BERNSTEIN-SATO POLYNOMIAL AND RELATED INVARIANTS FOR MEROMORPHIC FUNCTIONS

JOSEP ÀLVAREZ MONTANER[1], MANUEL GONZÁLEZ VILLA[2], EDWIN LEÓN-CARDENAL[3], AND LUIS NÚÑEZ-BETANCOURT[4]

In memory of Professor Roberto Callejas-Bedregal

ABSTRACT. We develop a theory of Bernstein-Sato polynomials for meromorphic functions. As a first application we study the poles of Archimedean local zeta functions for meromorphic germs. We also present a theory of multiplier ideals for meromorphic functions from the analytic and algebraic point of view. It is also shown that the jumping numbers of these multiplier ideals are related with the roots of the Bernstein-Sato polynomials.

CONTENTS

1. INTRODUCTION

The theory of Bernstein-Sato polynomials was introduced independently by Bernstein [Ber72] to study the analytic continuation of the Archimedean zeta function and Sato [SS72] in the context of prehomogeneous vector spaces. Rapidly, Bernstein-Sato polynomials became an indispensable tool to study invariants of singularities of holomorphic or regular functions. In particular, the roots of the Bernstein-Sato polynomials are related to many other invariants such as poles of zeta functions, eigenvalues of the monodromy of the

2010 *Mathematics Subject Classification.* Primary TBA; Secondary TBA..

Key words and phrases. Bernstein-Sato polynomial, Zeta Functions, Singularities.

[1]Partially supported by grant PID2019-103849GB-I00 funded by MCIN/AEI/10.13039/501100011033.

[2]Partially supported by grant PID2020-114750GB-C33 funded by MCIN/AEI/10.13039/501100011033.

[3] The third author was partially supported by CONACyT Grant No. 286445.

[4] The fourth author was partially supported by CONACyT Grant 284598 and Cátedras Marcos Moshinsky.

Milnor fiber, jumping numbers of multiplier ideals, spectral numbers and F-thresholds, among others.

The aim of this paper is to develop a theory of Bernstein-Sato polynomials for the case of meromorphic functions, and to relate it to other invariants of such functions. This is inspired in previous work that has been done for singularities of meromorphic functions. For instance, Arnold proposed the study of singularities of meromorphic functions and began its classification [Arn98]. He remarked that even simple examples provide new interesting phenomena. Gussein-Zade, Luengo, and Melle [GZLMH98, GZLMH99a, GZLMH99b, GZLMH01] pursued a systematic research of the topology and monodromy of germs of complex meromorphic functions (see also works by Tibar and Siersma [Tib02, ST04], Bodin, Pichon, and Seade [BP07, PS08, BPS09]), and by Raibaut [Rai12, Rai13] studied motivic zeta functions and motivic Milnor fibers for meromorphic germs, also considered by Libgober, Maxim, and the second author [GVLM16, GVLM18]. Zúñiga-Galindo and the third author treated p-adic zeta functions for Laurent polynomials [LCZG13]. Lemahieu and the second author treated the case of topological zeta functions of meromorphic germs, and proved a generalization of the monodromy conjecture in the two variable case [GVL14]. Veys and Zúñiga-Galindo [VZG17] investigated meromorphic continuation and poles of zeta functions and oscillatory integrals for meromorphic functions defined over local fields of characteristic zero (see also [LC17, BGZG18, BG20]). Nguyen and Takeuchi [NT19] introduced meromorphic nearby cycle functors, and studied some of its applications to the monodromy of meromorphic germs.

In this paper, we prove the existence of meromorphic Bernstein-Sato polynomials for differentiably admissible $\mathbb{K}$-algebras, which is a class of rings including polynomial and power series rings over $\mathbb{K}$ and rings of holomorphic functions in a neighbourhood of the origin in $\mathbb{C}$ among other cases (see Section 2 for an overview).

