COMPUTING THE SUPPORT OF LOCAL COHOMOLOGY MODULES

JOSEP ÀLVAREZ MONTANER AND ANTON LEYKIN

Abstract. For a polynomial ring $R = k[x_1, ..., x_n]$, we present a method to compute the characteristic cycle of the localization R_f for any nonzero polynomial $f \in R$ that avoids a direct computation of R_f as a D-module. Based on this approach, we develop an algorithm for computing the characteristic cycle of the local cohomology modules $H^r_I(R)$ for any ideal $I \subseteq R$ using the Čech complex. The algorithm, in particular, is useful for answering questions regarding vanishing of local cohomology modules and computing Lyubeznik numbers. These applications are illustrated by examples of computations using our implementation of the algorithm in Macaulay 2.

1. Introduction

Let k be a field of characteristic zero and $R = k[x_1, \ldots, x_n]$ the ring of polynomials in n variables. For any ideal $I \subseteq R$, the local cohomology modules $H^r_I(R)$ have a natural finitely generated module structure over the Weyl algebra A_n. Recently, there has been an effort made towards effective computation of these modules by using the theory of Gröbner bases over rings of differential operators. Algorithms given by U. Walther [23] and T. Oaku and N. Takayama [20] provide a utility for such computation and are both implemented in the package `D-modules` [16] for `Macaulay 2` [10].

Walther's algorithm is based on the construction of the Čech complex in the category of A_n-modules. So it is necessary to give a description of the localization R_f at a polynomial $f \in R$. An algorithm to compute these modules was given by T. Oaku in [19]. The main ingredient of the algorithm is the computation of the Bernstein-Sato polynomial of f which turns out to be a major bottleneck due to its complexity.

To give a presentation of R_f or $H^r_I(R)$ as A_n-modules is out of the scope of this work. Our aim is to provide an algorithm to compute an invariant that can be associated to a finitely generated A_n-module, the characteristic cycle. This invariant gives a description of the support of the A_n-module as an R-module, so it is a useful tool to prove the vanishing of local cohomology modules. Moreover, the characteristic cycles of local cohomology modules also give us some extra information since their multiplicities are a set of numerical invariants of the quotient ring R/I (see [2]). Among these invariants we may find Lyubeznik numbers that were introduced in [17].

We will present an algorithm to compute the characteristic cycle of any local cohomology module. It comes naturally from the structure of the Čech complex and the additivity of the characteristic cycle with respect to short exact sequences. The requirement is that we have to compute first the characteristic cycle of the localizations appearing in the Čech complex. To do so, we present a method based on a geometric formula given by V. Ginsburg in [9] and reinterpreted by J. Briançon, P. Maisonobe and M. Merle in [4]. The advantage of this approach is that we will not have to compute the Bernstein-Sato polynomial of f and we will be operating in a commutative graded ring in $2n$ variables instead of operating in the Weyl algebra A_n. The algorithm we will present is an elaboration of [4, Thm. 3.4.2], which – we have to point

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out – is stated in the complex analytic context. For our computational purposes we are interested in the algebraic context since we will need (absolute) primary decomposition so we have to make sure that, at least for the examples we will develop, this result is also true in the algebraic counterpart. The complex algebraic case may already be found in [9] but the approach we will use in this work is through flat base change. It allows us to work over any field of characteristic zero for a large sample of examples since localization modules and local cohomology modules have a good behavior with respect to this operation. Since absolute primary decomposition is not implemented in `Macaulay 2`, we compute over the field of rational numbers. This replacement, however, is not an issue for the examples that we present.

The scripts of the source codes we will use in this work as well as the output in full detail of the examples are available at the web page http://www.ima.umn.edu/∼leykin/CC. In the future, we would explore the possibility of using numerical primary decomposition – technique for varieties over $\mathbb{C}$ which is being developed by the second author.

2. Basics on the theory of $\mathcal{D}$-modules

Let $X = \mathbb{C}^n$ be the complex analytic space with coordinate system $x_1, \dots, x_n$. Given the convergent series ring $R = \mathbb{C}\{x_1, \dots, x_n\}$ consider the associated ring of differential operators $D_n := R\langle \partial_1, \dots, \partial_n \rangle$, i.e. the ring extension generated by the partial derivatives $\partial_i = \frac{\partial}{\partial x_i}$, with the relations given by $\partial_i \partial_j = \partial_j \partial_i$ and $\partial_i r - r\partial_i = \frac{\partial}{\partial x_i}$, where $r \in R$. For any unexplained terminology concerning the theory of rings of differential operators we shall use [3], [7].

The ring D_n has a natural increasing filtration given by the total order; the corresponding associated graded ring $gr(D_n)$ is isomorphic to the polynomial ring $R[a_1, \dots, a_n]$. A finitely generated D_n-module M has an increasing sequence of finitely generated R-submodules such that the associated graded module $gr(M)$ is a finitely generated $gr(D_n)$-module. The characteristic ideal of M is the ideal in $gr(D_n) = R[a_1, \dots, a_n]$ given by the radical ideal $J(M) := \mathrm{rad}\,(\mathrm{Ann}_{\,gr(D_n)}(gr(M)))$. The ideal $J(M)$ is independent of the good filtration on M. The characteristic variety of M is the closed algebraic set given by:

$$C(M) := V(J(M)) \subseteq \mathrm{Spec}\,(gr(D_n)) = \mathrm{Spec}\,(R[a_1, \dots, a_n]).$$

The characteristic variety describes the support of a finitely generated D_n-module as R-module. Let $\pi : \mathrm{Spec}\,(R[a_1, \dots, a_n]) \longrightarrow \mathrm{Spec}\,(R)$ be the map defined by $\pi(x, a) = x$. Then $\mathrm{Supp}_R(M) = \pi(C(M))$.

We single out the important class of regular holonomic D_n-modules. Namely, a finitely generated D_n-module M is holonomic if $M = 0$ or $\dim C(M) = n$. It is regular if there exists a good filtration on M such that $\mathrm{Ann}_{\,gr(D_n)}(gr(M))$ is a radical ideal ([5], see also [9, §3], [7]).

The characteristic cycle of M is defined as:

$$CC(M) = \sum m_i \;\; \Lambda_i$$

where the sum is taken over all the irreducible components $\Lambda_i = V(\mathfrak{q}_i)$ of the characteristic variety $C(M)$, where $\mathfrak{q}_i \in \mathrm{Spec}\,(gr(D_n))$ and m_i is the multiplicity of $gr(M)$ at a generic point along each component Λ_i. These multiplicities can be computed via Hilbert functions (see [6],[15]). Notice that the contraction of $\mathfrak{q}_i$ to R is a prime ideal so the variety $\pi(\Lambda_i)$ is irreducible. These components can be described in terms of conormal bundles to $X_i := \pi(\Lambda_i)$ in X, i.e.

$$CC(M) = \sum m_i \;\; T^*_{X_i} X.$$

In particular, the support of M is $\mathrm{Supp}_R(M) = \bigcup X_i$. For details we refer, among others, to [21, §10], [14, §7.5].

