26 Feu servir la transformada de Laplace per resoldre les equacions integrals següents:

(a) $$f(t) + 2\underbrace{\int_0^t f(z)\cdot \cos(t-z)\,dz}_{f(t)*\cos(t)} = 4e^{-t} + \sin t$$

$F(s) = \mathcal{L}[f(t)]$ Complex:

$$F(s) + 2\underbrace{\mathcal{L}[f(t)*\cos(t)]}_{= F(s)\cdot\mathcal{L}[\cos t]} = \mathcal{L}[4e^{-t} + \sin t]$$

$$\underbrace{F(s) + 2F(s)\frac{s}{s^2+1}}_{F(s)\frac{(s+1)^2}{s^2+1}} = \frac{4}{s+1} + \frac{1}{s^2+1} \Rightarrow F(s) = \underbrace{\frac{4(s^2+1)}{(s+1)^3}}_{(*)} + \frac{1}{(s+1)^2}$$

Fem descomposició en fraccions simples de $(*)$

$$\frac{4(s^2+1)}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} \iff 4(s^2+1) = A(s+1)^2 + B(s+1) + C$$

$C = 8,\ B = -8,\ A = 4$

$$F(s) = \frac{4}{s+1} - \frac{7}{(s+1)^2} + \frac{8}{(s+1)^3}$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = e^{-t}\mathcal{L}^{-1}\left[\frac{4}{s} - \frac{7}{s^2} + \frac{8}{s^3}\right] = e^{-t}\left[4 - 7t + 4t^2\right]$$

(b) $$t - 2f(t) = \underbrace{\int_0^t (e^z - e^{-z})\, f(t-z)\,dz}_{2\sinh(t)*f(t)}$$

$F(s) = \mathcal{L}[f(t)]$ Complex: $\frac{1}{s^2} - 2F(s) = 2\frac{1}{s^2-1}F(s) \Rightarrow$

$$2\left[\frac{1}{s^2-1} + 1\right]F(s) = \frac{1}{s^2} \Rightarrow \frac{2s^2}{s^2-1}F(s) = \frac{1}{s^2} \Rightarrow F(s) = \frac{1}{2s^4}(s^2-1) = \frac{1}{2}\frac{1}{s^2} - \frac{1}{2}\frac{1}{s^4}$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2}t - \frac{1}{2}\frac{t^3}{3!} = \frac{t}{2} - \frac{t^3}{12}$$