Theorem A (Theorem 4.5)**.** *Let R be a differentiably admissible $\mathbb{K}$-algebra. Let $f, g \in R$ be nonzero elements. There exists $b(s) \in \mathbb{K}[s] \setminus \{0\}$ and $\delta(s) \in D_{R|\mathbb{K}}[s]$ such that*

$$\delta(s)\frac{f}{g}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} = b(s)\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}}.$$

More generally, we define a family of Bernstein-Sato polynomials $b^{\alpha}_{f/g}(s)$ indexed by $\alpha \in \mathbb{R}_{\geq 0}$ in Proposition 4.2. We also provide a more elementary proof of the functional equation for meromorphic functions of polynomial rings in Theorem 3.2. In the case of holomorphic functions we provide a version of Kashiwara's proof of the rationality of the roots of the Bernstein-Sato polynomial [Kas77] and the refinement given by Lichtin [Lic89].

Theorem B (Theorem 4.14, Corollary 4.15)**.** *Let $f, g \in R$ be nonzero holomorphic functions and $\alpha \in \mathbb{R}_{\geq 0}$. The roots of $b^{\alpha}_{f/g}(s)$ are negative rational numbers. Moreover, they are contained in the set*

$$\left\{\frac{k_i + 1 + \ell}{N_{f/g,i}}\ ;\ \ell \geq 0,\ i \in I_0\right\},$$

where the integers $N_{f/g,i}$ and k_i are extracted from the numerical data of a log resolution of f/g.

As an application, we study meromorphic continuation of Archimedean local zeta functions of meromorphic germs, extending the ideas of Bernstein [Ber72] in the classical case.

Our approach is complementary to the one done by Veys and Zúñiga-Galindo [VZG17], where they used resolution of singularities following the ideas of Bernstein and Gel'fand [BG69] and Atiyah [Ati70].

Theorem C (Theorem 5.4)**.** *Let f and g be nonzero holomorphic functions and take $\alpha = \operatorname{lct}_0 g$. The local zeta function $Z_\phi(s, f/g)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$, and its poles are contained in the set*

$$\Big\{\zeta - k\alpha \ ; \ k \in \mathbb{Z}_{\geq 0}\Big\}_{\zeta \text{ root of } b^{\alpha}_{f/g}(s)} \bigcup \Big\{k\alpha - \xi \ ; \ k \in \mathbb{Z}_{\geq 0}\Big\}_{\xi \text{ root of } b^{\alpha}_{g/f}(s)}.$$

In particular, the poles of $Z_\phi(s, f/g)$ are rational numbers.

Finally, we study multiplier ideals of meromorphic functions from the analytic and algebraic point of view.

Theorem D. *Let f and g be nonzero holomorphic functions. Then:*

- (Theorem 6.6) *The set of jumping numbers of f/g is a set of rational numbers with no accumulation points.*
- (Theorem 6.7) *Let λ be a jumping number of f/g such that $\lambda \in (1 - \operatorname{lct}_0(g), 1]$. Then, $-\lambda$ is a root of the Bernstein-Sato polynomial of f/g.*

We also provide a weaker version of Skoda's Theorem.

Proposition A. *Let f and g be nonzero holomorphic functions. Then:*

- (Proposition 6.5) $\mathcal{J}((\frac{f}{g})^{\lambda+\ell}) = (\frac{f^\ell}{g^\ell}(\mathcal{J}(\frac{f}{g})^{\lambda})) \bigcap R$ *for every $\ell \in \mathbb{N}$. In particular, if $\lambda + 1$ is a jumping number, then λ is a jumping number.*
- (Lemma 7.5) *For every $\lambda \in \mathbb{R}_{>0}$, we have $\mathcal{J}(f^\lambda) \subseteq \mathcal{J}((\frac{f}{g})^\lambda)$. In addition, we have $\mathcal{J}((\frac{f}{g})^n) = (f^n)$ for every $n \in \mathbb{Z}_{>0}$.*

Takeuchi has developed independently a theory of Bernstein-Sato polynomials for meromorphic functions [Tak21]. Both approaches differ slightly (see Remark 4.8) but complement each other. Takeuchi is interested in the relation between the roots of the Bernstein-Sato polynomial and the eigenvalues of the Milnor monodromies. Meanwhile, we pay attention to the meromorphic continuation of the Archimedean local zeta function and multiplier ideals.

