(23) Feu servir la Transformada de Laplace per resoldre els següents problemes de Cauchy:

(a) $y' + 2y = t \;,\; y(0) = -1.$

$$Y(s) = \mathcal{L}\{y(t)\}(s) \Rightarrow s\cdot Y(s) - \overbrace{y(0)}^{=-1} + 2Y(s) = \frac{1}{s^2} \Rightarrow$$

$$\Rightarrow (s+2)Y(s) = \frac{1}{s^2} - 1 = \frac{1-s^2}{s^2} \Rightarrow Y(s) = \frac{1-s^2}{s^2(s+2)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1-s^2}{s^2(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1/4}{s} + \frac{1/2}{s^2} - \frac{3/4}{s+2}\right\} = -\frac{1}{4} + \frac{1}{2}t - \frac{3}{4}e^{-2t}$$

(b) $y'' - 4y' + 4y = t^3 ,\; y(0) = 1 ,\; y'(0) = 0$

$$s^2Y(s) - s\cdot\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0} - 4\left[sY(s) - \underbrace{y(0)}_{=1}\right] + 4Y(s) = \frac{6}{s^4}$$

$$(s^2 - 4s + 4)Y(s) = \frac{6}{s^4} + s - 4 = \frac{s^5 - 4s^4 + 6}{s^4} \Rightarrow Y(s) = \frac{s^5 - 4s^4 + 6}{s^4(s^2 - 4s + 4)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{3/4}{s} + \frac{9/8}{s^2} + \frac{3/2}{s^3} + \frac{3/2}{s^4} + \frac{1/4}{s-2} - \frac{13/8}{(s-2)^2}\right\} =$$

$$= \frac{3}{4} + \frac{9}{8}t + \frac{3}{2}t^2 + \frac{1}{4}t^3 + \frac{1}{4}e^{2t} - \frac{13}{8}e^{2t}\cdot t$$

(c) $y''' + 2y'' - y' - 2y = \sin 3t ,\; y(0) = y'(0) = 0 ,\; y''(0) = 1.$

$$s^3Y(s) - s^2\overbrace{y(0)}^{=0} - s\,\overbrace{y'(0)}^{=0} - \overbrace{y''(0)}^{=1} + 2\left[s^2Y(s) - s\,\overbrace{y(0)}^{=0} - \overbrace{y'(0)}^{=0}\right] -$$

$$-\left[sY(s) - \underbrace{y(0)}_{=0}\right] - 2Y(s) = \frac{3}{s^2+9}$$

$$(s^3 + 2s^2 - s - 2)Y(s) = \frac{3}{s^2+9} + 1 = \frac{s^2+12}{s^2+9}$$

$$Y(s) = \frac{s^2+12}{(s^2+9)(s^3+2s^2-s-2)} = \frac{s^2+12}{(s^2+9)(s-1)(s+1)(s+2)} =$$

$$= \frac{\frac{3}{130}s - \frac{6}{130}}{s^2+9} + \frac{13/60}{s-1} - \frac{13/20}{s+1} + \frac{16/39}{s+2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{3}{130}\cos 3t - \frac{1}{65}\sin 3t + \frac{13}{60}e^{t} - \frac{13}{20}e^{-t} + \frac{16}{39}e^{-2t}$$

(d) $y'' - 5y' + 6y = u(t-1)$, $y(0) = 0$, $y'(0) = 1$

$$s^2Y(s) - s\overset{=0}{y(0)} - \overset{=1}{y'(0)} - 5\left[sY(s) - \overset{=0}{y(0)}\right] + 6Y(s) = e^{-s}/s$$

$$(s^2 - 5s + 6)Y(s) = \frac{e^{-s}}{s} + 1 \Rightarrow Y(s) = \frac{e^{-s}}{s\cdot(s^2-5s+6)} + \frac{1}{s^2-5s+6}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s-2)(s-3)} + \frac{1}{(s-2)(s-3)}\right\} =$$

$$= \mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1/6}{s} + \frac{1/3}{s-3} + \frac{-1/2}{s-2}\right) + \frac{1}{s-3} - \frac{1}{s-2}\right\} =$$

$$= u(t-1)\,\mathcal{L}^{-1}\left\{\frac{1/6}{s} + \frac{1/3}{s-3} + \frac{-1/2}{s-2}\right\} + e^{3t} - e^{2t} =$$

$$= u(t-1)\left(\frac{1}{6} + \frac{1}{3}e^{3(t-1)} - \frac{1}{2}e^{2(t-1)}\right) + e^{3t} - e^{2t}$$