Càlcul 1: Resum Propietats Funcions Bàsiques

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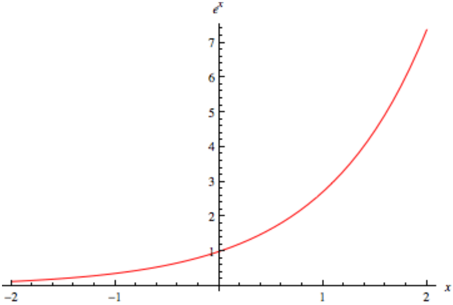
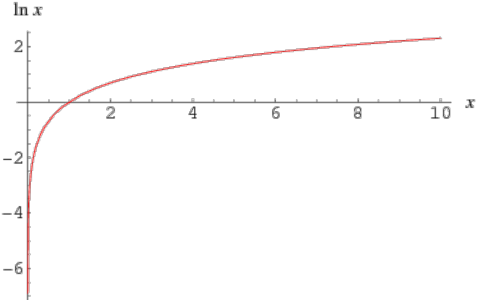
La funció exponencial $\exp(x) = e^x$ $(e \approx 2.71828)$



- $D_{\exp} = \mathbb{R}$ (domini), $\text{rang}_{\exp} = (0, +\infty)$ (rang), $e^0 = 1$.
- $e^{x+y} = e^x\, e^y$, $e^{-x} = \dfrac{1}{e^x}$, $(e^x)^y = e^{x\,y}$.
- $\dfrac{d}{dx} e^x = e^x$.
- $\lim\limits_{x \to +\infty} e^x = +\infty$, $\lim\limits_{x \to +-\infty} e^x = 0$.

Logarítme neperià (en base e / natural) $\ln(x)$



- $D_{\ln} = (0, +\infty)$, $\quad$ $\mathrm{rang}_{\ln} = \mathbb{R}$, $\quad$ $\ln(1) = 0$, $\quad$ $\ln(\mathrm{e}) = 1$.
- $\mathrm{e}^{\ln(x)} = x$, si $x > 0$; $\quad$ $\ln(\mathrm{e}^x) = x$, si $x \in \mathbb{R}$.
  (e^x i $\ln(x)$ són funcions inverses l'una de l'altre.)
- $\ln(x\,y) = \ln(x) + \ln(y)$, $\quad$ $\ln(x/y) = \ln(x) - \ln(y)$, $\quad$ $\ln(x^c) = c\ln(x)$.
- $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$.
- $\displaystyle\lim_{x\to+\infty} \ln(x) = +\infty$, $\qquad$ $\displaystyle\lim_{x\to 0^+} \ln(x) = -\infty$.

Logarítme en base a $\log_a(x)$ $(a > 0)$

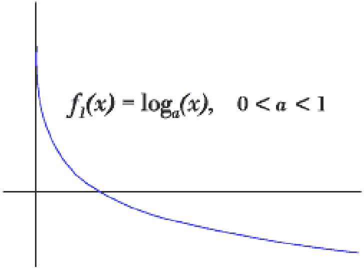


Figure 1

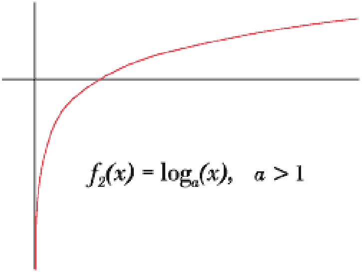


Figure 2

- $\log_a(x) = \dfrac{\ln(x)}{\ln(a)} = \dfrac{\log_b(x)}{\log_b(a)}$ $(b > 0)$.
- $D_{\log_a} = (0, +\infty)$, $\text{rang}_{\log_a} = \mathbb{R}$, $\log_a(a) = 1$, $\log_a(1) = 0$.
- $\dfrac{d}{dx} \log_a(x) = \dfrac{1}{x \ln(a)}$.