2. Background

2.1. **Rings of differential operators and its modules.** We start by recalling the basics on the theory of rings of differential operators as introduced by Grothendieck [Gro67, §16.8].

Definition 2.1. Let R be a Noetherian ring containing a field of characteristic zero $\mathbb{K}$. The ring of $\mathbb{K}$-linear differential operators of R is the subring $D_{R|\mathbb{K}} \subseteq \operatorname{Hom}_{\mathbb{K}}(R, R)$, whose elements are defined inductively as follows.

- Differential operators of order zero are defined by the multiplication by elements of R, and so, $D^0_{R|\mathbb{K}} \cong R$.
- An element $\delta \in \operatorname{Hom}_{\mathbb{Z}}(R, R)$ is an operator of order less than or equal to n if $[\delta, r] = \delta r - r\delta$ is an operator of order less than or equal to $n - 1$.

We denote by $D^n_{R|\mathbb{K}}$ the set of all differential operators of order less than or equal to n. We have a filtration $D^0_{R|\mathbb{K}} \subseteq D^1_{R|\mathbb{K}} \subseteq \cdots$ such that $D^m_{R|\mathbb{K}} D^n_{R|\mathbb{K}} \subseteq D^{m+n}_{R|\mathbb{K}}$. We denote this filtration by $D^\bullet_{R|\mathbb{K}}$. The ring of differential operators is defined as

$$D_{R|\mathbb{K}} = \bigcup_{n\in\mathbb{N}} D^n_{R|\mathbb{K}}.$$

We note that in general $D^1_{R|\mathbb{K}} \cong \mathrm{Der}_{R|\mathbb{K}} \bigoplus R$, but $D_{R|\mathbb{K}}$ is not always generated as R-algebra by derivations.

2.2. **Holonomic D-modules for polynomial rings.**

Definition 2.2. Let $R = \mathbb{C}[x_1, \ldots, x_d]$. We define the Bernstein filtration of R, $\mathcal{B}^\bullet_{R|\mathbb{C}}$ by

$$\mathcal{B}^i_{R|\mathbb{C}} = \mathbb{C}\{x^\alpha \delta^\beta \; ; \; |\alpha| + |\beta| \le i\}.$$

We note that

- $\dim_{\mathbb{C}} \mathcal{B}^i_{R|\mathbb{C}} = \binom{n+i}{i} < \infty$,
- $D_{R|\mathbb{C}} = \bigcup_{i\in\mathbb{N}} \mathcal{B}^i_{R|\mathbb{C}}$, and
- $\mathcal{B}^n_{R|\mathbb{C}} \mathcal{B}^j_{R|\mathbb{C}} = \mathcal{B}^j_{R|\mathbb{C}}$.

We observe that $\mathrm{gr}_{\mathcal{B}^\bullet_{R|\mathbb{C}}}(D_{R|\mathbb{C}})$ is a commutative ring isomorphic to $\mathbb{C}[x_1, \ldots, x_d, y_1, \ldots, y_d]$.

Definition 2.3. Given a left (right) $D_{R|\mathbb{C}}$-module, M, we say that a filtration $\Gamma^\bullet$ of $\mathbb{C}$-vector spaces is $\mathcal{B}^\bullet_{R|\mathbb{C}}$-compatible if

- $\dim_{\mathbb{C}} \Gamma^i < \infty$,
- $M = \bigcup_{i\in\mathbb{N}} \Gamma^i$,
- $\mathcal{B}^i_{R|\mathbb{C}} \Gamma^j \subseteq \Gamma^j$ ($\Gamma^j \mathcal{B}^i_{R|\mathbb{C}} \subseteq \Gamma^j$).

We say that $\Gamma^\bullet$ is a good filtration if $\mathrm{gr}_{\Gamma^\bullet}(M)$ is finitely generated as a $\mathrm{gr}_{\mathcal{B}^\bullet_{R|\mathbb{C}}}(D_{R|\mathbb{C}})$-module.

