It is known that hypercycles are sensitive to the so-called parasites and short-circuits. While the impact of parasites has been widely investigated for well-mixed and spatial hypercycles, the effect of short-circuits in hypercycles remains poorly understood. In this article we analyze the mean field and spatial dynamics of two small, asymmetric hypercycles with short-circuits. Specifically, we analyze a two-member hypercycle where one of the species contains an autocatalytic loop, as the simplest hypercycle with a short-circuit. Then, we extend this system by adding another species that closes a three-member hypercycle while keeping the autocatalytic short-circuit and the two-member cycle. The mean field model allows us to discard the presence of stable or unstable periodic orbits for both systems. We characterize the bifurcations and transitions involved in the dominance of the short-circuits i.e., in the reduction of the hypercycles’ size. The spatial simulations reveal a random-like and mixed distribution of the replicators in the all-species coexistence, ruling out the presence of large-scale spatial patterns such as spirals or spots typical of larger, oscillating hypercycles. A MonteCarlo sampling of the parameter space for the well-mixed and the spatial models reveals that the probability of finding stable hypercycles with short-circuits drastically diminishes from the two-member to the three-member system, especially at growing degradation rates of the replicators. These findings pose a big constrain in the increase of hypercycles’ size and complexity under the presence of inner cycles, suggesting the importance of a rapid growth of hypercycles able to generate spatial structures (e.g., rotating spirals) prior to the emergence of inner cycles. Our results can also be useful for the future design and implementation of synthetic cooperative systems containing catalytic short-circuits.

Keywords: Bifurcations; Complex systems; Cooperation; Nonlinear dynamics; Origins of Life; Short-circuits; Systems biology.

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I. INTRODUCTION

The precise description and prediction of the steps preceding the origin of life in our planet is one of the most challenging and hard problems in science. Despite this difficulty, several theories have been suggested in the last decades to describe possible scenarios behind the emergence of the first stable replicating systems, in what has been labelled as the crystallization of life in the framework of phase transitions phenomena [1]. The first groundbreaking discovery by Miller and Urey, which proved the Oparin and Haldane hypothesis, revealed that organic molecules (mainly two types of aminoacids) could be synthesized from inorganic ones in the Earth primitive conditions [2]. Miller’s approach opened a new era and since then, geological, physical, and chemical requirements for the beginning of life have been discussed with diligence [3–6]. For instance, the possibility of abiotic synthesis of the building blocks of RNA and DNA [7–9]. Subsequent research on the understanding of the origin of life was deepened substantially by the theory of Eigen and Schuster [10], who put evolution and heredity into the picture using biochemical and more formal mathematical arguments. They argued that at spontaneous error rates during replication of about $10^{-2}$ changes per nucleotide [11], the first faithfully reproducing molecules should be single-stranded RNAs not exceeding 50 to 100 nucleotides [10, 12]. This conjecture arose within the theoretical framework of quasispecies theory [13]. Quasispecies theory was conceived as a chemically-plausible mathematical formulation for the dynamics and evolution of information of macromolecules under large mutation rates (as expected in prebiotic evolution). Roughly, a quasispecies is a set of sequences that is generated by replication-mutation dynamics of an initial template (often named the master sequence). Differently to the concept of species, a quasispecies is a set of closely-related mutants which are (as a whole, and not individually) under the filter of natural selection.

Assuming RNAs as the building blocks of the first living forms, stability against hydrolysis [14] and replicability [15] might favor RNAs that display a loop and stem structure similar to that of modern tRNAs [12], which are about 76 – 90 nucleotides long. Indeed, some smaller, functional ribozymes (i.e., RNA with catalytic activity) of about 50 nucleotides have been described in viroids [16] and other RNAs [17]. Some of these hypothetic prebiotic RNAs were supposed to participate in ribosome-free translation of an appropriate messenger as suggested in [18]. In this sense, ligase reactions by RNA catalysts are known to occur even with small RNA sequences [19]. The RNA-first theory has some powerful arguments in its favour [1]. First of all, all life is based on RNA/DNA as the stable storage of genetic information. Second, RNA can act as informational carrier and enzyme at the same time, thus encoding in the same molecule the two characteristics needed for evolutionary potential: message and reproduction. Third, RNA is a beautiful example of Schrödinger’s aperiodic crystals, which are point-point local templating complements [1]. Also, it is known that certain introns can catalyze their own excision from single-stranded RNA (ssRNA). Furthermore, the same RNA sequences can catalyze transesterification reactions for elongation of one monomer [20], ligation of two independent ssRNAs [21, 22], and cleavage of RNA into smaller sequences [16, 20]. Despite the previous ribozyme reactions, self-replication through RNA-catalysed templated RNA synthesis seems to be quite limited. However, recent in vitro experiments evolving catalysts at sub-zero temperatures revealed that the combination of RNAs with cold-adaptive mutations with a previously described 5’ extension operating at ambient temperatures enabled catalysing the synthesis of an RNA sequence longer than itself (adding up to 206 nucleotides) [23]. Furthermore, a recent work by Vaidya and colleagues revealed that mixtures of RNA fragments that self-assemble into ribozymes spontaneously formed cooperative catalytic cycles, providing experimental evidences for the viability of spontaneous hypercycles (see below for the definition of hypercycles) formation from already synthesized RNAs [24].

The understanding of the properties of the first replicating molecules is crucial to disentangle the origin of life problem. Such replicating entities could carry compositional [26,28], or genetically-encoded [10, 13] information. A simple and thus plausible initial scenario concerning genetically-encoded systems could be given by a set of ribozymes...
replicating at extremely large error rates due to the lack of proof reading mechanisms. Such a system, similar to a quasi-species, could perform a wide and rapid exploration of the sequence space thus being able to find genotypes with catalytic phenotypes capable of forming the so-called hypercycles. Hypercycles are sets of macromolecules that are able to catalyze the reproduction of other macromolecules [10] (see also Ref. [29]). With autocatalysis, a given replicator type catalyzes its own replication, while when a given replicator type catalyzes the replication of a different macromolecule then the relation is called heterocatalytic (or cross-catalytic). Usually, the term hypercycle is used for sets of molecules with a cyclic and closed catalytic architecture, where the catalytic interactions from one member to the other (or to itself) are named catalytic connections or links (using the networks jargon). These hypercycles could contain several catalytically-connected molecules able to increase the information of the entire system while keeping the individual replicators below the critical length imposed by the error threshold [13, 30]. Since no molecule in the hypercycle can outcompete another because they are forced to cooperate, a large genetic message given by all the information of all templates could be attained, thus crossing the information threshold [13].

Following the previous scenario, it seems reasonable to envision the origin of life as a process starting from a small set of RNA molecules with hypercyclic organization. Such small sets could also become the nucleating agents towards more complex and disordered catalytic networks able to unleash the major transition towards more diverse, fully functional, self-replicating systems, as suggested by Kauffman [1, 31] (see also [32, 33]), beyond the cyclic architecture proposed by Eigen and Schuster. Such proto-genetic structures could have then grown in size and evolved towards more complex structures by means of spatial organisation [34, 35] or compartmentalization in protocells [12].

Hypercycles, however, due to the nature of their interactions, were criticized as stable systems involved in the origin of life since they are sensitive to the so-called parasites and short-circuits [10, 37]. Parasites are replicators that receive catalysis from another species but do not reciprocate the catalytic aid [10]. Short-circuits, which can be of different nature, are catalytic connections generating inner and smaller catalytic sub-cycles [13, 25]. Short-circuits have been suggested to pose a serious problem towards the growth of hypercycle systems, since smallest and thus fastest catalytic cycles may outcompete the larger and slower ones, thus constraining hypercycles’ size, complexity, and functionality. Despite this intuitive assertion, few literature has addressed the impact of short-circuits in hypercycles (see below).
Oppositely, multitude of articles have analysed the impact of parasites, showing that they can resist parasites by means of spatial self-structuring [33,36]. The effect of different configurations and sub-cycles in spatial hypercycles was addressed numerically in [36]. This approach revealed that hypercycles are able to coexist with inner cycles in an exclusive manner by means of spiral patterns [36]. According to [36], spatial self-structuring seems to play an important role in the stability of hypercycles with short-circuits, although independent communities are formed and thus there exists a physical separation between the sets of cycles forming the entire system. Actually, the simulations in [36] revealed that asymmetries in diffusion constants could involve the outcompetition of different cycles. Similarly, changes in the kinetic constants also involved processes of outcompetition between cycles, resulting in the loss of some of the hypercycle members (involving a loss of information). Alternatively, it could be possible that well-mixed hypercycles with short-circuits displayed stable coexistence, depending on the kinetic constants. Moreover, the exclusion of short-circuits driven by spatio-temporal phenomena has been described in hypercycles able to self-organize in rotating spiral waves, behavior that usually appears in hypercycles with five or more species (see e.g., [34,36]). However, the effect of short-circuits in smaller spatial hypercycles not able to form large-scale spatial structures have not been investigated. Such questions, crucial in early, small hypercycles remain thus unexplored. In this sense, a full description of the dynamics and transitions (i.e., bifurcations) in hypercycles with short-circuits seems necessary in order to hypothesize how these catalytic systems could circumvent this problem. Also, a comparison between the well-mixed system and the spatial counterpart seems necessary to reveal if space promotes small hypercycles with resistance to the short-circuits, as has been suggested in [36].

In this contribution we will focus on the dynamics of the smallest hypercycles with short-circuits. Here, by short-circuit we mean an inner closed cycle in which a given species establishes a single catalytic connection with another species (or with itself) inside a larger catalytic cycle, see Fig. 1(a). To do so, we will investigate a particular hypothetical scenario in the origin of life formed by small systems of catalytically-connected riboymes. Our scenario considers the presence of small functional RNA molecules able to promote the replication of other molecules that are not able to replicate without the catalytic support. This might correspond, for example, to a scenario where ligase reactions in a given replicator might depend on the catalytic activity of another different template. We are interested in exploring the impact of short-circuits in these primordial systems and their role in the bifurcations (transitions) between phases. We will first investigate a two-species hypercycle that contains an autocatalytic replicator as the minimal catalytic system with a short-circuit. Then, we will explore a three-member hypercycle that contains the previous system. The three-member system, albeit we are not explicitly modelling it, could arise from the growth (by means of replication and mutation) of the two-member one. Finally, we will quantify the likelihood of survival of the different catalytic cycles in the parameter space, comparing the outcomes of the well-mixed and the spatial systems.

II. MEAN FIELD MODEL

We consider two minimal hypercycle systems with short-circuits. First, we will investigate a two-member hypercycle where one of the replicators is autocatalytic. Then, we will extend this system to a three-member hypercycle by adding another species to the two-member architecture, see Fig. 1(a). Notice that our systems only consider auto- and heterocatalytic interactions, and the species are not able to reproduce alone i.e., in a Malthusian way. Hence, our systems are similar to Kauffman’s model of catalytic networks [31]. In our case we study the population dynamics of purely catalytic small networks, assuming an early appearance of short-circuits that could limit a stable persistence of the whole hypercycle. A model can be obtained for the three-member hypercycle (and the two-member one, see below), from:

\[
\begin{align*}
\dot{x}_1 &= x_1 (k_{11}x_1 + k_{12}x_2 + k_{13}x_3) \theta(x) - \varepsilon x_1, \\
\dot{x}_2 &= x_2 (k_{21}x_1) \theta(x) - \varepsilon x_2, \\
\dot{x}_3 &= x_3 (k_{32}x_2) \theta(x) - \varepsilon x_3,
\end{align*}
\]

where \( \theta(x) = 1 - (x_1 + x_2 + x_3) \) is a logistic growth term that introduces competition between the hypercycle elements. For short, we will also write system (13) as

\[
\dot{x} = F(x), \quad x = (x_1, x_2, x_3).
\]

The variables \( x_i \geq 0, i = 1, 2, 3 \), are the population numbers of the hypercycle species \( S_i \) (e.g., small riboymes). The parameters \( k_{ij} > 0 \) (also with \( j = 1, 2, 3 \) and \( j \neq i \)) are the heterocatalytic replication constants, i.e., the rate of replication of species \( i \) due to the catalytic aid from replicator \( j \). Constant \( k_{11} > 0 \) denotes, as we previously mentioned, the constant of autocatalytic growth of species \( S_1 \). Finally, the parameter \( \varepsilon > 0 \) represents the density-independent degradation rates of the species, which is taken to be equal in all species for the sake of simplicity.
We emphasize that \(k_{ij}\) and \(\varepsilon\) will be kept strictly positive. Notice that the dynamical system for the two-member hypercycle with the autocatalytic species can be obtained from system [13] setting \(x_3 = 0\). Defining

\[A = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & 0 & 0 \\ 0 & k_{32} & 0 \end{pmatrix} \quad \text{and} \quad D(x) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix},\]

the system above can be written as \(\dot{x} = F(x) = \theta(x)D(x)Ax - \varepsilon x\). It is clear from the equations that \((0,0,0)\) is a fixed point and that the coordinate planes \(\{x_j = 0\}\), for \(j = 1,2,3\), are invariant. As a consequence, no solutions can cross such planes.