Funció exponencial de base a $\quad a^x \quad (a > 0)$

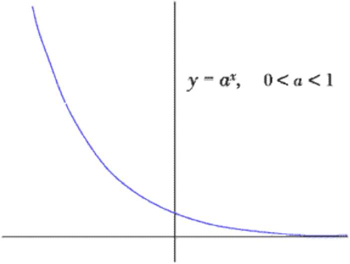


Figure 1

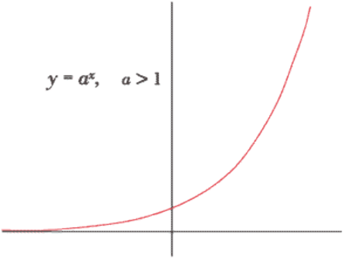


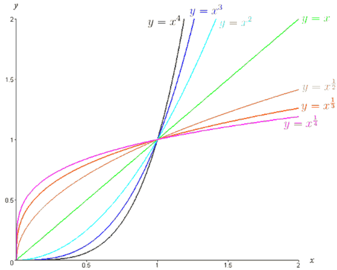
Figure 2

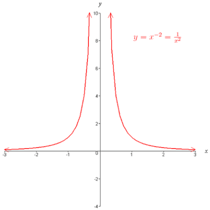
- $a^x = \mathrm{e}^{x\ln(a)}, \quad D_{a^x} = \mathbb{R}, \quad \mathrm{rang}_{a^x} = (0, +\infty), \quad a^0 = 1.$
- $\log_a(a^x) = x, \quad a^{\log_a(x)} = x, \quad a^{-x} = \dfrac{1}{a^x}, \quad (a^x)^y = a^{x\,y}.$
- $\dfrac{d}{dx} a^x = a^x \ln(a).$
- $\lim\limits_{x \to +\infty} a^x = +\infty$ i $\lim\limits_{x \to -\infty} a^x = 0$, si $a > 1$.
- $\lim\limits_{x \to +\infty} a^x = 0$ i $\lim\limits_{x \to -\infty} a^x = +\infty$, si $0 < a < 1$.

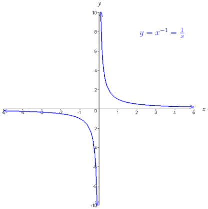
Funcions potència x^a $(a \in \mathbb{R})$

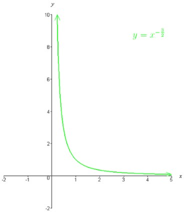
- $x^{-a} = \dfrac{1}{x^a}$, $\dfrac{d}{dx} x^a = a\,x^{a-1}$.
- *Cas* $a = n \in \mathbb{N} \cup \{0\}$*:* $D_{x^n} = \mathbb{R}$. (Si $a = 0 \implies x^0 = 1$).
- *Cas* $a = p/q \in \mathbb{Q}^+$ *amb mcd*$(p,q) = 1$ *(*p/q *fracció irreductible):*
  - $x^{p/q} = \sqrt[q]{x^p}$, $D_{x^{p/q}} = \mathbb{R}$ si q senar, $D_{x^{p/q}} = [0, +\infty)$ si q parell.
  - Si q és parell, cal vigila signe arrel q. S'entén: $x^{p/q} = +\sqrt[q]{x^p}$.
  - $\lim\limits_{x \to +\infty} x^{p/q} = +\infty$, $\lim\limits_{x \to 0^+} x^{p/q} = 0$, $\lim\limits_{x \to 0^-} x^{p/q} = 0$ si q senar.
  - $\lim\limits_{x \to -\infty} x^{p/q} = +\infty$ si q senar i p parell.
  - $\lim\limits_{x \to -\infty} x^{p/q} = -\infty$ si q i p senars.
- *Cas* $a = -p/q$, $p/q \in \mathbb{Q}^+$ *amb mcd*$(p,q) = 1$*:*
  - $D_{x^{-p/q}} = \mathbb{R} \setminus \{0\}$ si q senar, $D_{x^{-p/q}} = (0, +\infty)$ si q parell.
  - $\lim\limits_{x \to 0^+} x^{-p/q} = +\infty$, $\lim\limits_{x \to +\infty} x^{-p/q} = 0$.
  - $\lim\limits_{x \to -\infty} x^{-p/q} = 0$ si q senar.
  - $\lim\limits_{x \to 0^-} x^{-p/q} = +\infty$ si q senar i p parell.
  - $\lim\limits_{x \to 0^-} x^{p/q} = +\infty$ si q i p senars.

