(25) Resolem l'equació: $y'' + 2y' + 2y = \cos(t)\cdot\delta(t-3\pi)$, $y(0)=1$, $y'(0)=-1$

Fem $Y(s) = \mathcal{L}[y(t)](s)$ i apliquem transformada de Laplace a l'equació, tot usant que si $g(t)$ és contínua en t_0 llavors $\mathcal{L}[g(t)\,\delta(t-t_0)] = e^{-s\cdot t_0}\, g(t_0)$. Així:

$$\mathcal{L}\{y''(t)\} + 2\mathcal{L}\{y'(t)\} + 2\mathcal{L}\{y(t)\} = \mathcal{L}\{\cos(t)\cdot\delta(t-3\pi)\}$$

$$s^2 Y(s) - s\underset{=1}{y(0)} - \underset{=-1}{y'(0)} + 2\left(sY(s) - \underset{=1}{y(0)}\right) + 2Y(s) = \cos(3\pi)\cdot e^{-s\cdot 3\pi}$$

$$(s^2+2s+2)Y(s) = s+1 - e^{-3\pi s}$$

$$Y(s) = \frac{s+1}{s^2+2s+2} - \frac{e^{-3\pi s}}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1^2} - \frac{e^{-3\pi s}}{(s+1)^2+1^2}. \text{ Llavors:}$$

- $$\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1^2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\} = e^{-t}\cos(t)$$

- $$\mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{(s+1)^2+1^2}\right\} = e^{3\pi}\mathcal{L}^{-1}\left\{\frac{e^{-3\pi(s+1)}}{(s+1)^2+1^2}\right\} = e^{3\pi}e^{-t}\mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{s^2+1^2}\right\} =$$

$$= e^{3\pi}e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}_{t\to t-3\pi} u(t-3\pi) = e^{3\pi}e^{-t}\left(\sin(t)\right)_{t\to t-3\pi}\cdot u(t-3\pi) =$$

$$= e^{3\pi}e^{-t}\sin(t-3\pi)\cdot u(t-3\pi) = -e^{3\pi}e^{-t}\sin(t)\cdot u(t-3\pi)$$

Finalment:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{(s+1)^2+1^2}\right\} = e^{-t}\cos(t) + e^{3\pi}e^{-t}\sin(t)\,u(t-3\pi).$$