2.1. **Characteristic cycle of a localization.** Let M be a regular holonomic D_n-module. Then the localization M_f at a polynomial $f \in R$ is a regular holonomic D_n-module as well. A geometric formula that provides the characteristic cycle of M_f in terms of the characteristic cycle of M is given by V. Ginsburg in [9] and became known to us through the interpretation of J. Briançon, P. Maisonobe and M. Merle in [4].

First we will recall how to compute the conormal bundle relative to f. Let Y° be the smooth part of a subvariety $Y \subseteq X$ where $f|_Y$ is a submersion. Set:

$$W = \{(x,a) \in T^*X \ \mid \ x \in Y^\circ \ \text{ and } \ a \ \text{ annihilates } \ T_x(f|_Y)^{-1}(f(x))\}.$$

The conormal bundle relative to f, denoted by $T^*_{f|_Y}$, is then the closure of W in $T^*X|_Y$.

Theorem 2.1. ([4, Thm. 3.4.2]) *Let M be a regular holonomic D_n-module with characteristic cycle $CC(M) = \sum_i m_i \ T^*_{X_i}X$ and let $f \in R$ be a polynomial. Then*

$$CC(M_f) = \sum_{f(X_i)\neq 0} m_i(\Gamma_i + T^*_{X_i}X)$$

with $\Gamma_i = \sum_j m_{ij}\Gamma_{ij}$, where Γ_{ij} are the irreducible components of the divisor defined by f in $T^*_{f|_{X_i}}$ and m_{ij} are the corresponding multiplicities.

Remark 2.2. Assume for simplicity that M is a regular holonomic D_n-module such that $CC(M) = T^*_Y X$ and let $f \in R$ be a polynomial such that $f(Y) \neq 0$. By the formula above we have $CC(M_f) = T^*_Y X + \Gamma$. It is worthwhile to point out that the reduced variety associated to Γ is the characteristic variety of the local cohomology module $H^1_{(f)}(M)$.

Example 2.3. Set $R = \mathbb{C}\{x,y,z\}$, $M = H^1_{(x)}(R)$, $f = x$ and $g = y$.

We have $CC(R) = T^*_X X$. Then $T^*_{f|_X} = \{(x,y,z,a,b,c) \in T^*X \ \mid \ b = 0, c = 0\}$ and the divisor defined by f in $T^*_{f|_X}$ is $\Gamma = \{(x,y,z,a,b,c) \in T^*X \ \mid \ b = 0, c = 0, x = 0\} = T^*_{\{x=0\}}X$. Thus

$$CC(R_x) = T^*_X X + T^*_{\{x=0\}}X$$

We have $CC(M) = T^*_{\{x=0\}}X$. Then $T^*_{g|_{\{x=0\}}} = \{(x,y,z,a,b,c) \in T^*X \ \mid \ c = 0, x = 0\}$ and the divisor defined by g in $T^*_{g|_{\{x=0\}}}$ is $\Gamma = \{(x,y,z,a,b,c) \in T^*X \ \mid \ c = 0, x = 0, y = 0\} = T^*_{\{x=y=0\}}X$. Thus

$$CC(M_y) = T^*_{\{x=0\}}X + T^*_{\{x=y=0\}}X$$

The multiplicities m_{ij} appearing in the formula are the multiplicities of a generic point x along each component Γ_{ij} of Γ_i and can be computed via Hilbert functions as in [15].

Lemma 2.4. *Let $e(\Gamma, x)$ denote the multiplicity of the variety $\Gamma \subseteq T^*X$ defined by the ideal $I \subseteq R[a_1, \dots, a_n]$ at a point x. Then, the multiplicity m of a generic point x along Γ is*

$$m = e(\Gamma, x)/e(\Gamma^{red}, x),$$

where Γ^{red} is the variety defined by $\operatorname{rad}(I)$.

Proof. A reformulation of [11, Prop. 3.11] for the particular case of x being a generic point gives us the desired result, i.e. $e(\Gamma, x) = e(\Gamma^{red}, x) \cdot m$. □

2.2. **Algebraic $\mathcal{D}$-modules.** Let $X = \mathbb{C}^n$ be the complex affine space with coordinate system $x_1, \dots, x_n$. Given the polynomial ring $R = \mathbb{C}[x_1, \dots, x_n]$ consider the associated ring of differential operators $A_n := R\langle \partial_1, \dots, \partial_n\rangle$, i.e. the Weyl algebra. The theories of algebraic $\mathcal{D}$-modules and analytic $\mathcal{D}$-modules are very closely related. If one mimics the constructions given for the ring D_n, one can check that the results we have considered before, conveniently reformulated, remain true for A_n. In particular we may construct an algebraic characteristic cycle as a counterpart to the analytic characteristic cycle described before. Our aim is to explain how both cycles are related.

Set $\mathbb{C}\{x\} := \mathbb{C}\{x_1, \dots, x_n\}$ and $\mathbb{C}[x] := \mathbb{C}[x_1, \dots, x_n]$. Let M be a regular holonomic A_n-module. The D_n-module $M^{an} := \mathbb{C}\{x\} \otimes_{\mathbb{C}[x]} M$ is also regular holonomic. For a good filtration $\{M_i\}_{i\geq 0}$ on M the filtration $\{M_i^{an} := \mathbb{C}\{x\} \otimes_{\mathbb{C}[x]} M_i\}_{i\geq 0}$ is also good due to the fact that $\mathbb{C}\{x\}$ is flat over $\mathbb{C}[x]$. Therefore $gr(M^{an}) \simeq \mathbb{C}\{x\} \otimes_{\mathbb{C}[x]} gr(M)$ so the characteristic variety of M^{an} is the extension of the characteristic variety of M, i.e. $C(M^{an}) = C(M)^{an}$. However, we should notice that the components of the characteristic variety may differ depending on the ring we are considering. In particular we may have algebraically irreducible components that are analytically reducible.

The regular holonomic A_n-modules we will consider in this work, i.e. the polynomial ring $R = \mathbb{C}[x]$, the localization R_f for a polynomial $f \in R$, and the local cohomology modules $H_I^r(R)$, all have a good behavior with respect to flat base change. We state that, roughly speaking, the formulas of the algebraic and analytical characteristic variety of these modules are the same but the components and multiplicities of the corresponding characteristic cycle may differ.

Remark 2.5. The results of this section can be stated in general for X being any smooth algebraic variety over $\mathbb{C}$. It is worth to point out that $M \to M^{an}$ gives an equivalence between the category of regular holonomic $\mathcal{D}_X$-modules and the category of regular holonomic $\mathcal{D}_X^{an}$-modules when X is projective (see [9, §3]).

3. Algorithmic approach to Briançon-Maisonobe-Merle's Theorem

From now on we will assume that $R = \mathbb{C}[x_1, \dots, x_n]$ is the polynomial ring so we will be working in the algebraic context. Let M be a regular holonomic A_n-module with algebraic characteristic cycle $CC(M) = \sum m_i \; T^*_{X_i}X$ and let $f \in R$ be a polynomial. Our aim is to compute the characteristic cycle of the localization M_f operating in the commutative graded ring $gr(A_n) = R[a_1, \dots, a_n]$. We are going to provide two algorithms that are an elaboration of Theorem 2.1. The first one computes the part Γ_i of the formula in Theorem 2.1 corresponding to each irreducible component $T^*_{X_i}X$ in the characteristic cycle of M. The second one computes the components and the corresponding multiplicities of the varieties Γ_i.