If $\Gamma^\bullet$ is a good filtration for M, we have that $\dim_{\mathbb{C}} \Gamma^n$ is a polynomial function on n of degree equal to the Krull dimension of $\mathrm{gr}_{\Gamma^\bullet}(M)$ by Hilbert-Samuel theory. This degree does not depend of the choice of such good filtration, and it is called $\dim_{D_{R|\mathbb{C}}}(M)$. We note that every finitely generated $D_{R|\mathbb{C}}$-module has a good filtration.

Theorem 2.4 (Bernstein's Inequality)**.** *Let M be a finitely generated $D_{R|\mathbb{C}}$-module. Then,*

$$d \le \dim_{D_{R|\mathbb{C}}}(M) \le 2d.$$

Definition 2.5. A $D_{R|\mathbb{C}}$-module M, is holonomic if either $M = 0$ or $\dim_{D_{R|\mathbb{C}}}(M) = d$.

We observe that a $D_{R|\mathbb{C}}$-module M, with a good filtration $\Gamma^\bullet$ is holonomic if and only if there exists a polynomial q in one variable of degree d, such that $\dim_{\mathbb{C}} \Gamma^n \le q(n)$. We recall that every holonomic $D_{R|\mathbb{C}}$ has finite length.

2.3. **Rings differentiably admissible.** We now introduce a family of $\mathbb{K}$-algebras whose rings of differential operators satisfy good properties.

Definition 2.6. Let $\mathbb{K}$ be a field of characteristic zero. Let R be a Noetherian regular $\mathbb{K}$-algebra of dimension d. We say that R is differentiably admissible if

(1) $\dim(R_{\mathfrak{m}}) = d$ for every maximal ideal $\mathfrak{m} \subseteq R$,

(2) $R/\mathfrak{m}$ is an algebraic extension of $\mathbb{K}$ for every maximal ideal $\mathfrak{m} \subseteq R$, and
(3) $\mathrm{Der}_{R|\mathbb{K}}$ is a projective R-module of rank d such that the natural map

$$R_{\mathfrak{m}} \otimes_R \mathrm{Der}_{R|\mathbb{K}} \to \mathrm{Der}_{R_{\mathfrak{m}}|\mathbb{K}}$$

is an isomorphism.

This class of algebras were introduced by the third-named author [NB13] to obtain the existence of the Bernstein-Sato polynomial for algebras that do not have global coordinates. This later case was already studied by Narváez Macarro and Mebkhout [MNM91]. Examples of differentiably admissible algebras include polynomial rings over $\mathbb{K}$, power series rings over $\mathbb{K}$, the ring of convergent power series in a neighborhood of the origin over $\mathbb{C}$, Tate and Dwork-Monsky-Washnitzer $\mathbb{K}$-algebras [MNM91], the localization of complete regular rings of mixed characteristic at the uniformizer [NB13, Lyu00], and the localization of complete local domains of equal-characteristic zero at certain elements [Put18].

Theorem 2.7 ([NB13, Section 2])**.** *Let $\mathbb{K}$ be a field of characteristic zero. Let R be a differentiably admissible $\mathbb{K}$-algebra of dimension d. Then,*

(1) $D^n_{R|\mathbb{K}} = (\mathrm{Der}_{R|\mathbb{K}} + R)^n$*, and*
(2) $D_{R|\mathbb{K}} \cong R\langle \mathrm{Der}_{A|\mathbb{K}} \rangle$.
(3) $D_{R|\mathbb{K}}$ *is left and right Noetherian;*
(4) $\mathrm{gr}_{D^\bullet_{R|\mathbb{K}}}(D_{R|\mathbb{K}})$ *is a regular ring of pure graded dimension* $2d$*;*
(5) $\mathrm{gl.dim}(D_{R|\mathbb{K}}) = d$.

In particular, $D_{R|\mathbb{K}}$ is an R-algebra of differentiable type with the order filtration.