The system has biological meaning for non-negative values of the variables \(x_j\). Moreover, any solution in that domain will enter into

\[\Omega = \{x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 \leq 1\}\]

and remains in \(\Omega\) forever. Indeed, let \(\phi(x) = x_1 + x_2 + x_3\) and compute the scalar product \(F \cdot \grad \phi\) for \(x_j \geq 0, x_1 + x_2 + x_3 = \delta\), with \(\delta > 0\). We have

\[F \cdot \grad \phi = (x_1(k_{11}x_1 + k_{12}x_2 + k_{13}x_3) + x_2k_{21}x_1 + x_3k_{32}x_2)\theta(x) - \varepsilon(x_1 + x_2 + x_3).\]

If \(\delta \geq 1\), \(\theta(x) \leq 0\) and hence \(F \cdot \grad \phi < -\varepsilon\delta\) which means that the solutions must cross transversally all the planes \(\{x_1 + x_2 + x_3 = \delta\}\) with \(\delta \geq 1\), and finally enter into \(\Omega\).

Therefore we consider system [13] on the domain \(\Omega\) since it contains the long-term behaviour and the interesting dynamics.

**A. Equilibrium points**

As we have already mentioned the origin is always an equilibrium point and it is stable since \(\varepsilon > 0\). Thus, we concentrate on the nontrivial equilibrium points of system [13] in the domain \(\Omega\), which we classify in terms of the parameters. We introduce some definitions to simplify the notation in the forthcoming computations:

\[
\begin{align*}
\alpha_2 &:= \frac{k_{21} - k_{11}}{k_{12}}, & \alpha_3 &:= \frac{k_{21}}{k_{32}}, & \beta_3 &:= \frac{k_{12}}{k_{13}} \left(\frac{k_{21} - k_{11}}{k_{12}} - \frac{k_{21}}{k_{32}}\right), \\
\mu_2 &:= 1 + \alpha_2, & \mu_3 &:= 1 + \alpha_3 + \beta_3, & \Delta &:= \left(1 - \frac{k_{11}}{k_{21}}\right)^2 - 4 \left(1 - \frac{k_{12}}{k_{32}}\right), \\
\varepsilon_1 &:= \frac{k_{11}}{4}, & \varepsilon_2 &:= \frac{k_{21}}{4\mu_2}, & \varepsilon_3 &:= \frac{k_{21}}{4\mu_3}.
\end{align*}
\]

Note that it follows straightforwardly from these definitions that \(\beta_3 = \frac{k_{12}}{k_{13}}(\alpha_2 - \alpha_3)\) and that

\[\alpha_2 > 0 \iff k_{11} < k_{21}, \quad \beta_3 > 0 \iff \frac{k_{11}}{k_{21}} + \frac{k_{12}}{k_{32}} < 1.\]  

**Proposition 1** System [13] admits the following equilibrium points in \(\Omega\):

(i) Equilibrium points of type \((x_1,0,0)\) exist if and only if \(\varepsilon \leq \varepsilon_1\), and in that case they are given by

\[
\begin{align*}
P_{\pm} &= (p_{1\pm}^{\pm},0,0) & & \text{if } 0 < \varepsilon < \varepsilon_1, \\
P^0 &= (p_0^0,0,0) & & \text{if } \varepsilon = \varepsilon_1,
\end{align*}
\]

where \(p_{1\pm}^{\pm}\) are the two solutions of \(\xi^2 - \xi + \varepsilon/k_{11} = 0\), that is,

\[
p_{1\pm} = \frac{1}{2} \left(1 \pm \sqrt{1 - \frac{\varepsilon}{\varepsilon_1}}\right), \quad \text{and} \quad p_0^0 = \frac{1}{2}
\]

(7)
(ii) Equilibrium points of the form \((x_1, x_2, 0)\) exist in \(\Omega\) if and only if \(\varepsilon \leq \varepsilon_2\) and \(k_{11} \leq k_{21}\) (equivalently, \(\alpha_2 \geq 0\)). In this case, they are given by

\[
\begin{aligned}
Q^\pm &= (q_1^\pm, q_2^\pm, 0) \\
Q^0 &= (q_1^0, q_2^0, 0)
\end{aligned}
\]

if \(0 < \varepsilon < \varepsilon_2\), \(\varepsilon = \varepsilon_2\),

where \(q_1^\pm\) are the two solutions of \(\mu_2 \xi^2 - \xi + \varepsilon/k_{21} = 0\), that is,

\[
q_1^\pm = \frac{1}{2\mu_2} \left(1 \pm \sqrt{1 - \frac{\varepsilon}{\varepsilon_2}}\right), \quad q_1^0 = \frac{1}{2\mu_2}, \quad \text{and} \quad q_2^{\pm, 0} = \alpha_2 q_1^{\pm, 0}.
\]

In the case that \(k_{11} = k_{21}\) we have \(\alpha_2 = 0\) and \(\mu_2 = 1\). Consequently, \(\varepsilon_2 = \varepsilon_1\) and therefore \(Q^\pm = P^\pm\) and \(Q^0 = P^0\). If \(k_{11} > k_{21}\) (that is, \(\alpha_2 < 0\)) there are no fixed points of this type in \(\Omega\).

(iii) Equilibrium points of the form \((x_1, x_2, x_3)\) exist in \(\Omega\) if and only if \(\varepsilon \leq \varepsilon_3\) and \((k_{11}/k_{21}) + (k_{12}/k_{32}) \leq 1\) (equivalently, \(\beta_3 \geq 0\)). In this case, they are given by

\[
\begin{aligned}
R^\pm &= (r_1^\pm, r_2^\pm, r_3^\pm) \\
R^0 &= (r_1^0, r_2^0, r_3^0)
\end{aligned}
\]

if \(0 < \varepsilon < \varepsilon_3\), \(\varepsilon = \varepsilon_3\),

where \(r_1^\pm\) are the two solutions of \(\mu_3 \xi^2 - \xi + \varepsilon/k_{21} = 0\), that is,

\[
r_1^\pm = \frac{1}{2\mu_3} \left(1 \pm \sqrt{1 - \frac{\varepsilon}{\varepsilon_3}}\right), \quad r_1^0 = \frac{1}{2\mu_3}, \quad \text{and} \quad r_2^{\pm, 0} = \alpha_3 r_1^{\pm, 0}, \quad r_3^{\pm, 0} = \beta_3 r_1^{\pm, 0}.
\]

In the case that \((k_{11}/k_{21}) + (k_{12}/k_{32}) = 1\) (i.e. \(\beta_3 = 0\)) we have \(\mu_3 = \mu_2\), \(\varepsilon_3 = \varepsilon_2\) and therefore \(R^\pm = Q^\pm\) and \(R^0 = Q^0\). On the contrary, if \((k_{11}/k_{21}) + (k_{12}/k_{32}) > 1\) (that is \(\beta_3 < 0\)) there are no fixed points of this type in the domain \(\Omega\).

The proof of the proposition is elementary and has been omitted. The stability of these equilibria is provided by Proposition 2. In Fig. 2 and Fig. 3 we show the relative positions of points \(P^{\pm, 0}, \beta^\pm, 0\), and \(R^{\pm, 0}\). For the sake of clarity, in Fig. 2 we only show the evolution of the first coordinate of \(P^{\pm, 0}\) and \(Q^{\pm, 0}\) in terms of the parameter \(\varepsilon\). It is worth noticing the straightforward relation

\[
4(\varepsilon_2 - \varepsilon_1) = (k_{21} - k_{11})(k_{12} - k_{11})/(k_{21} + k_{12} - k_{11}),
\]

which implies, under the hypothesis of existence of \(Q^{\pm, 0}\) \((k_{21} > k_{11})\), that \(\text{sign}(\varepsilon_2 - \varepsilon_1) = \text{sign}(k_{12} - k_{11})\).

Panels (a-c) in Fig. 2 illustrate the three different possibilities of sign(\(\varepsilon_2 - \varepsilon_1\)).
particular, that autocatalysis needs to be weaker than heterocatalysis. In other words, the hypercycle must dominate over the inner

Notice that in order to have nontrivial equilibria the collision of points which are located on them.

We also want to emphasize that the condition \( k_{11} > k_{21} \) does not lead to a disappearance of equilibria \( Q^\pm,0 \) but to the negativity of the second component, which does not make sense, biologically, in our model. In Fig. we show both the collision of points \( P^\pm,0 \) with the homologous \( Q^\pm,0 \) that occurs when \( k_{11} = k_{21} \) (panel (a)) and the coincidence of the first coordinates of \( P^\pm,0 \) and \( Q^\pm,0 \) together with the vanishing of \( q^\pm,0 \) (panel (b)). In Section we further explore this seeming bifurcation. Similar comments are applicable to the condition \( (k_{11}/k_{21}) + (k_{12}/k_{32}) < 1 \) and the points \( Q^\pm,0 \) and \( R^\pm,0 \). The biological implications of these observations are commented in the next remark.

Another way to visualize the evolution of points \( Q^\pm \), that will be complemented in Section when we study bifurcations, is given in Fig. where we show the number of equilibria on the \( (k_{11}, k_{21}) \)-plane. On this plane, the condition \( \varepsilon = \varepsilon_2 \) beyond which the points \( Q^\pm \) vanish writes as \( k_{21} = 4 \varepsilon (k_{12} - k_{11})/(k_{12} - 4 \varepsilon) \).

**Remark 1** Notice that in order to have nontrivial equilibria \( Q^\pm,0 \) the condition to be satisfied is \( k_{11} < k_{21} \), that is, autocatalysis needs to be weaker than heterocatalysis. In other words, the hypercycle must dominate over the inner short-circuit. On the other hand, the condition to have the equilibria \( R^\pm,0 \), namely \( k_{11}/k_{21} + k_{12}/k_{32} < 1 \), implies, in particular, that \( k_{11}/k_{21} < 1 \) and \( k_{12}/k_{32} < 1 \). In biological terms, the second inequality means that species \( S_2 \) must invest more in the 3-hypercycle than in the former hypercycle of two species to maintain the nontrivial equilibria with population \( S_3 \) active. Taking into account the whole inequality, we appreciate that the existence of these nontrivial points implies also a balance between the two restrictions: if the autocatalytic activity in \( S_1 \) is close to the catalysis activity from \( S_1 \) to \( S_2 \), then the catalysis from \( S_2 \) to \( S_1 \) must be very weak compared to the catalysis from \( S_2 \) to \( S_3 \), and vice versa.

**B. Invariant lines and stability**

Determining the invariant lines of system is very useful in our study: first, because of its dynamical consequences; second, and not less important, because they help in the computation of the eigenvectors and eigenvalues of the equilibrium points which are located on them.

The next result establishes which are these invariant lines.

**Lemma 1** System has the following invariant lines through the origin.

(i) \( L_1 = \{ t v_1 \mid t \in \mathbb{R} \} \), with \( v_1 = (1, 0, 0) \).

(ii) \( L_2 = \{ t v_2 \mid t \in \mathbb{R} \} \), with \( v_2 = (1, \alpha_2, 0) \) and \( \alpha_2 \) as defined in .

(iii) \( L_3 = \{ t v_3 \mid t \in \mathbb{R} \} \), with \( v_3 = (1, \alpha_3, \beta_3) \) and \( \alpha_3, \beta_3 \) as defined in .

(iv) \( L_4 = \{ t v_j \mid t \in \mathbb{R} \}, j = 4, 5 \), with \( v_4 = (0, 1, 0) \) and \( v_5 = (0, 0, 1) \).

(v) \( L_6 = \{ t v_6 \mid t \in \mathbb{R} \} \), with \( v_6 = (1, 0, \beta_6) \) and \( \beta_6 = -k_{11}/k_{13} < 0 \).