Theorem 2.1 is a geometric reformulation of a result given by V. Ginsburg [9, Thm. 3.3]. Even though it is stated in the analytic context me may find in Ginsburg's paper the algebraic counterpart to the same result, see [9, Thm. 3.2]. We may interpret it as in Section 2.2. through flat base change.

Algorithm 3.1. *(Divisor defined by f in $T^*_{f|_Y}$, the conormal relative to f)*

Input: Generators $g_1, ..., g_d$ of an ideal $I \subset R$ defining the algebraic variety $Y = V(I) \subseteq X$ and a polynomial $f \in R$.
Output: Divisor defined by f in the conormal $T^*_{f|_Y}$ relative to f.

Compute the smooth part Y° of Y where $f|_Y$ is a submersion:

(0a) Compute $\nabla f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$

(0b) Compute $Y^\circ = Y \setminus V(I^\circ)$, where $I^\circ \subset R$ is the defining ideal of $\{x \in Y \mid \nabla f(x) = 0\}$ and the singular locus of Y.

Compute the conormal relative to f

(1a) Compute $K = \ker \phi$, where the $\phi : R^n \to R^{d+1}/I$ sends

$$s \mapsto (\nabla f, \nabla g_1, ..., \nabla g_d) \cdot s \in R^{d+1}/I.$$

(1b) Let $J \subset gr(A_n) = R[a_1, ..., a_n]$ be the ideal generated by $\{(a_1, ..., a_n) \cdot b \mid b \in K\}$.

(1c) Compute $J_{sat} = J : (gr(A_n)I^\circ)^\infty$. (Note: $I(T^*_{f|_Y}) = \sqrt{J_{sat}}$.)

Compute the divisor defined by f in $T^*_{f|_Y}$

(2a) Compute $K_f = \ker \phi_f$, where the map $\phi_f : R^n \to R^{d+1}/(I+(f))$ sends

$$s \mapsto (\nabla f, \nabla g_1, ..., \nabla g_d) \cdot s \in R^{d+1}/(I+(f)).$$

(2b) Let $J_f \subset gr(A_n) = R[a_1, ..., a_n]$ be the ideal generated by $\{(a_1, ..., a_n) \cdot b \mid b \in K_f\}$.
(2c) $C = J_{sat} + (f) + J_f \subset gr(A_n)$.

Return: The ideal C that defines the divisor f in $T^*_{f|_Y}$

Proof. (Correctness of the algorithm) The steps (0a), (0b) follow from the definition of $f|_Y$ being a submersion. The relative conormal $T^*_{f|_Y}$ is the closure of

$$W = \{(x,a) \in T^*X \mid x \in Y^\circ, \forall s \in K,\ a(s) = 0\}.$$

For every point $x \in Y^\circ$, the tangent space T_xY° is a specialization of $V(K)$, where K is computed in step (1a). A defining ideal of W is produced in (1b) and, finally, taking the closure amounts to the saturation in (1c). In order to restrict to $f = 0$, it is not enough to compute $J_{sat} + (f)$. However, step (2a) and (2b) that follow closely the idea of (1a) and (1b) provide the necessary correction term in (2c).

Recall that the analytic extension of the ideal C we obtain with the algorithm is what we would obtain applying Theorem 2.1 in order to compute the analytic characteristic cycle of the localization module (see Section 2.2). In our case, the ideal C will give us the components of the algebraic characteristic variety.

□

Algorithm 3.2. *(Components and multiplicities of the characteristic cycle)*

Input: The characteristic cycle $CC(M) = \sum m_i\ T^*_{X_i}X$ of a regular holonomic A_n-module M and a polynomial $f \in R$.
Output: The characteristic cycle $CC(M_f) = \sum_{f(X_i)\neq 0} m_i(\Gamma_i + T^*_{X_i}X)$.

For every component $Y = X_i$ we have to compute the ideal C_i corresponding to the divisor defined by f in $T^*_{f|_Y}$ using Algorithm 3.1. Then:

Compute the components of C_i

(1a) Compute the associated primes C_{ij} of C_i.
(1b) Compute $I_{ij} = C_{ij} \cap R$ (if you need to know the defining ideal of $X_{ij} = \pi(\Gamma_{ij})$ in Theorem 2.1).

Compute the multiplicities

(2) Compute the multiplicity m_{ij} in Theorem 2.1 as the multiplicity of a generic point x along each component C_{ij} of C_i as in Lemma 2.4 via Hilbert functions.

Return: The components of $CC(M_f)$ and their corresponding multiplicities.

Proof. The correctness of the algorithm is straightforward and follows from Lemma 2.4. □

The algorithm we propose requires the computation of the associated primes of an ideal; primary decomposition is also needed in the implementation if we want to avoid choosing generic points when computing the multiplicities (see Lemma 2.4). Therefore, we have to restrict ourselves to computations in the polynomial ring $R = \mathbb{Q}[x_1, \ldots, x_n]$ as we implemented the algorithm in the computer system `Macaulay 2`. What we are going to construct is the characteristic cycle of a regular holonomic A_n-module where now A_n stands for the Weyl algebra with rational coefficients. By flat base change we can extend the ideal C we obtain with Algorithm 3.1 to any ring of polynomials over a field of characteristic zero or to the convergent series ring over $\mathbb{C}$. As we stated in Section 2.2, the primary components may differ depending on the ring we are considering.

In order to construct the algebraic characteristic cycle over $\mathbb{Q}$ we would need to find the absolute primary decomposition of the ideal C we obtain with Algorithm 3.1. Even though the `Macaulay 2` command for primary decomposition is not implemented over the algebraic closure of $\mathbb{Q}$, it suffices for the examples we treat in the next section.

Another fine point in the implementation is the treatment of embedded components of the ideal C outputted in Algorithm 3.1. The ideal C contains the complete information about maximal components of the divisor f on $T^*_{f|_Y}$, in particular, we can compute their multiplicities. However, the primary ideals in the decomposition of C that correspond to an embedded component may not lead to the correct multiplicity due to the global nature of our computations. In order to obtain this multiplicity we restrict the divisor to the embedded component, which amounts to rerunning Algorithm 3.1 'modulo' its defining ideal. The top-level routine in our implementation processes components recursively "descending" to, i.e., localizing at, the embedded components when needed.