Theorem 2.8 ([NB13, Section 2])**.** *Let $\mathbb{K}$ be a field of characteristic zero, and $\mathbb{K}(s) =$ $\mathrm{Frac}(\mathbb{K}[s])$. Let R be a differentiably admissible $\mathbb{K}$-algebra of dimension d, and $R(s) =$ $R \otimes_{\mathbb{K}} \mathbb{K}(s)$. Then, $D(s) = D_{R|\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{K}(s)$ is a $R(s)$-algebra of differentiable type with the order filtration.*

Definition 2.9. Let A be an R-algebra with 1. By a filtration $F^\bullet = \{F^i\}_{i\in\mathbb{N}}$ on A we mean an ascending filtration of finitely generated R-modules F^i such that $A = \bigcup_{i\in\mathbb{N}} F^i$ and $F^iF^j \subseteq F^{i+j}$. In this case we say that $(A, F^\bullet)$ is filtered. We say that a left A-module M with a filtration $\Gamma^\bullet = \{\Gamma^i\}_{i\in\mathbb{N}}$ is $F^\bullet$-compatible if

(1) Γ^i is finitely generated,
(2) $M = \bigcup_{i\in\mathbb{N}} \Gamma^i$,
(3) $F^i\Gamma^j \subseteq \Gamma^j$.

We say that $\Gamma^\bullet$ is a good filtration if $\mathrm{gr}_{\Gamma^\bullet}(M)$ is finitely generated as a $\mathrm{gr}_{F^\bullet}(A)$-module. We also consider the analogue definitions for right A-modules.

Definition 2.10. We say that a filtered R-algebra $(A, F^\bullet)$ is an algebra of differentiable type if $\mathrm{gr}_{F^\bullet}(A)$ is a commutative Noetherian ring with 1 which is regular with pure graded dimension.

Remark 2.11. Let A be an R-algebra of differentiable type. We note that every two good filtrations, $\Gamma^\bullet$ and $\widetilde{\Gamma}^\bullet$, of M are shift cofinal. Specifically, there exists $a \in \mathbb{N}$ such that

$$\Gamma^{i-a} \subseteq \widetilde{\Gamma}^i \subseteq \Gamma^{i+a}.$$

Furthermore, M is finitely generated as A-module if and only if M has a good filtration.

Definition 2.12. Let A be a filtered R-algebra, and M be a left A-module. The dimension of M is defined by

$$\dim_A(M) = \dim_{\mathrm{gr}_{F^\bullet}(A)} \mathrm{gr}_{\Gamma^\bullet}(M),$$

where $\Gamma^\bullet$ is a good filtration of M.

Definition 2.13. Let A be an algebra of differentiable type. Let $M \neq 0$ be a finitely generated A-module. We define

$$\mathrm{grade}_A(M) = \inf\{j \mid \mathrm{Ext}^j_A(M, A) \neq 0\}.$$

We note that $\mathrm{grade}_A(M) \leq \mathrm{gl.}\dim(A) = \dim(A) - \mathrm{gl.}\dim(A)$.

Proposition 2.14 ([Bjö79, Chapter 2., Theorem 7.1])**.** *Let A be an algebra of differentiable type, and let $M \neq 0$ be a finitely generated A-module. Then,*

$$\dim_A(M) + \mathrm{grade}_A(M) = \dim(A).$$

In particular, $\dim_A(M) \geq \dim(A) - \mathrm{gl.}\dim(A)$.

Definition 2.15. Let A be a $\mathbb{K}$-algebra of differentiable type. Let M be a finitely generated left (right) A-module. We say that M is in the *left (right) Bernstein class* if either $M = 0$ or $\dim_A(M) = \dim(A) - \mathrm{gl.}\dim(A)$.

This class of Bernstein modules is an analogue of the class of holonomic modules. In particular, it is closed under submodules, quotients, extensions, and localizations [MNM91, Proposition 1.2.7]).