Moreover, each equilibrium point is on one of these lines. We also have

**FIG. 3:** Equilibrium points for different relations among parameters. (a) \( \varepsilon_1 < \varepsilon_2 < \varepsilon_3 \). We use the set of parameters \( k_{11} = 0.15, k_{12} = 0.3, k_{21} = 0.5, k_{13} = 0.45, k_{32} = 0.75 \). Observe that \( k_{12} < k_{13} \) and \( k_{11} < k_{12} \). (b) \( \varepsilon_1 < \varepsilon_2 = \varepsilon_3 \). Setting the parameters to \( k_{11} = 0.22, k_{12} = 0.25, k_{21} = 0.5, k_{13} = 0.25, k_{32} = 0.75 \). Notice that \( k_{12} = k_{13} \) and \( k_{11} < k_{12} \). (c) \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 \). Using \( k_{11} = 0.25, k_{12} = 0.25, k_{21} = 0.5, k_{13} = 0.25, k_{32} = 0.75 \). Here \( k_{11} = k_{12} = k_{13} \). (d) \( \varepsilon_1 < \varepsilon_3 < \varepsilon_2 \). Fixing now \( k_{11} = 0.25, k_{12} = 0.3, k_{21} = 0.5, k_{13} = 0.25, k_{32} = 0.75 \). Observe that \( k_{12} > k_{13} \) and \( k_{11} < k_{12} \).
FIG. 4: Case $k_{11} = k_{21}$. (a) Evolution of the first coordinate of the fixed points $P^+$, $Q^+$ (solid lines) and $P^-$, $Q^-$ (dotted lines) when moving $\varepsilon$ from 0 to $\varepsilon_1$ and $\varepsilon_2$, respectively. We use $k_{11} = k_{21} = 0.4$, $k_{12} = 0.35$, which implies $\mu_2 = 1$ and, therefore $\varepsilon_1 = \varepsilon_2$. (b) Plot of the first coordinate of $P_{\pm,0}$, and the first and second coordinates of $Q_{\pm,0}$ in terms of $k_{11}$. Note that $p_{1,1}^{\pm,0} = q_{1,1}^{\pm,0}$ and $q_{2,2}^{\pm,0} = 0 (< 0)$ for $k_{11} = k_{21}$ ($k_{11} > k_{21}$).

- $L_2$ intersects $\Omega \setminus \{ 0 \}$ if and only if $k_{21} \geq k_{11}$ or, equivalently, $\alpha_2 \geq 0$. If $k_{21} = k_{11}$ then $L_2 = L_1$.
- $L_3$ intersects $\Omega \setminus \{ 0 \}$ if and only if $\beta_3 \geq 0$. If $\beta_3 = 0$ then $L_3 = L_2$. If $\alpha_2 = \beta_3 = 0$ then $L_1 = L_2 = L_3$.
- $L_6$ never intersects $\Omega \setminus \{ 0 \}$.

**Proof.** We evaluate the vector field $F$ at the points of the line $t v$ for some unknown vector $v$, and we look for $v$ such that on this line the vector field has the direction of $v$, that is $F(t v) \propto v$. We begin by looking for $v$ of the form $v = (1, \alpha, \beta)$. The proportionality condition reads

$$F(t v) = t \begin{pmatrix} (k_{11} + \alpha k_{12} + \beta k_{13})(1 - (1 + \alpha + \beta)t)(t - \varepsilon) \\ \alpha (k_{21} - (1 + \alpha + \beta)t)(t - \varepsilon) \\ \beta (\alpha k_{32} - (1 + \alpha + \beta)t)(t - \varepsilon) \end{pmatrix} \propto \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}.$$  

(11)

- If $\alpha = 0$ and $\beta = 0$, $v$ has to be a multiple of $v_1 = (1, 0, 0)$.
- If $\alpha \neq 0$ and $\beta = 0$ the condition (11) implies $k_{11} + \alpha k_{12} = k_{21}$, which gives (ii).
- If $\alpha \neq 0$ and $\beta \neq 0$ the condition (11) implies $k_{11} + \alpha k_{12} + \beta k_{13} = k_{21} = \alpha k_{32}$, which has the unique solution for $\alpha$ and $\beta$ given in (iii).
- If $\alpha = 0$ and $\beta \neq 0$ the condition (11) implies $k_{11} + \beta k_{13} = 0$, which gives $\beta = -k_{11}/k_{13} < 0$. This provides an invariant line but it does not cut the interior of $\Omega$.

Finally we look for $v$ of the form $v = (0, \alpha, \beta)$. The proportionality condition now reads as

$$F(t v) = \begin{pmatrix} 0 \\ \alpha (-\varepsilon t) \\ \beta (-\varepsilon t + \alpha k_{32} t^2(1 - (\alpha + \beta)t)) \end{pmatrix} \propto \begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix},$$

which gives the further invariant lines generated by $v_4 = (0, 1, 0)$ and $v_5 = (0, 0, 1)$. The statement concerning the equilibrium points follows from a direct check.

We denote by $DF(x)$ the Jacobian matrix of $F$ at a point $x$.

**Proposition 2** Consider the equilibrium points $P_{\pm,0}$, $Q_{\pm,0}$ and $R_{\pm,0}$ described in Proposition 1. Then:
FIG. 5: Representation, on the \((k_{11}, k_{21})\) plane, of the existence of points \(Q^\pm\) in Proposition\(^{ii}\). The other parameters are kept constant: \(k_{12} = 0.5, k_{13} = 0.5, k_{32} = 0.95\). (a) Number of fixed points for \(\varepsilon = 0.05\). On the red line (with negative slope), a saddle-node bifurcation occurs; more precisely, this is the line given by \(k_{21} = 4 \varepsilon (k_{12} - k_{11})/(k_{12} - 4 \varepsilon)\). The blue line (positive slope) is not a bifurcation curve indeed, but the line on which some component of \(Q^\pm\) vanishes, thus loosing its biological meaning. (b) Position of the saddle-node bifurcation line for different values of \(\varepsilon\). The slope is not a bifurcation curve indeed, but the line on which some component of \(Q^\pm\) vanishes; more precisely, this is the line given by \(k_{21} = 4 \varepsilon (k_{12} - k_{11})/(k_{12} - 4 \varepsilon)\). The blue line (positive slope) is not a bifurcation curve indeed, but the line on which some component of \(Q^\pm\) vanishes, thus loosing its biological meaning. (c) Stability diagram for the fixed points \(x^*_1\) (up), \(x^*_1\) (centre) and \(x = 0\). Green colour denotes attractor and red repeller.

(i) The eigenvalues \(\lambda_j^{\pm,0}\) of \(DF\) at the points \(P_j^{\pm,0}\) are given by

\[
\lambda_1^\pm = 2k_{11}p_1^\pm \left( \frac{1}{2} - p_1^\pm \right), \quad \lambda_2^\pm = \frac{k_{21} - k_{11}}{k_{11}} \varepsilon, \quad \lambda_3^\pm = -\varepsilon < 0.
\]

Moreover, \(\lambda_1^+ < 0\) and \(\lambda_1^- > 0\). In particular, at the point \(P^0\), the eigenvalues become \(\lambda_1^0 = 0, \lambda_2^0 = (k_{21} - k_{11})/4\) and \(\lambda_3^0 = -\varepsilon < 0\).

(ii) The equilibrium points \(Q_j^{\pm,0}\) lie on the invariant line \(L_2\) and \(v_2 = (1, \alpha_2, 0)\) is an eigenvector of \(DF\). The eigenvalues \(\lambda_j^{\pm,0}\) of \(DF\) at the points \(Q_j^{\pm,0}\) are:

\[
\lambda_1^{\pm,0} = \frac{k_{13}}{k_{12}} \frac{\beta_3}{\alpha_3} \varepsilon, \quad \lambda_2^{\pm,0} = \varepsilon - k_{21} \mu_2 \left( q_1^{\pm,0} \right)^2, \quad \lambda_3^{\pm,0} = -\alpha_2 \varepsilon < 0,
\]

where \(q_1^{\pm,0}\) is the first component of \(Q_j^{\pm,0}\), respectively. In particular, \(\text{sign}(\lambda_1^{\pm,0}) = \text{sign}(\beta_3); \lambda_2^{+} < 0, \lambda_2^{-} > 0\) and \(\lambda_2^{0} = 0; \lambda_3^{\pm,0} < 0\).

In the case \(k_{11} = k_{21}\) the points \(Q_j^{\pm,0}\) coincide with the points \(P_j^{\pm,0}\).

(iii) The equilibrium points \(R_j^{\pm,0}\) lie on the invariant line \(L_3\) and \(v_3 = (1, \alpha_3, \beta_3)\) is an eigenvector of \(DF\). The eigenvalues \(\lambda_j^{\pm,0}\) of \(DF\) at the points \(R_j^{\pm,0}\) are given by:

\[
\lambda_1^{\pm,0} = \varepsilon - \mu_3 k_{21} \left( r_1^{\pm,0} \right)^2, \quad \lambda_2^{\pm,0} = -\frac{1}{2} \left( 1 - \frac{k_{11}}{k_{21}} \right) + \frac{\varepsilon}{2} \sqrt{\Delta}, \quad \lambda_3^{\pm,0} = -\frac{1}{2} \left( 1 - \frac{k_{11}}{k_{21}} \right) - \frac{\varepsilon}{2} \sqrt{\Delta},
\]

where \(\Delta = \left( \varepsilon - 2k_{21} \right)^2 - 4 \mu_3^2 k_{21}^{\pm,0}\).
Moreover, $\lambda_1^+ < 0$, $\lambda_1^- > 0$, $\lambda_2^+ = 0$, $\lambda_2^\pm < 0$ if $\Delta \geq 0$, and $\text{Re} \lambda_2^\pm < 0$ if $\Delta < 0$.

In the case $(k_{11}/k_{12}) + (k_{12}/k_{32}) = 1$ the points $R^{\pm,0}$ coincide with $Q^{\pm,0}$.

The proof of this proposition has been deferred to the Appendix VI.

C. Bifurcations

In this section we use the information provided by Proposition 1 and Proposition 2 to identify the different bifurcations and scenarios of stability in the space of parameters. We focus in the 2-dimensional space, the minimal dimension in which the most interesting bifurcation types appear. As stated above, the dynamics of the two-member hypercycle with the short-circuit is described by system (1)-(2) with $x_3 = 0$. In terms of parameters $k_{11}$, $k_{12}$, $k_{21}$ and $\varepsilon$, we have the cases given in Table I.

We note that bifurcations of equilibria in $\Omega$ are due to two reasons: either a saddle-node bifurcation when $\varepsilon$ overcomes the thresholds $\varepsilon_1$ and $\varepsilon_2$, or a transcritical bifurcation that makes at least one equilibrium to leave $\Omega$ and switch its stability character. According to Table I, these events occur whenever we have an equality (we mean, in $k_{21} - k_{11}$ or in $\varepsilon$). The number of equalities that meet simultaneously correspond to the so-called codimension of the bifurcation.

In Figs. 6 and 7, we illustrate all these bifurcations on the plane $(\varepsilon, k_{21})$ (degradation rate and catalysis from population 1 to population 2, respectively). The bifurcation diagram is shown on the four quadrants although only the first one makes sense in our model (since $\varepsilon_2, k_{21} > 0$). For the sake of clarity, we have divided the figures in three panels: in Fig. 6, we display the cases (a) $k_{11} < k_{12}$, (b) $k_{11} = k_{12}$ and, in Fig. 7, we display the case $k_{11} > k_{12}$. We remind that the sign of $k_{11} - k_{12}$ is equivalent to the sign of $\varepsilon_2 - \varepsilon_1$ as seen in (10). On these panels, we plot the bifurcation curves identified in Table I. Both $\varepsilon = \varepsilon_1$ and $k_{21} = k_{11}$ are straight lines on this plane and correspond, respectively, to the loci where the saddle-node bifurcation of points $P^\pm$ and the transcritical bifurcation occur. The curve where the points $Q^\pm$ collide, $\varepsilon = \varepsilon_2$, becomes a rational function in terms of $k_{21}$ (see the blue curves in the figures):

$$\varepsilon_2(k_{21}) := \frac{k_{21}k_{12}}{4(k_{21} + k_{12} - k_{11})}.$$

Observe that $\varepsilon_2(k_{11}) = k_{11}/4$ and that $\varepsilon_2(k_{21})$ is asymptotic to the lines $\varepsilon = k_{13}/4$ and $k_{21} = k_{11} - k_{12}$.

On the different regions determined by all these bifurcation curves, the existing $P^{\pm,0}, Q^{\pm,0}$ points have been written. Aside each equilibrium point $P^{\pm,0}$ or $Q^{\pm,0}$, triplets with symbols denote the signs of the corresponding eigenvalues, which are always real; for instance, $(-0+)$ would indicate $\lambda_1 > 0$, $\lambda_2 = 0$ and $\lambda_3 < 0$. The eigenvalues of $Q^{\pm,0}$ depend as well on $\beta_3$, which involves other parameters; the stability notation, thus, also considers this circumstance.