- *Cas* $a \in (\mathbb{R} \setminus \mathbb{Q})^+$ *(a irracional positiu):* $x^a = \mathrm{e}^{a \ln(x)}$,
  $D_{x^a} = [0, +\infty)$, $\lim_{x \to +\infty} x^a = +\infty$, $\lim_{x \to 0^+} x^a = 0$.
- *Cas* $a \in (\mathbb{R} \setminus \mathbb{Q})^-$*:* $D_{x^a} = (0, +\infty)$, $\lim_{x \to +\infty} x^a = 0$, $\lim_{x \to 0^+} x^a = +\infty$.





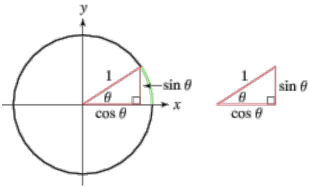


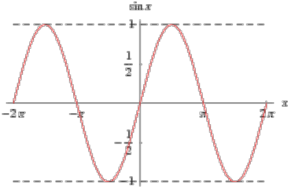


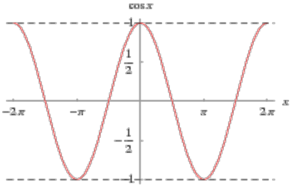
Funcions trigonomètriques bàsiques (*x* en radiants)

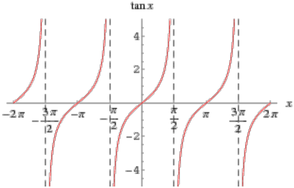
$\sin(x)$ (sinus), $\cos(x)$ (cosinus), $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ (tangent).

- $D_{\sin} = D_{\cos} = \mathbb{R}$, $D_{\tan} = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + 2\pi k : k \in \mathbb{Z} \right\}$.
- $\text{rang}_{\sin} = \text{rang}_{\cos} = [-1, 1]$, $\text{rang}_{\tan} = \mathbb{R}$.
- $\sin(x + 2\pi k) = \sin(x)$, $\cos(x + 2\pi k) = \cos(x)$ (2π-periòdiques).
- $\sin(-x) = -\sin(x)$ (senar), $\cos(-x) = \cos(x)$ (parell).
- $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$.
- $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$.
- $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$, $\sin^2(x) = \dfrac{1 - \cos(2x)}{2}$.
- $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$.
- $\dfrac{d}{dx}\sin(x) = \cos(x)$, $\dfrac{d}{dx}\cos(x) = -\sin(x)$.
- $\dfrac{d}{dx}\tan(x) = 1 + \tan^2(x) = \dfrac{1}{\cos^2(x)}$.
- $\lim\limits_{x \to \left(\frac{\pi}{2}\right)^-} \tan(x) = +\infty$, $\lim\limits_{x \to \left(\frac{\pi}{2}\right)^+} \tan(x) = -\infty$.









x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0

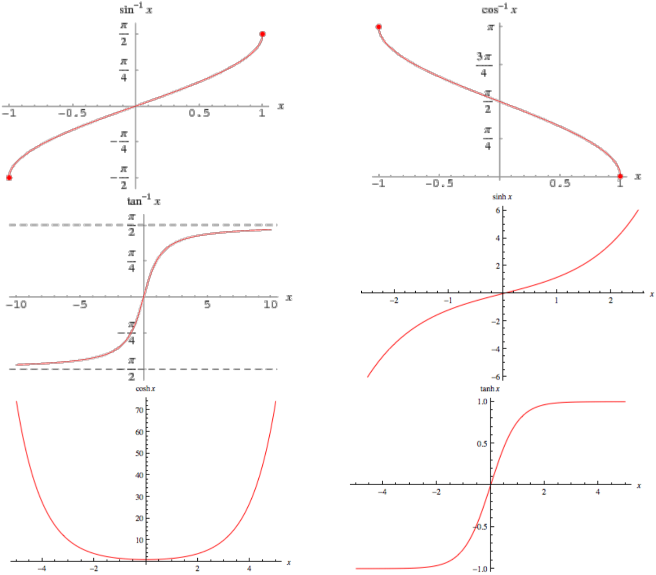
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

- *Altres funcions trigonomètriques:* $\text{cosec}(x) = \dfrac{1}{\sin(x)}$ (cosecant), $\sec(x) = \dfrac{1}{\cos(x)}$ (secant), $\text{cotan}(x) = \dfrac{1}{\tan(x)}$ (cotangent).