Let A be a $\mathbb{K}$-algebra of diferentiable type and global dimension d, and M be a finitely generated A-module. If M is in the Bernstein class of A, then $\mathrm{Ext}^i_A(M, A) \neq 0$ if and only if $i = d$ [Bjö79]. Then, the functor that sends M to $\mathrm{Ext}^d_A(M, A)$ is an exact contravariant functor that interchanges the left and the right Bernstein class. Furthermore, $M \cong \mathrm{Ext}^d_A(\mathrm{Ext}^d_A(M, A), A)$ for modules in the Bernstein class. Since A is left and right Noetherian, the modules in the Bernstein class are both Noetherian and Artinian. We conclude that the modules in the Bernstein class have finite length as A-modules [MNM91, Proposition 1.2.5])

2.4. **Log-resolution of meromorphic functions.** Let R be a differentiably admissible $\mathbb{K}$-algebra. Given nonzero elements $f, g \in R$ we are interested in the function f/g. Most likely $f, g : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ are germs of holomorphic functions and we refer to $f/g : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ as a germ of a meromorphic function. Taking local coordinates we will assume $f, g \in R = \mathbb{C}\{x_1, \ldots, x_n\}$. Recall that f/g and f'/g' define the same germ if there exist a unit $u \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $f = uf'$ and $g = ug'$.

Definition 2.16. Let X be a n-dimensional smooth analytic manifold, U a neighborhood of $0 \in \mathbb{C}^n$ and $\pi : X \to U$ a proper analytic map. We say that π is log resolution of the meromorphic germ f/g if:

- π is a log resolution of the hypersurface $H = \{f_{|_U} = 0\} \cup \{g_{|_U} = 0\}$, i.e. π is an isomorphism outside a proper analytic subspace in U;
- there is a normal crossing divisor F on X such that $\pi^{-1}(H) = \mathcal{O}_X(-F)$;
- the lifting $\tilde{f}/\tilde{g} = (f/g) \circ \pi = \frac{f \circ \pi}{g \circ \pi}$ defines a holomorphic map $\tilde{f}/\tilde{g} : X \to \mathbb{P}^1$.

One can obtain a log resolution π of f/g from a log resolution π' of $f \cdot g$ by blowing up along the intersections of irreducible components of ${\pi'}^{-1}\{f \cdot g = 0\}$ until the irreducible components of the strict transform in $\pi^{-1}\{f = 0\}$ and $\pi^{-1}\{g = 0\}$ are separated by a dicritical component, i.e. an exceptional divisor E of π for which $(\tilde{f}/\tilde{g})_{|_E} : E \to \mathbb{P}^1$ is a surjective map.

Let $\{E_i\}_{i \in I}$ be the irreducible components of F. We denote the relative canonical divisor, defined by the Jacobian determinant of π, as the divisor

$$K_\pi = \sum k_i E_i.$$

We also denote the strict transforms of f and g as

$$\tilde{f} = \pi^* f = \sum N_{f,i} E_i, \quad \tilde{g} = \pi^* g = \sum N_{g,i} E_i.$$

Moreover, we define

$$N_{f/g,i} = N_{f,i} - N_{g,i}.$$

Notice that E_i is a dicritical component if and only if $N_{f/g,i} = 0$. It is then natural to associate with the meromorphic germ f/g the divisor

$$\tilde{F} = \sum_{i \in I} N_{f/g,i} E_i.$$

For $i \in I$ we write $i \in I_0$ if $N_{f,i} > N_{g,i}$, $i \in I_\infty$ if $N_{g,i} > N_{f,i}$, and $i \in I_d$ if $N_{f_i} = N_{g,i}$. We have the decomposition $\tilde{F} = \tilde{F}_0 + \tilde{F}_\infty + \tilde{F}_d$ where

$$\tilde{F}_0 = \sum_{i \in I_0} N_{f/g,i} E_i, \quad \tilde{F}_\infty = \sum_{i \in I_\infty} N_{f/g,i} E_i, \quad \text{and} \quad \tilde{F}_d = \sum_{i \in I_d} N_{f/g,i} E_i,$$

with $\tilde{F}_d = 0$ by definition.