4. Characteristic cycle and Čech complex

Let $I = (f_1, \ldots, f_s) \subseteq R = \mathbb{C}[x_1, \ldots, x_n]$ be an ideal and M be a holonomic A_n-module. In this section we are going to compute the characteristic cycle of the local cohomology modules $H^r_I(M)$ using the Čech complex

$$\check{C}^\bullet(f_1, \ldots, f_s; M): \qquad 0 \longrightarrow M \xrightarrow{d_0} \bigoplus_{i=1}^{s} M_{f_i} \xrightarrow{d_1} \cdots \longrightarrow M_{f_1 \cdots f_s} \longrightarrow 0.$$

For simplicity we will assume from now on that M is indecomposable. Otherwise, if $M = M_1 \oplus M_2$, then

$$\check{C}^\bullet(f_1, \ldots, f_s; M) = \check{C}^\bullet(f_1, \ldots, f_s; M_1) \oplus \check{C}^\bullet(f_1, \ldots, f_s; M_2)$$

and $H^r_I(M) = H^r_I(M_1) \oplus H^r_I(M_2)$ for all r, so we can compute the characteristic cycle of both local cohomology modules separately. Sometimes we will denote the localization modules appearing in the Čech complex M_{f_α}, where $f_\alpha = \prod_{\alpha_i=1} f_i$ for all $\alpha \in \{0,1\}^s$. We will also denote $|\alpha| = \alpha_1 + \cdots + \alpha_s$ and $\varepsilon_1, \ldots, \varepsilon_s$ will be the natural basis of $\mathbb{Z}^s$.

From the characteristic cycle of the localization modules in the complex we develop an algorithm to extract the precise information needed to describe the characteristic cycles of the local cohomology modules. The algorithm comes naturally from the structure of the Čech complex and the additivity of the characteristic cycle with respect to short exact sequences. However the following assumption will be required:

(†) For all $\alpha \in \{0,1\}^s$ such that $\alpha_i = 0$, the localization map $M_{f_\alpha} \longrightarrow M_{f_{\alpha+\varepsilon_i}}$ is either a natural inclusion, i.e., M_{f_α} is saturated with respect to f_{ε_i}, or $M_{f_{\alpha+\varepsilon_i}} = 0$.

For unexplained terminology on the theory of complexes we refer to [22]. To shed some light on the process we first present the case of I being generated by one and two elements.

4.1. **The case** $s = 1$.

We have the short exact sequence

$$0 \longrightarrow H^0_I(M) \longrightarrow M \xrightarrow{d_0} M_{f_1} \longrightarrow H^1_I(M) \longrightarrow 0$$

Under the assumption (†) either $CC(H^1_I(M)) = CC(M_{f_1}) - CC(M)$ if M is f_1-saturated or $CC(H^0_I(M)) = CC(M)$ if $M_{f_1} = 0$. The algorithm we propose to compute the characteristic cycle works in both cases and boils down to the following step:

Prune the characteristic cycles of M_{f_1} and M, where 'prune' means remove the components (counting multiplicities) that both modules have in common.

The characteristic cycle of $H^0_I(R)$ (resp. $H^1_I(R)$) is the formal sum of components of M (resp. M_{f_1}) that survived this process.

4.2. **The case** $s = 2$**.** The vertical sequences of the following diagram are exact

$$\begin{tikzcd}
\check{C}^\bullet(f_1;M): & 0 \arrow[r] & M \arrow[r] & M_{f_1} \arrow[rr] & & 0 \\
\check{C}^\bullet(f_1,f_2;M): & 0 \arrow[r] & M \arrow[r] \arrow[u] & M_{f_1}\oplus M_{f_2} \arrow[r] \arrow[u] & M_{f_1f_2} \arrow[r] & 0 \\
\check{C}^\bullet(f_1;M_{f_2})[-1]: & & 0 \arrow[r] & M_{f_2} \arrow[r] \arrow[u] & M_{f_1f_2} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

so we have an exact sequence of Čech complexes

$$(i) \qquad 0 \longrightarrow \check{C}^\bullet(f_1;M_{f_2})[-1] \longrightarrow \check{C}^\bullet(f_1,f_2;M) \longrightarrow \check{C}^\bullet(f_1;M) \longrightarrow 0$$

where $[-1]$ stands for the result of shifting the complex one place to the right. Analogously, i.e. switching the roles played by f_1 and f_2, we also have

$$(ii) \qquad 0 \longrightarrow \check{C}^\bullet(f_2;M_{f_1})[-1] \longrightarrow \check{C}^\bullet(f_1,f_2;M) \longrightarrow \check{C}^\bullet(f_2;M) \longrightarrow 0$$

Notice that the vanishing of any localization module reduces the Čech complex to the case $s = 1$. So we are going to consider the only case remaining under the assumption (†), i.e. M is saturated with respect to f_1f_2. Consider the long exact sequence of cohomology modules associated to (i)

$$0 \longrightarrow H^{-1}_{(f_1)}(M_{f_2}) \longrightarrow H^0_I(M) \longrightarrow H^0_{(f_1)}(M) \xrightarrow{\delta^0} H^0_{(f_1)}(M_{f_2}) \longrightarrow H^1_I(M) \longrightarrow \cdots$$

where δ^j are the connecting maps. Non-vanishing may occur only in the sequence

$$0 \longrightarrow H^1_I(M) \longrightarrow H^1_{(f_1)}(M) \xrightarrow{\delta^1} H^1_{(f_1)}(M_{f_2}) \longrightarrow H^2_I(M) \longrightarrow 0$$

which breaks down into two short exact sequences with $C_1 = \operatorname{Coker}\delta^1$:

$$0 \longrightarrow H^1_I(M) \longrightarrow H^1_{(f_1)}(M) \longrightarrow C_1 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow C_1 \longrightarrow H^1_{(f_1)}(M_{f_2}) \longrightarrow H^2_I(M) \longrightarrow 0$$

In order to get $CC(H^1_I(M))$ and $CC(H^2_I(M))$ we only have to compute the characteristic cycle of C_1 since we already know that $CC(H^1_{(f_1)}(M)) = CC(M_{f_1}) - CC(M)$ and $CC(H^1_{(f_1)}(M_{f_2})) = CC(M_{f_1f_2}) - CC(M_{f_2})$.

Claim: $CC(C_1) = \sum m_i\, T^*_{X_i}X$ where the sum is taken over the components (counting multiplicities) that $CC(H^1_{(f_1)}(M))$ and $CC(H^1_{(f_1)}(M_{f_2}))$ have in common.

Proof. Assume that there is a component $T^*_{X_i}X$ in $CC(H^1_{(f_1)}(M))$ and $CC(H^1_{(f_1)}(M_{f_2}))$ not appearing in $CC(C_1)$, i.e. $T^*_{X_i}X$ is a component of $CC(H^1_I(M))$ and $CC(H^2_I(M))$. This component shows up in the computation of the characteristic cycle of the cohomology of the subcomplex $\check{C}^\bullet(f_2;M_{f_1})[-1]$, since it is in fact a component of $CC(M_{f_1})$ and $CC(M_{f_1f_2})$, but is not a component of $CC(M)$ and $CC(M_{f_2})$. Consider the long exact sequence of cohomology modules associated to the complex (ii):

$$\cdots \longrightarrow H^0_I(M) \longrightarrow H^0_{(f_2)}(M) \xrightarrow{\delta^0} H^0_{(f_2)}(M_{f_1}) \longrightarrow H^1_I(M) \longrightarrow H^1_{(f_2)}(M) \longrightarrow \cdots$$

It follows that the component $T^*_{X_i}X$ should belong to $CC(H^0_{(f_2)}(M_{f_1}))$ and $CC(H^1_{(f_2)}(M_{f_1}))$ in order to fulfill the hypothesis of being a component of $CC(H^1_I(M))$ and $CC(H^2_I(M))$. Thus we get a contradiction since $H^0_{(f_2)}(M_{f_1}) = 0$ as M_{f_1} has no f_2-torsion. □

To summarize the computation of the characteristic cycle of $H^r_I(M)$ using the sequence of Čech complexes (i) we propose the following algorithm:

(1) *Prune* the characteristic cycle of M_{f_1} and M.