In Figs. 6 and 7, the cases described in Table I have been overlayed as well. This allows to match these classes with the above described bifurcations. For instance, transitions from case I to case II (see Fig. 6(a), Fig. 6(b) and Fig. 7), from III to IV (see Fig. 6(a)) or from V to VI (see Fig. 7), induce a saddle-node bifurcation involving the equilibria $P^\pm$ that merge into $P^{\pm}$ when $\varepsilon = \varepsilon_1$ and then disappear. Analogous bifurcations occur for equilibria $Q^{\pm,0}$ when $\varepsilon = \varepsilon_2$ from IV to VI (see Fig. 6(a)) or from III to V (see Fig. 7).

On the other hand, equilibria $Q^{\pm}$ undergo transcritical bifurcations when $k_{21} = k_{11}$, that is, from III to I (see Fig. 6(a), Fig. 6(b) and Fig. 7). In these cases, the second component of $Q^\pm$, i.e. $Q^2\varepsilon$, becomes negative (see Fig. 6(b)) thus leaving $\Omega$ and, at the same time, one eigenvalue changes sign. Depending on the sign of $\beta_3$, the transcritical "exchange" is made with points $P^{\pm}$ or $R^{\pm}$.

<table>
<thead>
<tr>
<th>Case</th>
<th>$k_{21} - k_{11}$</th>
<th>$\varepsilon$</th>
<th>$P^-$</th>
<th>$P^+$</th>
<th>$Q^-$</th>
<th>$Q^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$&lt; 0$</td>
<td>$\varepsilon &lt; \varepsilon_1$</td>
<td>Saddle</td>
<td>Attractor</td>
<td>$\beta_1$</td>
<td>$\beta_1$</td>
</tr>
<tr>
<td>II</td>
<td>$&lt; 0$</td>
<td>$\varepsilon &gt; \varepsilon_1$</td>
<td>$\beta_1$</td>
<td>$\beta_1$</td>
<td>$\beta_1$</td>
<td>$\beta_1$</td>
</tr>
<tr>
<td>III</td>
<td>$&gt; 0$</td>
<td>$\varepsilon &lt; \min{\varepsilon_1, \varepsilon_2}$</td>
<td>Saddle</td>
<td>Saddle</td>
<td>Saddle</td>
<td>Attractor</td>
</tr>
<tr>
<td>IV</td>
<td>$&gt; 0$</td>
<td>$\varepsilon_1 &lt; \varepsilon &lt; \varepsilon_2$</td>
<td>$\beta_1$</td>
<td>$\beta_1$</td>
<td>$\beta_1$</td>
<td>$\beta_1$</td>
</tr>
<tr>
<td>V</td>
<td>$&gt; 0$</td>
<td>$\varepsilon_2 &lt; \varepsilon &lt; \varepsilon_1$</td>
<td>Saddle</td>
<td>Saddle</td>
<td>$\beta_1$</td>
<td>$\beta_1$</td>
</tr>
<tr>
<td>VI</td>
<td>$&gt; 0$</td>
<td>$\varepsilon &gt; \max{\varepsilon_1, \varepsilon_2}$</td>
<td>$\beta_1$</td>
<td>$\beta_1$</td>
<td>$\beta_1$</td>
<td>$\beta_1$</td>
</tr>
</tbody>
</table>

Table I: Existence and stability of equilibria in $\Omega$ in the planar case. Each case is displayed in the parameter spaces of Figs. 6 and 7 with different colors: Case I (light green); II (blue); III (pink); IV (white, Fig. 6(a)); V (dark green); and VI (white).
FIG. 6: $PQ$-bifurcation 2-dimensional diagrams. The diagrams show (i) the regions of existence and non-existence of the equilibrium points $P_{\pm}$ and $Q_{\pm}$ together with (ii) the signs of the three (real) eigenvalues of the associated Jacobian matrix. The bifurcation curves appearing correspond to a pass through 0 of one or more of these eigenvalues. Observe that the appearance of equilibria $R_{\pm}$, related to $\beta_3 > 0$, causes a change in the stability of the points $Q_{\pm}$. The sign ($\beta_3$) has been denoted by $\square$ in the diagram. The cases, represented with different colors, correspond to those listed in Table I. From top to bottom, (a) $k_{11} < k_{12}$ and (b) $k_{11} = k_{12}$.

All other bifurcations are, at least, of codimension two. For instance, when $\varepsilon = \varepsilon_1 = \varepsilon_2$ or the two conditions $\varepsilon = \varepsilon_1$ and $k_{11} = k_{21}$ hold simultaneously, as those observed from III to VI (see Fig. 6(b)) and from IV to V (moving from Fig. 6(a) to Fig. 7).

The analysis of bifurcations in the phase space $(x_1, x_2, x_3)$ becomes more cumbersome just because the increase in the number of parameters and conditions to be taken into account, but it does not show other codimension-1 bifurcations qualitatively different from those described in the phase space $(x_1, x_2)$. More precisely, in the 3D case we have to take into account the new parameters $k_{13}$ and $k_{23}$ and the conditions $k_{11}/k_{21} + k_{12}/k_{32} = 1$ (related to
FIG. 7: PQ-bifurcation 2-dimensional diagrams. Same situation as presented in Fig. 6 but in the case \( k_{11} > k_{12} \). Remind that the symbol \( \square \) denotes \( \text{sign}(\beta_3) \) and that the cases represented with different colors correspond to those listed in Table I.

the transcritical bifurcation between points \( R^\pm \) and points \( Q^\pm \) and \( \varepsilon = \varepsilon_3 \) (saddle-node bifurcation of points \( R^\pm \)). We think that these comments give already a fair idea of the bifurcations in the three dimensional case, so to avoid unnecessary intricacy in the manuscript we do not present the exhaustive study of bifurcations in this case. Figure 15 in the Appendix Section includes a biparametric diagram displaying the different dynamics for the three-member hypercycle in terms of parameters \( k_{11}/k_{21} \) and \( k_{12}/k_{32} \).

D. Non-existence of periodic orbits

In this Section we will explore both analytically and numerically whether periodic orbits (stable or unstable) can be found in the parameter space of the models.

1. Two-member system

The two-member system can be seen as the three-member system restricted to \( \{x_3 = 0\} \), which is invariant. We denote by \( F_{12} \) the vector field \( F \) restricted to \( \{x_3 = 0\} \). We take as domain of \( F_{12} \) the set

\[
\Omega_2 = \Omega \cap \{x_3 = 0\} = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}.
\]

Note that, in the same way as in the three-member system, the solutions in \( \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0\} \) enter into \( \Omega_2 \) and then remain there because \( \Omega_2 \) is positively invariant. We have the following properties

**Proposition 3**

(i) The vector field \( F_{12} \) has no periodic orbits in \( \Omega_2 \).

(ii) The origin is a global attractor for \( F_{12} \) in \( \Omega_2 \) if and only if either \( k_{21} \leq k_{11} \) and \( \varepsilon > \varepsilon_1 \) or \( k_{21} > k_{11} \) and \( \varepsilon > \max\{\varepsilon_1, \varepsilon_2\} \).
Proof. Since $F_{12}$ is a planar vector field we can easily prove (i) using the Poincaré-Bendixson theorem. We already know that the domain $\Omega_2$ is positively invariant. Let $(x_1, x_2) \in \Omega_2$ and consider its $\omega$-limit. It must be an equilibrium point, a periodic orbit or a graph formed by equilibrium points and homo/heteroclinic orbits connecting them. If there is a periodic orbit, by a Poincaré-Hopf theorem there should be an equilibrium point inside it, but this possibility can not happen in our case since all equilibrium points are contained in invariant lines, and then the periodic orbit should intersect one of such invariant lines. This is impossible by uniqueness of solutions.

From Proposition 1 we know that the conditions in (ii) imply that $F_{12}$ has no equilibrium points in $\Omega_2$ except the origin. Then the $\omega$-limit set of every solution must be the origin.

Conversely, if the conditions on the parameters do not hold there are other equilibrium points in $\Omega_2$ and hence the origin cannot be a global attractor. □

2. Three-member system

Concerning the 3D system we provide a quite complete analytical description of the dynamics when $\beta_3 \leq 0$. For $\beta_3 > 0$ we have to proceed numerically (numerical integrations along the work have been performed with a Runge-Kutta-Fehlberg (RKF78) method, with automatic step size control, using a step size $10^{-4} \leq \Delta t \leq 10^{-1}$ and the local relative tolerance $10^{-14}$).

Proposition 4 Assume $\beta_3 \leq 0$. Then:

(i) The vector field $F$ has no periodic orbits in $\Omega$.

(ii) If either $k_{21} \leq k_{11}$ and $\varepsilon > \varepsilon_1$ or $k_{21} > k_{11}$ and $\varepsilon > \max \{\varepsilon_1, \varepsilon_2\}$, the origin is a global attractor for $F$.

Before proving Proposition 4 we state an auxiliary result. If $\alpha_2 > 0$ we introduce

$$
\Omega^+_{\alpha_2} = \{x \in \Omega \mid x_2 \geq \alpha_2 x_1\}, \quad \Omega^-_{\alpha_2} = \{x \in \Omega \mid x_2 < \alpha_2 x_1\}.
$$

If $\alpha_2 \leq 0$, for any $\eta > 0$ we introduce

$$
\Omega^+_{\eta} = \{x \in \Omega \mid x_2 \geq \eta x_1\}, \quad \Omega^-_{\eta} = \{x \in \Omega \mid x_2 < \eta x_1\}.
$$

Clearly $\Omega = \Omega^+_\alpha \cup \Omega^-_\alpha$ and $\Omega = \Omega^+_\eta \cup \Omega^-_\eta$, respectively.

Lemma 2 (i) If $\alpha_2 > 0$ then, given any initial condition in $\Omega^+_{\alpha_2}$, the solution enters into $\Omega^-_{\alpha_2}$ or converges to a fixed point in $\{x \in \Omega \mid x_2 = \alpha_2 x_1, \ x_3 = 0\}$.

(ii) If $\alpha_2 \leq 0$ then, for any $\eta > 0$, given any initial condition in $\Omega^+_{\eta}$, the solution enters into $\Omega^-_{\eta}$ or converges to a fixed point in $\{x \in \Omega \mid x_2 = 0, \ x_3 = 0\}$ if $\alpha_2 = 0$ or converges to the origin if $\alpha_2 < 0$.

Proof. We recall that the coordinate axes are invariant and that $\Omega$ is positively invariant. We introduce the functions $\phi_\gamma(x) = -\gamma x_1 + x_2$ for $\gamma \geq \alpha_2$. The set $\{\phi_\gamma = 0\}$ represents a plane for which $\text{grad} \phi_\gamma$ is an orthogonal vector. We compute the scalar product $F \cdot \text{grad} \phi_\gamma$ at points of $\{\phi_\gamma = 0\} \cap (\Omega \setminus \partial \Omega)$ (in particular, points with strictly positive coordinates and such that $\theta(x) > 0$). We have

$$
F \cdot \text{grad} \phi_\gamma = -\gamma x_1 + \dot{x}_2 = x_1 \theta(x) (-\gamma (k_{11} x_1 + k_{12} x_2 + k_{13} x_3) + k_{21} x_2)
\geq \gamma x_1 \theta(x) ((k_{21} - k_{11} - \gamma k_{12}) x_1 - k_{13} x_3) < 0.
$$

Notice that we have used that $\varepsilon (x_1 - x_2) = 0$ on $\{\phi_\gamma = 0\}$. The above inequality indicates that the solutions starting in $\Omega^+_{\alpha_2} \setminus \partial \Omega^+_{\alpha_2}$ cross all planes $\{\phi_\gamma = 0\}$, with $\gamma > \alpha_2 = (k_{21} - k_{11})/k_{12}$, transversally. In particular, $\Omega^+_{\alpha_2}$ is positively invariant. Then any solution either arrives at $\{\phi_{\alpha_2} = 0\} \cap \partial (\{\phi_{\alpha_2} = 0\} \cap \Omega)$ in which case it crosses this set transversally and enters into $\Omega^-_{\alpha_2}$ or tends to $\partial (\{\phi_{\gamma} = 0\} \cap \Omega)$ for some $\gamma \geq \alpha_2$. In the latter case the solution must follow the dynamics on this boundary (by continuity), which is one-dimensional and hence the solution converges to an equilibrium point lying on $\{x \in \Omega \mid x_2 = \alpha_2 x_1, \ x_3 = 0\}$.