Remark 2.17. Around a given point $p \in X$ we may consider a local system of coordinates $(z_1, \ldots, z_n)$ such that the components E_i containing p are given by the equations $z_i = 0$. Assume that $p \in E_i$, for $i = 1, \ldots, m$, then we have the local equations:

$$\tilde{f} = u z_1^{N_{f,1}} \cdots z_m^{N_{f,r}}, \quad \text{and} \quad \tilde{g} = v z_1^{N_{g,1}} \cdots z_m^{N_{g},r},$$

where $u, v \in \mathcal{O}_{X,p}$ are units. Moreover, the local equations of the relative canonical divisor are

$$\tilde{\omega} = \pi^*(dx_1 \wedge \cdots \wedge dx_n) = w(z_1^{k_1} \cdots z_m^{k_m})\, dz_1 \wedge \cdots \wedge dz_n,$$

where $w \in \mathcal{O}_{X,p}$ is a unit as well. Furthermore, applying some extra blow-ups if necessary, we may assume that either $\tilde{f}$ divides $\tilde{g}$ or the other way around, so we have either

$$\frac{\tilde{f}}{\tilde{g}} = \frac{u}{v}\, z_1^{N_{f/g,1}} \cdots z_m^{N_{f/g,m}} \quad \text{or} \quad \frac{\tilde{f}}{\tilde{g}} = \frac{u}{v}\, \frac{1}{z_1^{|N_{f/g,1}|} \cdots z_m^{|N_{f/g,m}|}}.$$

3. Meromorphic Bernstein-Sato polynomial for polynomial rings

Our first goal is to introduce the theory of Bernstein-Sato polynomials for quotients of polynomials. We single out this case for the convenience of the reader since it follows the same lines of reasoning as in the classical case.

Definition 3.1. Let $f, g \in R = \mathbb{C}[x_1, \ldots, x_d]$ be nonzero elements. We set $M_{f/g} = R_{fg}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}}$. This free module has an structure of $D_{R|\mathbb{C}}[s]$-modulo given by

$$\partial \cdot \frac{h}{f^a g^b}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} = \partial\left(\frac{h}{f^a g^b}\right)\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} + s\frac{h}{f^{a+1}g^{b-1}}\partial\left(\frac{f}{g}\right)\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}}.$$

where ∂ is a $\mathbb{C}$-linear derivative on R.

Theorem 3.2. *Let $f, g \in R = \mathbb{C}[x_1, \ldots, x_d]$ be a nonzero elements. There exists $b(s) \in \mathbb{C}[s] \setminus \{0\}$ and $\delta(s) \in D_{R|\mathbb{C}}[s]$ such that*

$$\delta(s)\frac{f}{g}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} = b(s)\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}}.$$

Proof. Let $R(s) = R \otimes_{\mathbb{C}[s]} \mathbb{C}(s)$ and $D(s) = D_{R(s)|\mathbb{C}(s)}$, where $\mathbb{C}(s)$ denotes the fraction field of $\mathbb{C}[s]$. Let $M = M_{f/g} \otimes_{\mathbb{C}[s]} \mathbb{C}(s)$. We claim that M is a holonomic $D(s)$-module. Let $\theta = \deg(f) + \deg(g)$. We set a filtration of finite dimensional $\mathbb{C}$-vector spaces.

$$\Gamma^n = \frac{1}{f^n g^n}\left\{h\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} \mid \deg(h) \leq (\theta+1)n\right\}.$$

We have that $x_i\Gamma^n \subseteq \Gamma^{n+1}$. We now observe that $\partial_i\Gamma^n \subseteq \Gamma^{n+1}$. Given $\frac{h}{f^n g^n}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} \in \Gamma^n$, we have that

$$\begin{aligned}\partial_i\frac{h}{f^n g^n}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} &= \frac{f^n g^n \delta_i(h) - nhf^{n-1}g^{n-1}\partial_i(fg)}{f^{2n}g^{2n}}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} + \frac{sh}{f^{n+1}g^{n-1}}\left(\frac{g\delta_i(f) - f\delta_i(g)}{g^2}\right)\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} \\ &= \frac{fg\delta_i(h) - nh\partial_i(fg)}{f^{n+1}g^{n+1}}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} + \frac{sh(g\delta_i(f) - f\delta_i(g))}{f^{n+1}g^{n+1}}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}}.\end{aligned}$$

By considering the degrees of the polynomials appearing in the numerators, we have that $\partial_i\Gamma^n \subseteq \Gamma_{n+1}$. Then, $\Gamma^\bullet$ is a filtration compatible with the Bernstein filtration. We observe that $\dim_{\mathbb{C}(s)} \Gamma^n \leq \dim_{\mathbb{C}}[R]_{\leq(\theta+1)n}$, where the latter is a polynomial on n of degree d. Then, M is a holonomic $D(s)$-module.