Prune the characteristic cycle of $M_{f_1f_2}$ and M_{f_2}.

(2) *Prune* the characteristic cycle of M_{f_2} and M.
Prune the characteristic cycle of $M_{f_1f_2}$ and M_{f_1}.

'Prune' means remove the components (counting multiplicities) such that both characteristic cycles still have in common in that step of the algorithm. The characteristic cycle of $H^0_I(M)$ (resp. $H^1_I(M)$, $H^2_I(M)$) is the formal sum of components of M (resp. M_{f_1} and M_{f_2}, $M_{f_1f_2}$), i.e., components of the characteristic cycles of $\check{C}^0(f_1,f_2;M)$ (resp. $\check{C}^1(f_1,f_2;M)$, $\check{C}^2(f_1,f_2;M)$) that survived to the process.

Remark 4.1. Naturally, one may permute the steps (1) and (2) in the above algorithm.

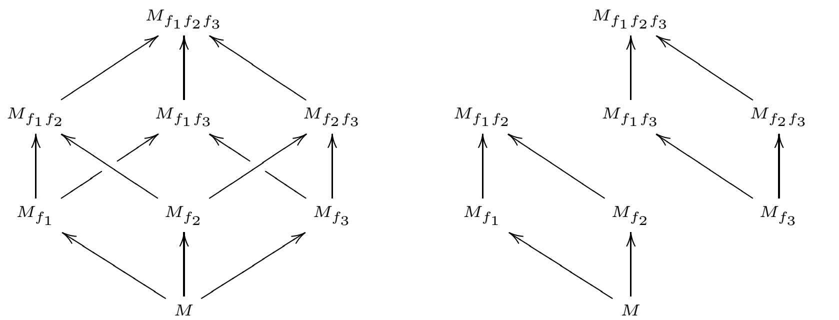
4.3. **The general case.** Let $I=(f_1,\dots,f_s)\subseteq R$ be an ideal and M be a holonomic A_n-module satisfying

(†) For all $\alpha\in\{0,1\}^s$ such that $\alpha_i=0$, the localization map $M_{f_\alpha}\longrightarrow M_{f_{\alpha+\varepsilon_i}}$ is a natural inclusion, i.e. M_{f_α} is saturated with respect to f_{ε_i}, or $M_{f_{\alpha+\varepsilon_i}}=0$.

Our aim is to proceed inductively in order to compute the characteristic cycle of the local cohomology modules $H^r_I(M)$. To this purpose it is useful to visualize the Čech complex $\check{C}^\bullet(f_1,\dots,f_s;M)$ as a s-hypercube where the edges are the localization maps that, with the corresponding sign, describe the differentials of the complex. The exact sequence of complexes

$$0\longrightarrow \check{C}^\bullet(f_1,\dots,f_{s-1};M_{f_s})[-1]\longrightarrow \check{C}^\bullet(f_1,\dots,f_s;M)\longrightarrow \check{C}^\bullet(f_1,\dots,f_{s-1};M)\longrightarrow 0$$

can be easily identified in the s-hypercube. For the case $s=3$ we visualize the Čech complexes $\check{C}^\bullet(f_1,f_2,f_3;M)$, $\check{C}^\bullet(f_1,f_2;M)$ and $\check{C}^\bullet(f_1,f_2;M_{f_3})[-1]$ as follows



Via a diagram chase, one may check that the localization map with respect to f_s, i.e., the edges $M_{f_\alpha}\longrightarrow M_{f_{\alpha+\varepsilon_s}}$ in the s-hypercube, induces the connecting maps δ^j in the long exact sequence of cohomology modules

$$0\longrightarrow H^{-1}_{(f_1,\dots,f_{s-1})}(M_{f_s})\longrightarrow H^0_I(M)\longrightarrow H^0_{(f_1,\dots,f_{s-1})}(M)\xrightarrow{\delta^0} H^0_{(f_1,\dots,f_{s-1})}(M_{f_s})\longrightarrow H^1_I(M)\longrightarrow\cdots$$

We are not going to give a precise description of the connecting maps, since we are only interested in the data given by the characteristic cycle. The formula we obtain in Theorem 4.4 for the characteristic cycle of the local cohomology modules $H^r_I(M)$ is given just in terms of the components of the characteristic cycle of the localizations M_{f_α}. The precise information we need to extract is given by the following algorithmic procedure.

Algorithm 4.2. *(Characteristic cycle and Čech complex)*

INPUT: Characteristic cycles $CC(M_{f_\alpha})=\sum m_{\alpha,i}\,T^*_{X_i}X$ for $\alpha\in\{0,1\}^s$.
OUTPUT: (Pruned) characteristic subcycles $\overline{CC}(M_{f_\alpha})\subseteq CC(M_{f_\alpha})$ for $\alpha\in\{0,1\}^s$.

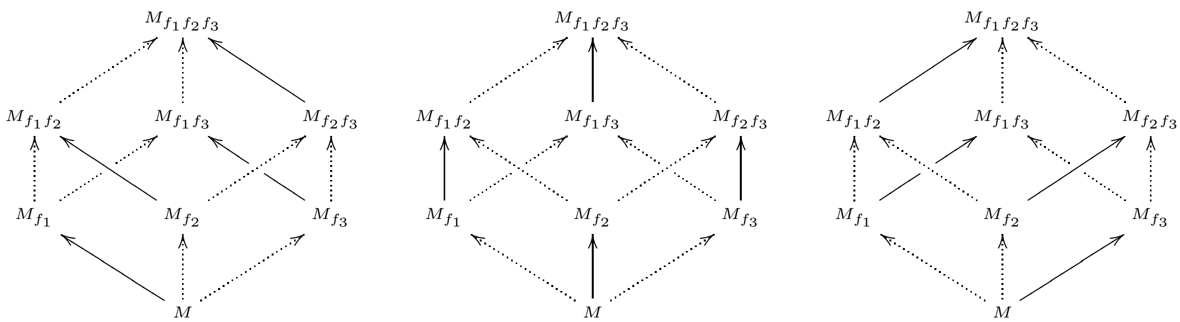
Prune the extra components

For j from 1 to s, incrementing by 1

(j) *Prune* the localizations M_{f_α} and $M_{f_{\alpha+\varepsilon_j}}$ for all $\alpha \in \{0,1\}^s$ such that $\alpha_j = 0$, where 'prune' means remove the components (counting multiplicities) such that both modules still have in common after step $(j-1)$.

Return: The components and the corresponding multiplicities of a characteristic subcycle of $CC(M_{f_\alpha})$ for $\alpha \in \{0,1\}^s$.

For the case $s = 3$ we visualize the steps of the algorithm as follows



The solid arrows indicate the modules we must prune at each step.

Remark 4.3. As in the case $s = 2$, the order we propose in the algorithm depends on the Čech subcomplexes we consider when computing the characteristic cycle of the cohomology of the Čech complex. We can obtain equivalent algorithms permuting the generators of the ideal I. It is also worth mentioning that the algorithm can be used in the algebraic context over any field of characteristic zero and in the analytic context.