Funcions trigonomètriques inverses

$\arcsin(x) = \sin^{-1}(x)$ (arc sinus),
$\arccos(x) = \cos^{-1}(x)$ (arc cosinus),
$\arctan(x) = \tan^{-1}(x)$ (arc tangent).

- $\sin(\arcsin(x)) = x$, $\arcsin(\sin(x)) = x$ (ídem per les altres).
- $D_{\text{arcsin}} = D_{\text{arccos}} = [-1, 1]$, $D_{\text{arctan}} = \mathbb{R}$.
- $\text{rang}_{\text{arccos}} = [0, \pi]$, $\text{rang}_{\text{arcsin}} = \text{rang}_{\text{arctan}} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- $\dfrac{d}{dx}\arcsin(x) = \dfrac{1}{\sqrt{1-x^2}}$, $\dfrac{d}{dx}\arccos(x) = -\dfrac{1}{\sqrt{1-x^2}}$.
- $\dfrac{d}{dx}\arctan(x) = \dfrac{1}{1+x^2}$.
- $\lim_{x\to+\infty} \arctan(x) = \dfrac{\pi}{2}$, $\lim_{x\to-\infty} \arctan(x) = -\dfrac{\pi}{2}$.
- *Altres funcions trigonomètriques inverses:* $\text{arcsec}(x) = \sec^{-1}(x)$, $\text{arccosec}(x) = \cos^{-1}(x)$, $\text{arccotan}(x) = \text{cotan}^{-1}(x)$.



sin⁻¹ x
cos⁻¹ x
tan⁻¹ x
sinh x
cosh x
tanh x
x
π/2
π/4
−π/4
−π/2
π
3π/4
−1
−0.5
0.5
1
−10
−5
5
10
1.0
−1.0

Funcions hiperbòliques

$\sinh(x)$ (sinus hiperbòlic), $\cosh(x)$ (cosinus hiperbòlic), $\tanh(x)$ (tangent hiperbòlica).

- $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$, $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$, $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$.
- $D_{\sinh} = D_{\cosh} = D_{\tanh} = \mathbb{R}$.
- $\text{rang}_{\sinh} = \mathbb{R}$, $\text{rang}_{\cosh} = [0, +\infty]$, $\text{rang}_{\tanh} = [-1, 1]$.
- $\sinh(0) = 0$, $\cosh(0) = 1$, $\cosh^2(x) - \sinh^2(x) = 1$.
- $\sinh(-x) = -\sinh(x)$, $\cosh(-x) = \cosh(x)$.
- $\dfrac{d}{dx}\sinh(x) = \cosh(x)$, $\dfrac{d}{dx}\cosh(x) = \sinh(x)$.
- $\dfrac{d}{dx}\tanh(x) = \dfrac{1}{\cosh^2(x)}$.
- $\lim\limits_{x\to+\infty} \tanh(x) = 1$, $\lim\limits_{x\to-\infty} \tanh(x) = -1$.

Funcions hiperbòliques inverses

$\text{arcsinh}(x) = \sinh^{-1}(x)$ (argument del sinus hiperbòlic),
$\text{arccosh}(x) = \cosh^{-1}(x)$, $\text{arctanh}(x) = \tanh^{-1}(x)$.

- $D_{\sinh^{-1}} = \mathbb{R}$, $D_{\cosh^{-1}} = [0, +\infty]$, $D_{\tanh^{-1}} = [-1, 1]$.
- $\text{rang}_{\sinh^{-1}} = \mathbb{R}$, $\text{rang}_{\cosh^{-1}} = [0, +\infty]$, $\text{rang}_{\tanh^{-1}} = \mathbb{R}$.
- $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$, $\dfrac{d}{dx}\sinh^{-1}(x) = \dfrac{1}{\sqrt{x^2+1}}$.
- $\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$, $\dfrac{d}{dx}\cosh^{-1}(x) = \dfrac{1}{\sqrt{x^2-1}}$.
- $\tanh^{-1}(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$, $\dfrac{d}{dx}\tanh^{-1}(x) = \dfrac{1}{1-x^2}$.