If $\alpha_2 \leq 0$ the previous computations also give that the solutions either arrive at $\{\phi_{\eta} = 0\} \cap \Omega$ and cross it transversally or go to $\partial (\{\phi_{\eta} = 0\} \cap \Omega)$. □

Proof of Proposition 4

(i) From Lemma 2 we know that the long term dynamics is in either $\Omega^-_{\alpha_2}$ if $\alpha_2 > 0$ or in $\Omega^-_\eta$, for all $\eta > 0$, if $\alpha_2 \leq 0$. Now we introduce $\psi_\gamma(x) = -\gamma x_2 + x_3$, for $\gamma > 0$, and we compute the scalar product $F \cdot \nabla \psi_\gamma$ on $\{\psi_\gamma = 0\} \cap \Omega^-_{\alpha_2}$ or $\{\psi_\gamma = 0\} \cap \Omega^-_\eta$. We have

$$F \cdot \nabla \psi_\gamma = -\gamma (x_2 k_{21} x_1 \theta(x) - \varepsilon x_2) + x_3 k_{32} x_2 \theta(x) - \varepsilon x_3 = x_2 \theta(x) (-\gamma k_{21} x_1 + x_3 k_{32}) = \gamma k_{32} x_2 \theta(x) (-\alpha_3 x_1 + x_2),$$

where we have used that $\varepsilon (\gamma x_2 - x_3) = 0$ on $\{\psi_\gamma = 0\}$. Note that $\beta_3 \leq 0$ is equivalent to

$$\alpha_2 = \frac{k_{21} - k_{11}}{k_{12}} \leq \frac{k_{21}}{k_{32}} = \alpha_3.$$

Then, under this assumption,

$$F \cdot \nabla \psi_\gamma \leq \gamma k_{32} x_2 \theta(x) (-\alpha_2 x_1 + x_2) < 0,$$

and hence a solution starting in $\Omega^-_{\alpha_2}$ must cross all planes $\{\psi_\gamma = 0\} \cap \Omega^-_{\alpha_2}$, $\gamma > 0$, and tend to $\{x_3 = 0\}$. Since in this plane there are no periodic orbits, as it is proved in Proposition 3, we conclude that there are no periodic orbits.

(ii) According to Lemma 3 the conditions on the parameters imply that $F$ has a unique equilibrium point at the origin, which is asymptotically stable. By Proposition 3 the set $\{x \in \Omega \mid x_3 = 0\}$ is contained in the basin of attraction of the origin. Since the basin is an open set it also contains a set $\{x \in \Omega \mid 0 \leq x_3 < \delta\}$ for some $\delta > 0$. In (i) we have proved that all solutions tend to $\{x_3 = 0\}$. Then they first enter into $\{x \in \Omega \mid 0 \leq x_3 < \delta\}$ and then converge to the origin.

When $\beta_3 > 0$ we have to deal with the non-existence of periodic orbits numerically. To facilitate the study we first provide some features of the dynamics. We introduce the planes through the origin generated by one of the coordinate axes and the vector $v_3 = (1, \alpha_3, \beta_3)$ which generates the line $L_3$, see Lemma 4. More precisely, let $\Pi_i$ be the plane $\Pi_i = \{t e_i + s v_3 \mid t, s \in \mathbb{R}\}$, where $\{e_i\}_i$ are the vectors of the canonical basis. These planes divide $\Omega$ into three subdomains $\Omega = \Omega_{ij} \cup \Omega_{ik} \cup \Omega_{jl}$. Taking into account the equations of the planes we can write

$$\Omega_{12} = \{(x_1, x_2, x_3) \in \Omega \mid \alpha_3 x_3 \leq \beta_3 x_2, x_3 \leq \alpha_3 x_1\},$$
$$\Omega_{23} = \{(x_1, x_2, x_3) \in \Omega \mid x_3 \geq \beta_3 x_1, x_2 \geq \alpha_3 x_1\},$$
$$\Omega_{31} = \{(x_1, x_2, x_3) \in \Omega \mid \alpha_3 x_3 \geq \beta_3 x_2, \alpha_3 x_1 \geq x_2\}.$$

We denote by $\Sigma_i \subset \Pi_i$ the common boundary of $\Omega_{ij}$ and $\Omega_{ik}$ with $j, k \neq i, j \neq k$. We can check that at each such boundaries $\Sigma_i$ the vector field is transversal and crosses from $\Omega_{ki}$ to $\Omega_{ij}$. 

FIG. 8: An example of the set of 5000 initial conditions on $\Sigma$ whose flow has been integrated numerically to check the non-existence of periodic orbits for $\beta_3 > 0$. The parameter values randomly selected for this particular simulation were $k_{11} = 0.227510, k_{12} = 0.140495, k_{13} = 0.378144, k_{21} = 0.403362, k_{32} = 0.432082$, and $\varepsilon = 0.041691$. 

\[\begin{align*}
\Omega_{12} &= \{(x_1, x_2, x_3) \in \Omega \mid \alpha_3 x_3 \leq \beta_3 x_2, x_3 \leq \alpha_3 x_1\}, \\
\Omega_{23} &= \{(x_1, x_2, x_3) \in \Omega \mid x_3 \geq \beta_3 x_1, x_2 \geq \alpha_3 x_1\}, \\
\Omega_{31} &= \{(x_1, x_2, x_3) \in \Omega \mid \alpha_3 x_3 \geq \beta_3 x_2, \alpha_3 x_1 \geq x_2\}.
\end{align*}\]
An equation representing $\Pi_1$ is $\phi_1(x_1, x_2, x_3) = -\beta_3 x_2 + \alpha_3 x_3 = 0$. The vector $\text{grad} \phi_1 = (0, -\beta_3, \alpha_3)$ points from $\Omega_{12}$ to $\Omega_{31}$. We compute the scalar product $F \cdot \text{grad} \phi_1$ and use the conditions that the points belong to $\Pi_1$ and $\Omega_{31}$, that is, $x_3 = (\beta_3/\alpha_3)x_2$ and $x_2 \leq \alpha_3 x_2$ respectively. Then, we have

$$F \cdot \text{grad} \phi_1 = -\beta_3 (x_2 k_{21} x_1 \theta(x) - \varepsilon x_2) + \alpha_3 (x_3 k_{32} x_2 \theta(x) - \varepsilon x_3)$$

$$= x_2 \theta(x) [-\beta_3 k_{21} x_1 + \alpha_3 k_{32} x_3]$$

$$= x_2 \theta(x) [-\beta_3 k_{21} x_1 + \beta_3 k_{32} x_2]$$

$$= \beta_3 x_2 \theta(x) k_{32} [-\alpha_3 x_1 + x_2] \leq 0.$$ (12)

The inequality is strict except at the boundary of $\Sigma_1$. Completely analogous computations check the result for $\Sigma_2$ and $\Sigma_3$. For $\Sigma_2$ we represent $\phi_2$ as $\phi_2(x_1, x_2, x_3) = \beta_3 x_1 - x_3 = 0$ and we compute $F \cdot \text{grad} \phi_2$ using $\beta_3 x_1 = x_3$ and $\alpha_3 x_1 < x_2$. For $\Sigma_3$ we represent $\phi_3(x_1, x_2, x_3) = -\alpha_3 x_1 + x_2 = 0$ and we compute $F \cdot \text{grad} \phi_3$ using $\alpha_3 x_1 = x_2$ and $x_3 > \beta_3 x_1$.

This suggests that some solutions may turn around the axis generated by $v_3$ visiting the domains $\Omega_{12}, \Omega_{23}$ and $\Omega_{31}$ in the indicated order and eventually tend to the origin or to the stable fixed point on $L_3$ (when it exists).

The calculation in (12) shows that $\Sigma_1$ is a good Poincaré section for computational purposes. Accordingly, we have used it to compute the Poincaré map, $P$, numerically and check the non-existence of periodic orbits for a huge set of parameter values and initial conditions. We have chosen random initial conditions on $\Sigma_1$ and integrated the system (13) in forward time. We have checked that all the trajectories we have considered (50000 initial conditions for each of the 100000 parameter sets (see one instance in Fig. 5), never lead to a periodic orbit. Parameter values were randomly chosen from the set $\{k_{11}, k_{12}, k_{13}, k_{21}, k_{22}, c\} \in [10^{-3}, 0.5] \times [10^{-3}, 0.5] \times [10^{-1}, 0.5] \times [10^{-1}, 0.5] \times [10^{-1}, 0.5] \times [10^{-2}, 0.2]$). To gain control and reduce the computing time, we additionally implemented several exit conditions to discard initial conditions: either after 2500 time units, or when they reached the basin of attraction of the origin, or the basin of attraction of the line $L_3$ (see Lemma 1 (iii)) or escaped from the simplex $\Omega$, we stopped the computation and took another initial condition. We checked as well that the Poincaré map cannot have periodic orbits of periods 2, 3 or 6, that is, fixed points of $P^2$, $P^3$ or $P^6$.

### III. STOCHASTIC SPATIAL DYNAMICS

It has been suggested that mineral surfaces would have played a key role during the emergence of life because the assembly of complex biomolecules in a three-dimensional environment is implausible [25]. In this sense, honeycombed feldspar mineral surfaces could have provided a suitable organized environment for the synthesis of complex molecules, also protecting them from dispersion and hydrolysis. Following these ideas, previous researches have used computational models to investigate the spatial dynamics of hypercycles [34, 38–40]. In this section we explore the behaviour of the hypercycles with short-circuits previously characterised, now considering space in an explicit way. To do so we build stochastic cellular automata (CA) models. To simulate the hypercycle interactions on a surface, we define a state space given by a $L \times L$ lattice, $\Gamma \in \mathbb{Z}^2$, with periodic boundary conditions. The automaton has $L^2$ cells and $\nu+1$ states, $S_n$, $n = 1, \ldots, \nu$, being the $n$-th hypercycle member (see Fig. 4(c)), and the other state, $S_0$, corresponding to empty sites. As in the previous sections, we will focus on hypercycles of size $\nu = 2$ and $\nu = 3$, keeping the catalytic interactions previously analyzed. The CA works as follows. At each generation $\tau$, we asynchronously choose $L^2$ random cells (this updating procedure ensures that, on average, each cell is updated once per generation). Every time we choose a random cell, say $\Gamma(x, y)$ (where $(x, y)$ is a spatial coordinate of $\Gamma$), we also choose two different random neighbours of $\Gamma(x, y)$, named $\Gamma(k, l)$ and $\Gamma(m, p)$, considering a Moore neighborhood (i.e. 8 nearest cells). If $\Gamma(x, y)$ is empty nothing will happen. If $\Gamma(x, y)$ contains a replicator, then we apply the following state-transition rules:

1. **Autocatalytic replication**: If both cells $\Gamma(x, y)$ and $\Gamma(k, l)$ are occupied by $S_1$, $S_1$ will replicate with probability $r_{11} \in [0, 1]$ towards $\Gamma(m, p)$ (if empty). If $\Gamma(k, l)$ is empty or $\Gamma(m, p)$ is occupied, nothing will happen. This process is represented by the next reaction:

\[ S_1(x, y) + S_1(k, l) + S_0(m, p) + s \xrightarrow{r_{11}} S_1(x, y) + S_1(k, l) + S_1(m, p), \]

hereafter $s$ are some available building blocks (i.e., nucleotides) needed for replication, which are not explicitly considered.

2. **Heterocatalytic replication**: If the cell $\Gamma(x, y)$ is occupied by a hypercycle species $S_j$, and the cell $\Gamma(k, l)$ is occupied by $S_j$ (where $j = i - 1$ and using the convention that if $i = 1$ then $j = \nu$, thus introducing the cyclic architecture) $S_i$ will replicate with probability $r_{ij} \in [0, 1]$ to $\Gamma(m, p)$ (provided it is empty). The parameters
\(r_{ij}\) are the probabilistic versions of the constants \(k_{ij}\) presented in the mean field equations. This process is represented by the following reactions:

\[
S_i(x, y) + S_j(k, l) + S_0(m, p) + s \xrightarrow{r_{ij}} S_i(x, y) + S_j(k, l) + S_1(m, p).
\]

The previous reaction for \(n = 2\) considers the two-member hypercycle structure. However, for the 3-member hypercycle it does not consider the short-circuit between species \(S_1\) and \(S_2\). Hence, for this larger hypercycle we need to introduce the reaction:

\[
S_1(x, y) + S_2(k, l) + S_0(m, p) + s \xrightarrow{r_{12}} S_1(x, y) + S_2(k, l) + S_1(m, p).
\]

After replication rule 1 or 2 is applied to \(\Gamma(x, y)\), the replicator in this cell will decay following rule 3:

3. **Degradation**: The replicator \(S_n(x, y)\) will decay with probability \(\varepsilon \in [0, 1]\), according to reaction:

\[
S_n(x, y) \xrightarrow{\varepsilon} S_0(x, y) + s,
\]

with \(n = 1, \ldots, \nu\). Here we will also consider equal degradation rates for all the species, as we did for the mean-field model.