Theorem 4.4. *Let $I = (f_1, \dots, f_s) \subseteq R$ be an ideal and M be an indecomposable holonomic A_n-module satisfying (†). Then*
$$CC(H_I^r(M)) = \sum_{|\alpha|=r} \overline{CC}(M_{f_\alpha}),$$
where the pruned characteristic subcycles are obtained with Algorithm 4.2.

Proof. We proceed by induction on the number of generators s of the ideal. The cases $s = 1, 2$ have been already done. For $s > 2$ we have an exact sequence of complexes
$$0 \longrightarrow \check{C}^\bullet(f_1, \dots, f_{s-1}; M_{f_s})[-1] \longrightarrow \check{C}^\bullet(f_1, \dots, f_s; M) \longrightarrow \check{C}^\bullet(f_1, \dots, f_{s-1}; M) \longrightarrow 0$$
Splitting the corresponding associated long exact sequence of cohomology modules
$$0 \longrightarrow H^{-1}_{(f_1,\dots,f_{s-1})}(M_{f_s}) \longrightarrow H^0_I(M) \longrightarrow H^0_{(f_1,\dots,f_{s-1})}(M) \xrightarrow{\delta^0} H^0_{(f_1,\dots,f_{s-1})}(M_{f_s}) \longrightarrow H^1_I(M) \longrightarrow \cdots$$
into short exact sequences we obtain
$$0 \longrightarrow A_r \longrightarrow H_I^r(M) \longrightarrow B_r \longrightarrow 0$$
$$0 \longrightarrow B_r \longrightarrow H^r_{(f_1,\dots,f_{s-1})}(M) \longrightarrow C_r \longrightarrow 0$$
$$0 \longrightarrow C_r \longrightarrow H^r_{(f_1,\dots,f_{s-1})}(M_{f_s}) \longrightarrow A_{r+1} \longrightarrow 0$$

The characteristic cycle of $H^r_{(f_1,\dots,f_{s-1})}(M)$ (resp. $H^r_{(f_1,\dots,f_{s-1})}(M_{f_s})$) is the formal sum of components of M_{f_α} satisfying $\alpha_s = 0$ and $|\alpha| = r$ (resp. $\alpha_s = 1$ and $|\alpha| = r + 1$) that survived to step $(s - 1)$ of the algorithm. Thus, for every r, in order to get $CC(H_I^r(M))$ we only have to compute $CC(C_r)$ and use additivity of the characteristic cycle with respect to short exact sequences.

Claim: $CC(C_r) = \sum m_i\, T^*_{X_i} X$ where the sum is taken over the components (counting multiplicities) that $CC(H^r_{(f_1,\dots,f_{s-1})}(M))$ and $CC(H^r_{(f_1,\dots,f_{s-1})}(M_{f_s}))$ have in common.

Assume that there is a component $T^*_{X_i} X$ in $CC(H^r_{(f_1,\dots,f_{s-1})}(M))$ and $CC(H^r_{(f_1,\dots,f_{s-1})}(M_{f_s}))$ not appearing in $CC(C_r)$, i.e. it has not been pruned in step (s). In particular, this component belongs to the characteristic cycle of some localization modules M_{f_α} and $M_{f_{\alpha+\varepsilon_s}}$ satisfying $\alpha_s = 0$ and $|\alpha| = r$. Then, this component is not pruned in the computation of the characteristic cycle of the cohomology of a convenient proper Čech subcomplex of $\check{C}^\bullet(f_1,\dots,f_s;M)$ containing M_{f_α} and $M_{f_{\alpha+\varepsilon_s}}$. Thus we get a contradiction.

□

If M is not indecomposable we only have to apply Theorem 4.4 to each component.

Example 4.5. Consider $R = \mathbb{C}[x]$ and the holonomic A_1-module $M = R \oplus H^1_{(x)}(R)$. We have:

$CC(M) = T^*_X X + T^*_{\{x=0\}} X$
$CC(M_x) = T^*_X X + T^*_{\{x=0\}} X.$

Applying the pruning algorithm to each component, that satisfy (†), we get

$$CC(H^0_{(x)}(M)) = CC(H^1_{(x)}(M)) = T^*_{\{x=0\}} X.$$

One may be tempted to apply the pruning algorithm to M, however, it misleads us resulting in the seeming vanishing of the local cohomology modules $H^r_{(x)}(M)$. Notice that M is not saturated with respect to $f = x$.

The question whether Theorem 4.4 still holds for indecomposable A_n-modules not satisfying (†) is open. For some examples we may give an affirmative answer.

Example 4.6. Set $R = \mathbb{C}[x,y]$. The holonomic A_2-module $M = H^1_{(xy)}(R)$ is not saturated with respect to $f = y$. We have:

$CC(M) = T^*_{\{x=0\}} X + T^*_{\{y=0\}} X + T^*_{\{x=y=0\}} X$
$CC(M_y) = T^*_{\{x=0\}} X + T^*_{\{x=y=0\}} X.$

Pruning the components they have in common we get $CC(H^0_{(y)}(M)) = T^*_{\{y=0\}} X$. It agrees with the fact that $H^0_{(y)}(M) \cong H^1_{(y)}(R)$.

Remark 4.7. Let $I = (f_1,\dots,f_s) \subseteq R$ be an ideal and M be an indecomposable holonomic A_n-module saturated with respect to $f_1 \cdots f_s$. The first step of Algorithm 4.2 also comes from the fact that the Čech complex $\check{C}^\bullet(f_1,\dots,f_s;M)$ is quasi-isomorphic to

$$0 \longrightarrow 0 \xrightarrow{d_0} M_{f_1}/M \xrightarrow{d_1} \bigoplus_{i=2}^{s} M_{f_1 f_i}/M_{f_i} \xrightarrow{d_2} \cdots \longrightarrow M_{f_1\cdots f_s}/M_{f_2\cdots f_s} \longrightarrow 0.$$

It would be interesting to continue the pruning process for the whole complex (not just for the components of the characteristic cycle) in order to find a complete description of a minimal complex quasi-isomorphic to the Čech complex.

A canonical Čech complex is introduced in [18] for the case of I being a monomial ideal and $M = R$. This complex is associated to a minimal free resolution of R/I in the same way as the usual Čech complex is associated to the Taylor resolution of R/I. The difference with the pruned Čech complex we propose is that, roughly speaking, they only prune when the localization map $R_{f_\alpha} \longrightarrow R_{f_{\alpha+\varepsilon_i}}$ is the identity.

5. Examples

In this section we want to present some examples where we will apply our algorithm to compute the characteristic cycle of local cohomology modules. First we have to study localizations R_f of the polynomial ring $R = \mathbb{Q}[x_1, \dots, x_n]$ at a polynomial $f \in R$. To compute its characteristic cycle directly one needs to:

· Construct a presentation of the A_n-module R_f,
· Compute the characteristic ideal $J(R_f)$,
· Compute the primary decomposition of $J(R_f)$ and its corresponding multiplicities.

