\cos y - \cos x + \frac{1}{y} \underset{\substack{\uparrow \\ \text{integrem resp. } y}}{\Longrightarrow} U(x,y) = x\sin y - y\cos x -$
$-\ln|y| + \varphi(x)$

- $\frac{\partial U}{\partial x} = \sin y + y\sin x + \varphi'(x) = P = \sin y + y\sin x + \frac{1}{x} \Rightarrow$

$\Rightarrow \varphi'(x) = \frac{1}{x} \Rightarrow \varphi(x) = \ln|x| + \text{const.}$

Així: $U(x,y) = x\sin y - y\cos x + \ln|x| + \ln|y|$. Solució $U(x,y) = C$