After replication and decay rules are applied, we will apply the rule of diffusion, explained below.

4. **Diffusion**: We choose a random position, say \(\Gamma(q, u)\). If \(\Gamma(q, u)\) contains a replicator, it will diffuse towards a randomly chosen neighbor, \(\Gamma(w, z)\), with probability \(D \in [0, 1]\), or remain in the same place with probability \((1 - D)\). Following Ref. [38], if the replicator diffuses, it will interchange the position with the replicator living in the neighboring cell. If the neighbor is empty, it will just move towards the neighbor with probability \(D\). The reaction is

\[
S_c(q, u) + S_d(w, z) \xrightarrow{D} S_d(q, u) + S_c(w, z), \quad \text{with } c, d = 0, \ldots, \nu.
\]

Hereafter, we will denote

\[
N_i = \frac{1}{L^2} \sum_{r=1}^{L} \sum_{p=1}^{L} S_i(r, p), \quad \text{with } i = 1, \ldots, \nu,
\]

as the normalized population of replicators \(S_i\) in \(\Gamma\), \(N_0\) being the normalized number of empty sites. In all of our simulations we will fix \(L = 200\), if not otherwise specified. The initial conditions for the two-member hypercycle will be \(N_{1,2}(\tau = 0) = 1/3\), while for the three-member hypercycle will be \(N_{1,2,3}(\tau = 0) = 1/4\). All other remaining sites will be empty (i.e., \(N_0(\tau = 0) = 1/3\) for the two-member and \(N_0(\tau = 0) = 1/4\) three-member hypercycles). Other initial configurations will be used to compute the survival probabilities in the parameter space (see Section 4). Next, we present the results of this computational model for the two-member hypercycle containing an autocatalytic species and for the three-member hypercycle containing the previous system, see Fig. [1a]. This will allow us to compare the mean field dynamics with its spatial counterpart, characterizing the critical probability values calculated from the mean field model, and thus unveiling the changes in the survival and extinction patterns introduced by space.

**A. Spatio-temporal dynamics: the impact of diffusion**

Here we investigate the spatio-temporal dynamics for the hypercycle with an inner autocatalytic short-circuit (Fig. [1a]). Firstly, we compute the equilibrium population values of species \(i = 1, 2\) tuning the CA probabilities. We run several replicates and plot the mean equilibrium population values (± standard deviation), \(N_i(±SD)\), for several values of the catalysis and degradation probabilities. In Fig. [2a] we display the population equilibria tuning the probability of autocatalytic growth of \(S_1\) averaging, for each probability value, 10 independent replicas after discarding \(\tau = 5000\) generations. The populations \(S_1\) and \(S_2\) are shown with black and red circles, respectively. Notice that at increasing \(r_{11}\), the equilibrium populations of \(S_1\) and \(S_2\) increase and decrease, respectively, since the first species undergoes a faster growth. This tendency at increasing \(r_{11}\) holds up to a critical value of \(r_{11}\) at which the second species of the hypercycle becomes extinct while the population of the first species continues increasing, although more slowly. In these analyses we also plot the equilibrium populations of both replicators obtained numerically from the mean field model previously analyzed using as kinetic constants the same values of the probabilities of the spatial simulations (we also plot the unstable branches of equilibria; we use superscripts \(s\), for stable, and \(u\), for unstable, to
FIG. 9: Spatial and temporal dynamics of the two-member hypercycle with a short-circuit. Populations $S_1$ and $S_2$ in all panels are represented, respectively, in black and red. Mean population values ($N \pm SD$, represented with circles and computed averaging over 10 independent replicas at $\tau = 5000$ time generations) are shown together with the equilibria obtained from the mean-field model (solid lines), both the stable (superscript $s$) and unstable ones (superscript $u$): (a) tuning $r_{11}$ with $r_{12,21} = 0.5$; (b) tuning $r_{21}$ setting $r_{11} = r_{12} = 0.5$; (c) tuning $r_{12}$ with $r_{11} = 0.25$ and $r_{21} = 0.5$. In (d) we change $\varepsilon$ setting $D = 0.25$, $r_{11} = 0.25$, and $r_{12,21} = 0.5$. In (a)-(c) we display the results using $D = 0.25$ and $\varepsilon = 0.05$, using the probabilities $r_{ij}$ as the parameters governing the mean-field model, that is, taking $k_{ij} = r_{ij}$. The insets display an enlarged view of the stable equilibria for $S_2$ near the transition points. (a.1) Time series using $r_{11} = 0.3$, $r_{12,21} = 0.5$ and $\varepsilon = 0.05$. Here we display, overlapped, 3 runs of the spatial dynamics and the dynamics obtained numerically from the mean field model (solid lines, with $k_{11} = 0.3$, $k_{12,21} = 0.5$, and $\varepsilon = 0.05$). (c.1) Time series with $r_{12} = 0.2$, $r_{11} = 0.25$, $r_{21} = 0.5$, and $\varepsilon = 0.05$. Panels (a.2) and (c.2) display, respectively, the spatial distribution of replicators at the end of one of the replicates in (a.1) and (c.1), respectively. In all of the simulations we used as initial conditions $N_1(\tau = 0) = N_2(\tau = 0) = 1/3$.

distinguish both branches). Notice that the results at increasing $r_{11}$ match the results obtained from the mean-field model (displayed with a solid black (for $S_1$) and red ($S_2$) line). The critical value of $r_{11}$ causing the extinction of $S_2$ populations is slightly lower for the spatial system (see the inset in Fig. 9(a)). For this particular case, $S_2$ disappears at a lower value of $r_{11}$ compared to the well-mixed system. We note that in panels (a-d) of Fig. 9 we used a diffusion probability of $D = 0.25$. As we will comment on below, diffusion typically affects the critical parameter values causing the transitions in the spatial model (which are homologous to the bifurcations described in the previous sections, and which share the same nature). For instance, the transition of Fig. 9(a) is a continuous and smooth one, corresponding to the transcritical bifurcation.

A smooth transition corresponding also to a transcritical bifurcation is found at decreasing $r_{21}$. In this case, there exists a critical value of the catalytic aid from species 1 to species 2 that involves the outcompetition of the second hypercycle element by the autocatalytic one. Here the spatial simulations also reveal a slight difference between the bifurcation value predicted by the mean field model and the one obtained with the CA. Further analyses using $r_{12}$ and $\varepsilon$ as control parameters reveal abrupt transitions that correspond to the saddle-node bifurcations predicted by the mean field model. For these two cases, differences between the bifurcation values obtained from the mean field model and the CA simulations are also obtained (the effect of diffusion on the critical parameter values will be analyzed below).

As mentioned above, the dynamics of the CA model matches the dynamics predicted by the mean field model. For example, in Fig. 9(a.1) we display the time dynamics of the two replicators using the same parameter values
than in Fig. 9(a) fixing $r_{11} = 0.3$. For this particular parameters combination we plot three runs of the CA model, overlapping the mean field dynamics (solid line). Also, the time dynamics near the transition point in Fig. 9(c) is displayed in Fig. 10(c.1). Here it can be seen that the dynamics predicted by the mean field model matches again the spatial dynamics. This result is general for our spatial model. Finally, in Fig. 9(a.2) and Fig. 9(c.2) we show the spatial patterns for these runs reveal a well-mixed distribution of the two species. Other simulations in the coexistence scenarios revealed these type of spatial patterns (for the two- and three-member system analyzed below, results not shown).

The dynamics of the three-member hypercycle containing the two-member short-circuit and the autocatalytic species are displayed in Figs. 10 and 11. Here, similarly to the two-member system, the equilibrium populations are very close to the values predicted by the mean-field model given by Eqs. (1-3). This can be seen in panels (a-f) in Fig. 10. The nature of the transitions involved in the extinctions of the different hypercycle members in the spatial model coincide with the bifurcations of the three-member system. For example, the transitions displayed in Fig. 10 are smooth (i.e., transcritical bifurcations) for parameters $r_{11}$ and $r_{21}$ (as we previously described), as well as for $r_{12}$ and $r_{32}$. Here, the increase of $r_{11}$ beyond the critical value involves the outcompetition of the two- and three-member hypercycle by the autocatalytic species. However, the increase of $r_{21}$ displays two effects, a first transition involving the survival of the two-member hypercycle and a second one (for a large enough value of $r_{21}$) that allows the persistence of the three-member system. The parameter $r_{32}$ (i.e., the catalytic aid from species $S_2$ to $S_3$) causes, above a critical value, the survival of the three-member system. Below this critical value, the aid of $S_2$ to $S_3$ is not large enough to ensure the survival of the whole system. Notice that this parameter has no effect on the population equilibrium of $S_1$. As we...
previously did, in all the panels we overlap the equilibrium values for each species predicted by the mean field model (solid lines). As we discussed for the two-member system, there exists small variations in the critical parameter values (compare with Fig. 11(a)). For each of the simulations we display the patterns at different times.

In Fig. 11(a) we display an example of coexistence of the three hypercycle members. For this particular parameter combination, species $S_3$ achieves low population numbers. The spatial patterns of the replicators at $\tau = 0$ (initial random conditions) and $\tau = 5000$ are displayed. Here also, well-mixed (i.e., random-like) spatial patterns are observed at the end of the simulation. Figure 11(b) displays the spatio-temporal dynamics near the critical value of $\varepsilon$ causing the extinction of all the species. Here, while the dynamics for the mean field model under these parameters combination does not involve the extinction of the whole hypercycle ($\varepsilon$ is below its critical value causing the saddle-node bifurcation), the spatial dynamics involves the extinction. Interestingly, the time trajectory, which achieves extinction at $\tau \approx 19000$ generations, undergoes a long and flat plateau before collapsing. This dynamical effect, which has been described for two-member hypercycles (for both mean field and spatial stochastic dynamics), is due to the transient behavior near a saddle-node bifurcation, which is extremely long causing the so-called delayed transition. The asymptotic state for this scenario is an absorbing state where the full hypercycle becomes extinct, as reported in [41]. The simulation displayed in Fig. 11(b) falls in the extinction regime, where the flat plateau preceding extinction can be clearly seen.

Up to now, we have discussed the impact and the transitions tied to the autocatalytic and heterocatalytic probabilities among replicators, having evidences that diffusion slightly changes the values of these parameters causing the bifurcations when compared to the values predicted by the mean field models. In order to analyze how diffusion affects these transition values, we computed the critical probability values causing the transitions in the populations at increasing values of the diffusion probability $D$. For the two-member system we display the results in Fig. 12. For each value of $D$ we ran 10 simulations (replicas) during $\tau = 2 \times 10^4$ generations for several values of the investigated probabilities (e.g., $r_{11}$ in Fig. 12(b)). From all these replicas we computed the mean probability value causing a change of behavior. For example, separating the survival from the extinction of $S_2$. Then we plotted the mean critical values against probability $D$. Figure 12(a) displays the results for the mean critical probability of autocatalytic growth, $r_{11}(\pm SD)$, causing the transition towards $S_2$ extinction. As expected, for maximum values of $D$, the critical value approaches to the one predicted by the mean field model (this result is observed in all of our analyses, see Figs. 12 and 13). Low values of $D$ promote (under the analyzed parameters combination) the extinction of $S_2$, while increasing $D$
The critical values of the mean field model. 
Fig. 13. We notice that in all of the panels displayed in Figs. 12 and 13, we also plot (using a dashed line) the 
outcompete the hypercycles. The critical values of the other parameters as a function of diffusion are displayed in
the transition takes place, favouring the persistence of all replicators since a higher value of $r$ means critical values of
values, different probability values have been sampled to find their critical values. For instance, Fig. 13(a) displays the
increase of $r$ for $S$ value causing $r$ the population dynamics for $S$ 
changes in $S$ for $r$ replicator to the second one promotes the survival of the whole hypercycle. The changes in the critical probabilities
that involve the survival of the whole three-member system, of the two-member hypercycle, $11$ in (a), $12$ in (b), $12$ in (c),
and $S$ in (d)) averaged over 10 independent runs. The dashed
lines indicate the transition values predicted by the mean field model, using the probabilities as parameters. (e) Dynamics for
different diffusion probabilities driving to extinction or survival scenarios: (upper) we set $D = 0.05$ ($S_2$ extinction) and $D = 0.8$ (hypercycle survival), here we only plot the population of $S_2$ (red). (lower) Simulations using $r_{11} = 0.25, r_{12} = 0.2, r_{21} = 0.5$, with $D = 0.3$ (hypercycle extinction) and $D = 0.9$ (hypercycle survival), here we display
the population dynamics for $S_1$ (black) and $S_2$ (red). In each panel in (e) we display 3 different runs for each value of diffusion.
values make the value of $r_{11}$ larger. This indicates that mixing favours the survival of the two-member hypercycle. The effect of $D$ in the critical value of $r_{21}$ is similar to the previous case. The increase of the catalytic aid of the first replicator to the second one promotes the survival of the whole hypercycle. The changes in the critical probabilities for $r_{12}$ and $\varepsilon$ are displayed in Fig. 12(e) (c) and (d), respectively. In Fig. 12(e) we display several trajectories for which changes in $D$ involve a qualitative change in the dynamics i.e., survival or extinction depending on diffusion.