The first two steps require expensive computations in the Weyl algebra A_n since we have to compute the Bernstein-Sato polynomial of f. For some short examples we can do the job just using the `Macaulay2` commands `Dlocalize` and `charIdeal`.

Following the approach of this work we have developed some scripts written in `Macaulay 2` that compute and print out the list of components and the corresponding multiplicities showing up in the characteristic cycles of the localizations R_f in the examples we present in this section. In fact we develop two different strategies that we may use depending on the examples we want to treat.

· *Single localization:* Since the characteristic cycle of R is $CC(R) = T^*_X X$, the characteristic cycle of R_f is $CC(R_f) = T^*_X X + \Gamma$, where Γ is computed according to Theorem 2.1 so we may compute it in one step. Notice that the defining ideal of Γ may be quite large so computing its primary decomposition can be expensive.

· *Iterative localization:* We can apply Theorem 2.1 in an iterative way on the components of the polynomial f. This strategy is useful to treat large examples, since, usually, it leads to computing primary decompositions of ideals of lower degrees compared to the former strategy.

For the examples we present in this work both strategies can be applied.

5.1. **Local cohomology modules.** Consider the ideal $I \subset R = \mathbb{Q}[x_1, ..., x_6]$ generated by the minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}.$$

It is a nontrivial problem to show that the local cohomology module $H^3_I(R)$ is nonzero (see [12, Remark 3.13], [13]). For example, `Macaulay 2` runs out of memory before computing this module with the command `localCohom`. U. Walther [23, Example 6.1] gives a complete description of this module using a tailor-made implementation of his algorithm which is based on the construction of the Čech complex. The difference with the implementation of the `Macaulay 2` command is that he uses iterative localization to reduce the complexity in the computation of Bernstein-Sato polynomials.

Our method makes it possible to prove algorithmically that $H^3_I(R) \neq 0$ from the computation of the characteristic cycles of the localization modules in the Čech complex which for this particular example looks like

$$(\star) \qquad 0 \to R \to R_{f_1} \oplus R_{f_2} \oplus R_{f_3} \to R_{f_1 f_2} \oplus R_{f_1 f_3} \oplus R_{f_2 f_3} \to R_{f_1 f_2 f_3} \to 0,$$

where $f_1 = x_1x_5 - x_2x_4$, $f_2 = x_1x_6 - x_3x_4$ and $f_3 = x_2x_6 - x_3x_5$.

Remark 5.1. By flat base change we can also deduce the non-vanishing of the local cohomology module $H^3_I(R)$ where $R = k[x_1, ..., x_6]$ is the polynomial ring over any field k of characteristic zero.

The list of components and their corresponding multiplicities showing up in the characteristic cycles of the chains in the Čech complex $(\star)$ and different from the whole space X that we get with our script contains 14 elements. A sample entry is as follows:

```
Component = V(ideal (x x  - x x , x x  - x x , x x  - x x ))
                      3 5    2 6   3 4    1 6   2 4    1 5
entries-> HashTable{{0, 1, 2} => 2}
                    {0, 1} => 1
                    {0, 2} => 1
                    {0} => 0
                    {1, 2} => 1
                    {1} => 0
                    {2} => 0
```

Namely, the component corresponding to the ideal I is present with multiplicity one in $R_{f_1f_2}$, $R_{f_2f_3}$, $R_{f_1f_3}$ and with multiplicity two in $R_{f_1f_2f_3}$. The following is the complete list of 14 components:

$$\begin{array}{lll}
A_1 = V(f_1), & A_2 = V(f_2), & A_3 = V(f_3), \\
B_1 = V(x_3, x_6), & B_2 = V(x_2, x_5), & B_3 = V(x_1, x_4), \\
C_1 = V(x_3, x_6, f_1), & C_2 = V(x_2, x_5, f_2), & C_3 = V(x_1, x_4, f_3), \\
D_1 = V(x_1, x_2, x_4, x_5), & D_2 = V(x_1, x_3, x_4, x_6), & D_3 = V(x_2, x_3, x_5, x_6), \\
E = V(x_1, x_2, x_3, x_4, x_5, x_6), & F = V(I). &
\end{array}$$

Piecing the results of our computation together we can draw the 3-hypercube in Figure 1.

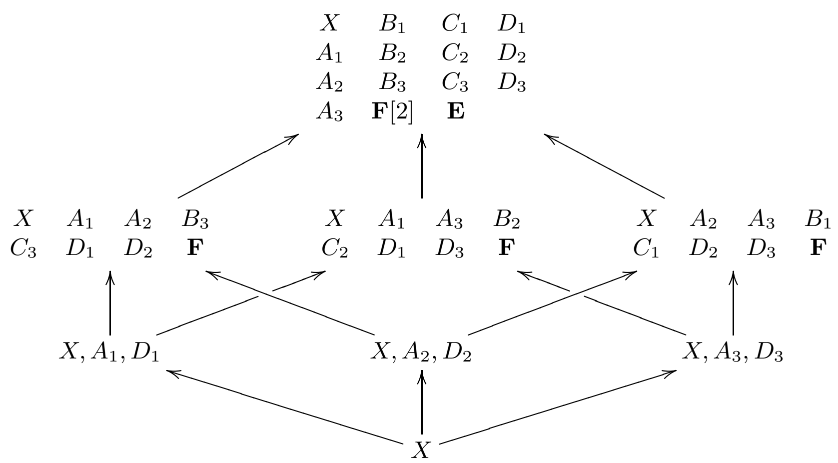


FIGURE 1. Components of characteristic cycles for the Čech complex ($\star$) (multiplicity $>$ 1 is specified in square brackets).

To compute the characteristic cycle of the cohomology modules we have to apply Theorem 4.4 that has been implemented in the routine `PruneCechComplexCC`. According to the output

```
{} => {}
{0} => {}
{1} => {}
{2} => {}
{0, 1} => {ideal (x x  - x x , x x  - x x , x x  - x x ) => 1}
                  3 5    2 6   3 4    1 6   2 4    1 5
{0, 2} => {}
{1, 2} => {}
{0, 1, 2} => {ideal (x , x , x , x , x , x ) => 1}
                      6   5   4   3   2   1
```

we get $CC(H^2_I(R)) = T^*_F X$ and $CC(H^3_I(R)) = T^*_E X$. Finally, it is worth to point out that the obtained result is coherent with the fact that the local cohomology module $H^3_I(R)$ is isomorphic to the injective hull of the residue field $E_R(R/(x_1, ..., x_6))$.

5.2. **Lyubeznik numbers.** Let $R = k[x_1, ..., x_n]$ be the polynomial ring over a field k of characteristic zero. Let $I \subseteq R$ be an ideal and $\mathfrak{m} = (x_1, ..., x_n)$ be the homogeneous maximal ideal. G. Lyubeznik [17] has defined a new set of numerical invariants of the quotient ring R/I by means of the Bass numbers

$$\lambda_{p,i}(R/I) := \mu_p(\mathfrak{m}, H_I^{n-i}(R)) := \dim_k \mathrm{Ext}_R^p(k, H_I^{n-i}(R)).$$

These invariants can be described as the multiplicities of the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{m}}(H_I^{n-i}(R))$ (see [2]). Namely,

$$CC(H^p_{\mathfrak{m}}(H_I^{n-i}(R))) = \lambda_{p,i}\ T^*_E X$$

Lyubeznik numbers carry interesting topological information of the quotient ring R/I as it is pointed in [17] and [8]. To compute them for a given ideal $I \subseteq R$ and arbitrary i, p we refer to U. Walther's algorithm [23, Algorithm 5.3] even though it has not been implemented yet. When I is a squarefree monomial ideal, a description of these invariants is given in [1]. Some other particular computations may also be found in [8] and [24].