The same analyses have been carried out for the three-member system. The results can be found in Fig. 13. Here we must notice that several bifurcations can be found tuning a single parameter. For instance, as shown in Fig. 10(a), the increase of $r_{11}$ involves a first bifurcation that causes the extinction of $S_2$ and then there exists a second critical value causing $S_2$ extinction. Both transitions are governed by transcritical bifurcations. To account for these different values, different probability values have been sampled to find their critical values. For instance, Fig. 13(a) displays the mean critical values of $r_{11}$ that involve the survival of the whole three-member system, of the two-member hypercycle, and of the autocatalytic replicator. Diffusion, for both transitions, is shown to increase the values of $r_{11}$ at which the transition takes place, favouring the persistence of all replicators since a higher value of $r_{11}$ is needed for $S_1$ to outcompete the hypercycles. The critical values of the other parameters as a function of diffusion are displayed in Fig. 13. We notice that in all of the panels displayed in Figs. 12 and 13, we also plot (using a dashed line) the bifurcation values predicted by the mean field model.

IV. HYPERCYCLE PERSISTENCE IN PARAMETER SPACE

In our previous analyses we have focused on the dynamics and the transitions obtained from both mean field and stochastic spatial models for the hypercycles with the short-circuits. However, it is also interesting to characterize how probable is to find all of the characterized scenarios in the parameter spaces of both systems. For instance, how probable is the coexistence of all the hypercycle species or how probable is the persistence of only the first
autoacatalytic replicator, thus obtaining a simpler system with a lower potential information. In this Section we will estimate the likelihood of survival of each of the replicators in the parameter space of the two model approaches: mean field and spatial stochastic models. To do so, we built a MonteCarlo (MC) algorithm, which works as follows for the two modelling approaches. At each MC iteration, $M$, we randomly select the model parameters (probabilities in the CA) from a uniform distribution. For the two-member system we select $\{k_{11}, k_{21}, k_{12}\} \in \mathcal{U}(0,1)$ for the mean field model (and $\{r_{11}, r_{21}, r_{12}, D\} \in \mathcal{U}(0,1)$ for the CA). For the three-member hypercycle, we proceed in a similar way, selecting randomly the parameters $\{k_{11}, k_{21}, k_{12}, k_{32}, k_{31}\} \in \mathcal{U}(0,1)$ for the mean field model (and $\{r_{11}, r_{21}, r_{12}, r_{23}, r_{31}, D\} \in \mathcal{U}(0,1)$ for the CA). For each selected combination of random parameters we integrate numerically the mean field model (for which we study to final values of time $t = 2.5 \times 10^4$ and $t = 5 \times 10^4$) or run simulations for the CA (with $\tau = 10^4$). At these times the population of each hypercycle member is evaluated.

For the two-member hypercycle with the short-circuit two possible scenarios (predicted by the mean field model) are possible: only-survival of $S_1$ given by scenario (A), and scenario (B) where the hypercycle replicators $S_1$ and $S_2$ coexist. Similarly, scenarios (A) and (B) are also found for the three-member hypercycle. For this case, another scenario is possible: scenario (C) where the three replicator species survive. For the mean field model, the survival of species $i$ is considered when $x_i > 10^{-5}$ (with $i = 1, 2, 3$) and thus extinction is assumed with $x_i \leq 10^{-5}$. For the CA model, since replicators only replicate catalytically, we assume survival when $S_i \geq 2$ and extinction if $S_i \leq 1$. As a first approach (case (i) in Table II) we use the same initial population values: $x_{1,2}(0) = 1/3$ (two-member hypercycle) and $x_{1,2,3}(0) = 1/4$ (three-member system) in the mean field model; and $N_{1,2}(\tau = 0) = 1/3$ and $N_{1,2,3}(\tau = 0) = 1/4$ for the CA model. However, different initial configurations will be also analysed (see below and Table II).

First, we display those regions in the probabilities space of the CA where each scenario is found by means of two-dimensional projections (Fig. 14). For the two-member hypercycle we display the spaces $(r_{11}, r_{21})$ and $(r_{21}, D)$. The different scenarios in the probabilities space $(r_{11}, r_{21})$ are clearly separated, with scenario (A) dominating at increasing values of $r_{11}$. From the MC simulations we can estimate the survival probability $\Pi_S$ of the replicators for each of the scenarios above. Such a probability is computed as: $\Pi_S = N_S/M$, where $N_S$ is the number of probabilities (parameters) combinations fulfilling the survival condition of each scenario, and $M$ is the number of $MC$ iterations (we use $M = 2 \times 10^5$). We notice that larger values of $M$ gave qualitatively similar results (results not shown).

The results for the two-member hypercycle with the autocatalytic replicator are found in Fig. 14(b). Notice that...
FIG. 14: Monte Carlo (MC) sampling in the parameter spaces setting $\varepsilon = 0.05$, $\tau = 10^4$, and $L = 100$. (a) Projections on $(r_{11}, r_{21})$ and $(r_{12}, D)$, for the spatially-extended two-membered hypercycle. Black dots indicate those regions in each parameter space with only survival of $S_1$ (scenario (A)) while blue dots indicate persistence of the two species (scenario (B)). (b) Computation of the survival probability $\Pi_S$ for the scenarios (A) and (B) from the MC sampling for $\varepsilon = 0.025$ (white bars); $\varepsilon = 0.05$ (red bars); $\varepsilon = 0.075$ (gray bars); and $\varepsilon = 0.1$ (green bars). The thick lines inside the bars of the histogram (also represented in (d)) are the values of $\Pi_S$ obtained from the MC sampling of the mean field model. (c) Same analysis for the three-membered hypercycle with the short-circuits. As before, we plot those pairs of values giving place to scenario (A) (black dots); and scenario (B) (blue dots). Here another scenario given by the survival of the three-member hypercycle is possible (scenario (C), red dots). The projections are displayed in the probability spaces (a) $(r_{11}, r_{21})$, (b) $(r_{12}, r_{32})$, and (c) $(r_{13}, D)$. (d) Survival probability $\Pi_S$ for each of the three scenarios (A), (B), and (C) computed as in (b). The MC algorithm was ran over $M = 2 \times 10^5$ iterations for both mean field and CA simulations. In all the analyses we used $N_{1,2}(0) = 1/3$ for $n = 2$; $N_{1,2,3}(0) = 1/4$ for $n = 3$ as initial populations for the CA. For the mean field we set $x_{1,2}(0) = 1/3$ for $n = 2$; $x_{1,2,3}(0) = 1/4$ for $n = 3$ as initial conditions (case (i) in Table II).
TABLE II: Survival probabilities ($\Pi_S$) computed with the Monte Carlo (MC) algorithm (from $M = 2 \times 10^5$ iterations). We use four different values of degradation: $\varepsilon = 0.025, 0.05, 0.075, 0.1$, and three different initial conditions (cases (i)-(iii)). For the mean field ($x_i(0)$) and the CA model ($N_i(0)$): case (i) $s_i(0) = 1/3$ for $n = 2$; $s_i(0) = 1/4$ for $n = 3$. Case (ii): $s_1(0) = 0.6$ and $s_2(0) = 0.1$ for $n = 2$; $s_1(0) = 0.6$ and $s_3(0) = 0.1$ for $n = 3$. Case (iii): $s_1(0) = 0.1$ and $s_2(0) = 0.6$ for $n = 2$; $s_1(2,0) = 0.1$, and $s_3(0) = 0.6$ for $n = 3$, with $s_i(0) \in \{x_i(0), N_i(0)\}$. In the mean field we consider survival of replicator $i$ if $x_i > 10^{-5}$ and extinction if $x_i \leq 10^{-5}$ at $t = 25000$ (the values inside the parenthesis display the same simulations using $t = 5 \times 10^4$). For the CA we use $L = 100$, considering $S_i > 1$ (survival) and $S_i \leq 1$ (extinction) at $\tau = 10^4$.

The nature of the first self-replicating systems on Earth and how they might have grown in complexity remains an open question. However, compelling evidence indicates that ribozymes might be good candidates for primordial replicative systems. Ribozymes are catalytic RNA molecules that can be both functional and message carriers. In this sense, ribozymes could have been the building blocks of the so-called hypercycles. Pointing in this direction, recent experimental findings revealed that variants of bacteria ribozymes could assemble to form cooperative cycles, being able to outcompete smaller autocatalytic cycles. Hypercycles have been widely investigated since Eigen and Schuster proposed them. Hypercycles would integrate and ensure the coexistence of several, distinct cooperative cycles, being able to outcompete smaller autocatalytic cycles. Hypercycles have been widely investigated since Eigen and Schuster proposed them. Hypercycles would integrate and ensure the coexistence of several, distinct cooperative cycles, being able to outcompete smaller autocatalytic cycles. The results are also displayed in Table II. Similar results to case (i) have been obtained. However, $\Pi_S$ for case (iii) diminish for the mean field model, especially for $\varepsilon = 0.075$ and $\varepsilon = 0.1$, where the survival probabilities become smaller when compared to the other two cases.

V. CONCLUSIONS

The nature of the first self-replicating systems on Earth and how they might have grown in complexity remains an open question. However, compelling evidence indicates that ribozymes might be good candidates for primordial replicative systems. Ribozymes are catalytic RNA molecules that can be both functional and message carriers. In this sense, ribozymes could have been the building blocks of the so-called hypercycles. Hypercycles have been widely investigated since Eigen and Schuster proposed them. Hypercycles would integrate and ensure the coexistence of several, distinct cooperative cycles, being able to outcompete smaller autocatalytic cycles. The results are also displayed in Table II. Similar results to case (i) have been obtained. However, $\Pi_S$ for case (iii) diminish for the mean field model, especially for $\varepsilon = 0.075$ and $\varepsilon = 0.1$, where the survival probabilities become smaller when compared to the other two cases.
nature of the bifurcations involving the reduction of the hypercycles and the dominance of the inner short-circuits. Moreover, the spatial simulations reveal that the dynamics is very similar to the one predicted by the mean field models, especially at maximum diffusion probabilities. The mean field model also allowed us to discard the presence of periodic orbits for both two- and three-member systems. Both mean field and spatial models reveal scenarios where all the hypercycle species coexist in a stable manner, being the hypercycle able to coexist with the members establishing shorter, inner cycles. Although this coexistence scenario drastically reduces at increasing degradation rates.

It is known that oscillating hypercycles (hypercycles with \( n > 4 \) members) typically form self-organized spatial patterns. For instance, spiral waves [34, 36] or clusters [35]. Such spatial patterns have been suggested to be crucial for the survival of the hypercycle under the presence of parasites. Moreover, the presence of spiral waves also ensured the exclusive coexistence between hypercycles and inner catalytic cycles [36]. As mentioned, our results indicate that no oscillations are found in the mean field models, suggesting that no oscillatory behavior may be found in the spatial simulations. Indeed, the spatial patterns obtained in our simulations are given by well-mixed (random-like) patterns. Our results suggest that without self-structuring, small hypercycles are very sensitive to the short-circuits, especially at increasing degradation rates. Following our results and [36], it seems that a minimum number of hypercycle species would be required prior to the emergence of the short-circuits to ensure the formation of large-scale spatial patterns allowing a stable coexistence with the short-circuits in a stable manner. As mentioned, however, under appropriate parametric conditions (especially at low degradation rates), the hypercycles are able to persist with the short-circuits. These results have been found in both the mean field and spatial models.

Finally, as stated by Kauffman [1]: “The essential feature of autocatalysis is independent of its precise biochemical definition. Therefore, study on autocatalysis would also be applicable to several areas including ecosystems, immune system, and neural and social networks”. Hence, our results can be useful for other catalytic systems beyond the origins-of-life molecular hypercycles, in which inner cycles may arise. For instance, our results can help to build synthetic cooperative systems containing short-circuits. In this sense, recent articles have reported the experimental building of synthetic cooperative populations in bacteria [44] and yeast [45] populations.

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VI. APPENDIX

A. Proof of Proposition 2

To ease the reading the proof has been divided into three cases.

1. Case I: Fixed points of the form \((x_1, 0, 0)\)

The differential \(DF(x_1, 0, 0)\) is given by the following triangular matrix

\[
\begin{pmatrix}
  k_{11} (2x_1(1-x_1) - x_1^2) - \varepsilon & k_{12} x_1(1-x_1) - k_{11} x_1^2 & k_{13} x_1(1-x_1) - k_{11} x_1^2 \\
  0 & k_{21} x_1(1-x_1) - \varepsilon & 0 \\
  0 & 0 & -\varepsilon
\end{pmatrix},
\]

whose eigenvalues are

\[
\lambda_1 = k_{11} (2x_1(1-x_1) - x_1^2) - \varepsilon, \quad \lambda_2 = k_{21} x_1(1-x_1) - \varepsilon, \quad \lambda_3 = -\varepsilon.
\]

- Regarding points \(P^\pm\), for simplicity, let us denote, \(p_1\) to indicate the first component \(p_1^\pm\) of \(P^\pm\). From Proposition 1 \(p_1\) satisfies \(p_1^2 - p_1 + \varepsilon/k_{11} = 0\). So, the first eigenvalue satisfies

\[
\lambda_1 = k_{11} \left( (-p_1^2 + p_1 - \frac{\varepsilon}{k_{11}}) + (-2p_1^2 + p_1) \right) = 2k_{11}p_1 \left( \frac{1}{2} - p_1 \right).
\]

Using that \(p_1^+ > 1/2\) and \(p_1^- < 1/2\) it is clear that \(\lambda_1^+ < 0\) and \(\lambda_1^- > 0\). On the other hand, since \((1 - p_1^2)p_1^\pm = \varepsilon/k_{11}\) we have

\[
\lambda_2^\pm = k_{21}p_1(1 - p_1) - \varepsilon = k_{21} \frac{\varepsilon}{k_{11}} - \varepsilon = \frac{k_{21} - k_{11}}{k_{11}} \varepsilon,
\]

whose sign depends on the difference \(k_{21} - k_{11}\). The eigenvalues \(\lambda_3^\pm = -\varepsilon\) are always negative.

- At the point \(P^0 = (1/2, 0, 0)\) we have \(\varepsilon = \varepsilon_1 = k_{11}/4\) and \(x_1 = 1/2\). Thus, the previous formulae read

\[
\lambda_1^0 = 0, \quad \lambda_2^0 = \frac{k_{21} - k_{11}}{4}, \quad \lambda_3^0 = -\varepsilon_1 < 0.
\]

2. Case II: Fixed points of the form \((x_1, x_2, 0)\)

Remind that \(Q^\pm,0 = (q_1^\pm,0, \alpha_2 q_1^\pm,0, 0)\) and let us introduce \(\theta^\pm,0 = (1 - q_1^\pm,0 - \alpha_2 q_1^\pm,0) = (1 - \mu_2 q_1^\pm,0)\), and \(\Sigma^{\pm,0} = k_{11} q_1^{\pm,0} + k_{12} \alpha_2 q_1^{\pm,0}\). The condition of \(Q^\pm,0\) being equilibrium points gives \(\Sigma^{\pm,0} \theta^{\pm,0} = \varepsilon\), \(k_{21} q_1^{\pm,0} \theta^{\pm,0} = \varepsilon\). To simplify the notation we will not write the superscripts \((\pm, 0)\). Using the previous relations and notation we have

\[
DF(Q) = \begin{pmatrix}
  k_{11}q_1\theta - k_{21}q_1^2 & k_{12} q_1\theta - k_{21} q_1^2 & k_{13}q_1\theta - k_{21} q_1^2 \\
  k_{21} \alpha_2 q_1(\theta - q_1) & -k_{21} \alpha_2 q_1^2 \theta - k_{21} q_1^2 \theta & k_{23} \alpha_2 q_1\theta - \varepsilon \\
  0 & 0 & k_{23} \alpha_2 q_1\theta - \varepsilon
\end{pmatrix}.
\]

From the block structure of \(DF(Q)\) one gets that

\[
\lambda_1 = k_{21} \alpha_2 q_1 \theta - \varepsilon = \left( \frac{k_{32}}{k_{21}}, \frac{k_{21} - k_{11}}{k_{12}} - 1 \right) \varepsilon = k_{13} \frac{k_{32}}{k_{12}} \varepsilon
\]

is an eigenvalue. We know that \(Q^{\pm,0}\) are located on the line \(L_2\). Since it is an invariant line, \(v_2 = (1, \alpha_2, 0)\) is an eigenvector of \(DF(Q^{\pm,0})\). Computing \(DF(Q^{\pm,0})v_2\) will provide the corresponding eigenvalue. Indeed, \(DF(Q)v_1 = (\varepsilon - k_{21} \mu_2 q_1^2)v_1\), so it follows that

\[
\lambda_2 = \varepsilon - k_{21} \mu_2 q_1^2
\]
is another eigenvalue. Finally, from the trace of $DF(Q)$ we get the third eigenvalue,

$$
\lambda_3 = \text{tr} \ DF(Q) - \lambda_1 - \lambda_2 = k_{11}q\theta - k_{21}q(1 - \mu_2 q) = -\alpha_2 \varepsilon.
$$

Observe that the sign of the eigenvalues $\lambda_1$ and $\lambda_3$ does not depend on $\varepsilon$ but only on the values of the parameters $k_{ij}$. Precisely, we have $\lambda^\pm_1 = \text{sign}(\beta_1)$ and $\lambda^\pm_3 < 0$ since $k_{11} < k_{21}$. However, the sign of $\lambda_2$ is different for $\lambda^+_2$, $\lambda^-_2$ and $\lambda^0_2$. Let us show this by expressing $\lambda_2$ in a more suitable form. Thus,

$$
\lambda^\pm_2 = \varepsilon - k_{21}\mu_2 \left( q^\pm_1 \right)^2 = \varepsilon - \frac{k_{21}}{4\mu_2} \left( 1 \pm \sqrt{1 - \frac{\varepsilon}{\varepsilon_2}} \right)^2
$$

$$
= \varepsilon - \varepsilon_2 \left( 2 - \frac{\varepsilon}{\varepsilon_2} \pm 2 \sqrt{1 - \frac{\varepsilon}{\varepsilon_2}} \right) = 2 \left( -\varepsilon_2 - \varepsilon \mp \sqrt{\varepsilon_2 \varepsilon_2 - \varepsilon} \right)
$$

$$
= -2 \sqrt{\varepsilon_2} \varepsilon \left( \sqrt{\varepsilon_2} - \varepsilon \pm \sqrt{\varepsilon_2} \right),
$$

so $\lambda^+_2 < 0$, $\lambda^-_2 > 0$, and $\lambda^0_2 = 0$. If $\varepsilon = \varepsilon_2$, the other two eigenvalues are given by $\lambda^0_1 = \frac{k_{13}}{k_{12}} \frac{\beta_3}{\alpha_3} \varepsilon_2$, $\lambda^0_3 = -\alpha_2 \varepsilon_2$.

In the particular case $k_{11} = k_{21}$ the points $Q^{\pm,0}$ coincide with the points $P^{\pm,0}$.
3. Case III: Fixed points of the form \((x_1, x_2, x_3)\)

Assuming now that \(x_1x_2x_3 \neq 0\) leads, in \([13]\), to \(x_2 = k_{21}/k_{32}x_1 = \alpha_3x_1\). Thus, the differential matrix becomes

\[
DF(R^{\pm,0}) = \begin{pmatrix}
\frac{k_{11}}{k_{21}} \varepsilon - k_{21} r_1^{2} & \frac{k_{12}}{k_{21}} \varepsilon - k_{21} r_1^2 & \frac{k_{13}}{k_{21}} \varepsilon - k_{21} r_1^2 \\
\alpha_3(\varepsilon - k_{21} r_1^2) & -\frac{k_{22}}{k_{32}} r_2^2 & -\frac{k_{23}}{k_{32}} r_2^2 \\
-k_{21} \beta_3 r_3^2 & \beta_3 \left(\frac{k_{22}}{k_{32}} \varepsilon - k_{21} r_1^2\right) & -k_{21} \beta_3 r_1^2
\end{pmatrix},
\]

where \(r_i\) denotes \(r_i^{\pm,0}\), \(i = 1, 2, 3\), each of the components of the points \(R^{\pm,0}\) and it has been used that \(\varepsilon/\theta = k_{21} r_1 = k_{32} r_2\) and \(r_2 = \alpha_3 r_1\), \(r_3 = \beta_3 r_1\). The computation of its eigenvalues can be carried out using that the invariant line to where they belong provides the eigenvector \((1, \alpha_3, \beta_3)\) and, consequently, its associated eigenvalue. The other two eigenvalues are obtained using the determinant and the trace of the matrix. Like in the previous cases, we denote by \((\lambda_1^+, \lambda_2^+, \lambda_3^+)\) and \((\lambda_1^-, \lambda_2^-, \lambda_3^-)\) the associated eigenvalues to the equilibrium points \(R^+, R^0\), respectively.

We start dealing with \(R^+\). A first eigenvalue \(\lambda_1^+\) is given by the expression

\[
\lambda_1^+ = \varepsilon + k_{21} \left(\frac{k_{12}}{k_{32}} - \frac{k_{13}}{k_{32}} + k_{21} (k_{11} k_{32} - k_{12} k_{31}) / k_{32}\right) r_1^2
\]

This is,

\[
\lambda_1^+ = \varepsilon - \mu_3 k_{21} (r_1^-)^2, \quad \lambda_1^- = \varepsilon - \mu_3 k_{21} (r_1^+)^2.
\]

As we did for the eigenvalue \(\lambda_2\) of the points \(Q^\pm\), having in mind the definition of \(r_1^\pm\) (see \([3]\)) it is straightforward to check that

\[
\lambda_1^+ = -2 \sqrt{\varepsilon_3 - \varepsilon (\sqrt{\varepsilon_3 - \varepsilon + \varepsilon_3})},
\]

so then \(\lambda_1^+ < 0\) and \(\lambda_1^- > 0\). Concerning \(\lambda_2^+, \lambda_3^+\), one has that

\[
\lambda_2^+ = \lambda_2^- = \varepsilon - \frac{1}{2} \left(1 - \frac{k_{11}}{k_{21}}\right) + \frac{\varepsilon}{2} \sqrt{\Delta}, \quad \lambda_3^+ = \lambda_3^- = \varepsilon - \frac{1}{2} \left(1 - \frac{k_{11}}{k_{21}}\right) - \frac{\varepsilon}{2} \sqrt{\Delta},
\]

where \(\Delta\) has been already defined in \([3]\). Observe that:

- If \(\Delta \geq 0\) then \(\lambda_2^+, \lambda_3^+\) are real and since \(|\Delta| < 1 - k_{11}/k_{21}\) then \(\lambda_2^+, \lambda_3^+ < 0\). Moreover \(|\lambda_3^+| > |\lambda_2^+|\).

- If \(\Delta < 0\) then \(\lambda_2^+, \lambda_3^+\) are complex and \(\text{Re} \lambda_2^+, \lambda_3^+ = -\frac{1}{2} \left(1 - \frac{k_{11}}{k_{21}}\right) < 0\).

Regarding the point \(R^0\), corresponding to \(\varepsilon = \varepsilon_3\), the same expressions obtained for \(\lambda_{1,2,3}^\pm\) hold, that is

\[
\lambda_2^0 = -\frac{1}{2} \left(1 - \frac{k_{11}}{k_{21}}\right) + \frac{\varepsilon_3}{2} \sqrt{\Delta}, \quad \lambda_3^0 = -\frac{1}{2} \left(1 - \frac{k_{11}}{k_{21}}\right) - \frac{\varepsilon_3}{2} \sqrt{\Delta},
\]

and, last but not least, using that \(r_1^0 = 1/2 \mu_3\), it follows that \(\lambda_1^0 = 0\).

One can represent graphically the focus-node transition in the eigenvalues \(\lambda_{2,3}^{\pm,0}\). To do it, let us denote \(s = k_{11}/k_{21}\) and \(t = k_{12}/k_{32}\). From the assumptions \(k_{11} < k_{21}\) and \((k_{11}/k_{21}) + (k_{12}/k_{32}) < 1\) it follows that \((s, t) \in (0, 1)\) and satisfy \(s + t < 1\). The bifurcation curve \(\Delta = 0\) reads \(t = 1 - (1 - s)^2/4\) and separate both behaviours (see Fig. \([15]\)).