Let $I \subset R = \mathbb{Q}[x_1, ..., x_6]$ be the ideal generated by the minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}$$

considered above, i.e. $I = (x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_2x_6 - x_3x_5)$. We want to compute the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{m}}(H^i_I(R))$ for $i = 2, 3$ and $\forall p$ so we have to construct the Čech complex

$$(\star\star) \qquad 0 \to M \to \bigoplus_{i=1}^{6} M_{x_i} \to \cdots \to M_{x_1\cdots x_6} \to 0,$$

where M is either $H^2_I(R)$ or $H^3_I(R)$. Then we have to compute the characteristic cycles of the localization modules and use Theorem 4.4.

• For $M = H^3_I(R)$ we know that its characteristic cycle is $T^*_E X$ so, applying Theorem 2.1, the Čech complex $(\star\star)$ reduces to the first term. Then,

$$CC(H^0_{\mathfrak{m}}(H^3_I(R))) = T^*_E X$$

and the other local cohomology modules vanish.

• For $M = H^2_I(R)$ we obtain

$$CC(H^2_{\mathfrak{m}}(H^2_I(R))) = T^*_E X$$

$$CC(H^4_{\mathfrak{m}}(H^2_I(R))) = T^*_E X$$

and the other local cohomology modules vanish. We are not going to present the complete output with all the components as in Figure 1 for this case but at least we are going to show the multiplicities of the component $T^*_E X$ appearing in the Čech complex $(\star\star)$ in Figure 2. We point out that the components $T^*_E X$ that survive to Algorithm 4.2 belong to $CC(M_{x_1x_2})$ and $CC(M_{x_1x_4x_5x_6})$.

Using the properties that Lyubeznik numbers satisfy (see [17, Section 4]), we can collect the multiplicities in a triangular matrix as follows:

$$\Lambda(R/I) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 1 \\ & & & 0 & 0 \\ & & & & 1 \end{pmatrix}$$

$M:\emptyset$	$M_{x_1}:\emptyset$	$M_{x_1x_2}:E$	$M_{x_1x_2x_3}:E[2]$	$M_{x_1x_2x_3x_4}:E[3]$	$M_{x_1x_2x_3x_4x_5}:E[3]$	$M_{x_1x_2x_3x_4x_5x_6}:E[3]$
	$M_{x_2}:\emptyset$	$M_{x_1x_3}:E$	$M_{x_1x_2x_4}:E$	$M_{x_1x_2x_3x_5}:E[3]$	$M_{x_1x_2x_3x_4x_6}:E[3]$	
	$M_{x_3}:\emptyset$	$M_{x_1x_4}:\emptyset$	$M_{x_1x_2x_5}:E$	$M_{x_1x_2x_3x_6}:E[3]$	$M_{x_1x_2x_3x_5x_6}:E[3]$	
	$M_{x_4}:\emptyset$	$M_{x_1x_5}:E$	$M_{x_1x_2x_6}:E[3]$	$M_{x_1x_2x_4x_5}:E$	$M_{x_1x_2x_4x_5x_6}:E[3]$	
	$M_{x_5}:\emptyset$	$M_{x_1x_6}:E$	$M_{x_1x_3x_4}:E$	$M_{x_1x_2x_4x_6}:E[3]$	$M_{x_1x_3x_4x_5x_6}:E[3]$	
	$M_{x_6}:\emptyset$	$M_{x_2x_3}:E$	$M_{x_1x_3x_5}:E[3]$	$M_{x_1x_2x_5x_6}:E[3]$	$M_{x_2x_3x_4x_5x_6}:E[3]$	
		$M_{x_2x_4}:E$	$M_{x_1x_3x_6}:E$	$M_{x_1x_3x_4x_5}:E[3]$		
		$M_{x_2x_5}:\emptyset$	$M_{x_1x_4x_5}:E$	$M_{x_1x_3x_4x_6}:E$		
		$M_{x_2x_6}:E$	$M_{x_1x_4x_6}:E$	$M_{x_1x_3x_5x_6}:E[3]$		
		$M_{x_3x_4}:E$	$M_{x_1x_5x_6}:E[3]$	$M_{x_1x_4x_5x_6}:E[3]$		
		$M_{x_3x_5}:E$	$M_{x_2x_3x_4}:E[3]$	$M_{x_2x_3x_4x_5}:E[3]$		
		$M_{x_3x_6}:\emptyset$	$M_{x_2x_3x_5}:E$	$M_{x_2x_3x_4x_6}:E[3]$		
		$M_{x_4x_5}:E$	$M_{x_2x_3x_6}:E$	$M_{x_2x_3x_5x_6}:E$		
		$M_{x_4x_6}:E$	$M_{x_2x_4x_5}:E$	$M_{x_2x_4x_5x_6}:E[3]$		
		$M_{x_5x_6}:E$	$M_{x_2x_4x_6}:E[3]$	$M_{x_3x_4x_5x_6}:E[3]$		
			$M_{x_2x_5x_6}:E$			
			$M_{x_3x_4x_5}:E[3]$			
			$M_{x_3x_4x_6}:E$			
			$M_{x_3x_5x_6}:E$			
			$M_{x_4x_5x_6}:E[2]$			

FIGURE 2. Component $T^*_E X$ appearing in the Čech complex ($\star\star$) (multiplicity > 1 is specified in square brackets).

The complex variety V defined by I has an isolated singularity at the origin. The singular cohomology groups of V with complex coefficients and support at the origin can be described from Lyubeznik numbers (see [8]). In our case we get

$$
\begin{aligned}
1 &= \lambda_{4,4} = \dim_{\mathbb{C}} H^8_{\{0\}}(V,\mathbb{C}),\\
1 &= \lambda_{2,4} = \dim_{\mathbb{C}} H^6_{\{0\}}(V,\mathbb{C}),\\
1 &= \lambda_{0,3} = \dim_{\mathbb{C}} H^3_{\{0\}}(V,\mathbb{C}).
\end{aligned}
$$

6. Conclusion and possible developments

We have shown that characteristic cycles of local cohomology modules can be be computed by algorithm operating in commutative polynomial rings as opposed to the direct computation of these modules, which requires Gröbner bases technique in noncommutative Weyl algebras.

The computational engine of our method is primary decomposition, from which we extract only geometrical information. This prompts a natural interest in *numerical primary decomposition*, an algorithm that would produce just that – the descriptions of reduced components and their multiplicities – by means of numerical computations.

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Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Avinguda Diagonal 647, Barcelona 08028, SPAIN
E-mail address: `Josep.Alvarez@upc.es`

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 851 South Morgan (M/C 249), Chicago, IL 60607-7045, USA
E-mail address: `leykin@math.uic.edu`