Since M is holonomic, it has finite length, and the sequence of $D(s)$-submodules

$$D(s)\frac{1}{fg}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} \supseteq D(s)\frac{1}{f^2 g}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} \supseteq D(s)\frac{1}{f^3 g}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} \supseteq \ldots$$

stabilizes. There exists $m \in \mathbb{N}$ such that

$$D(s)\frac{1}{f^{m-1}g}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} = D(s)\frac{1}{f^m g}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}}.$$

Thus, there exists $\delta(s) \in D(s)$ such that

$$\delta(s)\left(\frac{1}{f^{m-1}g}\right)\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} = \left(\frac{1}{f^m g}\right)\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}}.$$

Then, there exists $b(s) \in \mathbb{C}[s] \setminus \{0\}$ such that $\widetilde{\delta}(s) = b(s)\delta(s) \in D_{R|\mathbb{C}}[s]$. Hence,

$$\widetilde{\delta}(s)\left(\frac{1}{f^{m-1}g}\right)\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} = b(s)\frac{1}{f^m g}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}}. \tag{3.0.1}$$

We have an isomorphism of $D_{R[s]|\mathbb{C}[s]}$-modules $\psi : M_{f/g} \to M_{f/g}$ defined by

$$\frac{p(s)h}{f^\alpha g^\beta}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} \mapsto \frac{p(s-m)h}{f^{\alpha+m}g^{\beta-m}}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}}.$$

After applying ψ to Equation 3.0.1, we obtain that

$$\widetilde{\delta}(s-m)\frac{f}{g}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} = b(s-m)\frac{1}{g}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}}.$$

Finally multiplying by g, we obtain the desired equation. □

4. Meromorphic Bernstein-Sato polynomial for differentibly admissible algebras

In this Section we will generalize the Bernstein-Sato theory to the case of differentibly admissible algebras. As in the polynomial case, the functional equation takes place in the following module.

Definition 4.1. Let R be a differentiably admissible $\mathbb{K}$-algebra. Let $f, g \in R$ be nonzero elements, take $\alpha \in \mathbb{R}_{\geq 0}$ and set $\mathscr{M}^{\alpha}_{f/g}[s] = R_{fg}[s]\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}}$. This free module has an structure of $D_{R|\mathbb{K}}[s]$-module given by

$$\partial \cdot \frac{h(s)}{f^a g^b}\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}} = \partial\left(\frac{h(s)}{f^a g^b}\right)\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}} + s\frac{\partial(f)h(s)}{f^{a+1}g^b}\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}} - (s+\alpha)\frac{h(s)\partial(g)}{f^a g^{b+1}}\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}},$$

where ∂ is a $\mathbb{K}$-linear derivative on R. If $\alpha = 0$, we just write $\mathscr{M}_{f/g}[s] = R_{fg}[s]\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}}$.

Proposition 4.2. *Let R be a differentiably admissible $\mathbb{K}$-algebra. Let $f, g \in R$ be nonzero elements and $\alpha \in \mathbb{R}_{\geq 0}$. There exists $b(s) \in \mathbb{K}[s] \setminus \{0\}$ and $\delta(s) \in D_{R|\mathbb{K}}[s]$ such that*

$$\delta(s)f\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}} = b(s)\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}}.$$

Proof. We have that the global dimensions of $D_{R(s)|\mathbb{K}(s)}$ and $D_{R|\mathbb{K}}$ are equal to $d = \dim(R)$ [MNM91, Proposition 2.2.3]. For the sake of simplicity we denote $D = D_{R|\mathbb{K}}$, $D(s) = D_{R(s)|\mathbb{K}(s)}$ $\mathscr{M}[s] = \mathscr{M}^{\alpha}_{f/g}[s]$, and $\mathscr{M}(s) = \mathscr{M}[s] \otimes_{\mathbb{K}} \mathbb{K}(s)$. We also consider R_{fg} with the usual action of $D_{R|\mathbb{K}}$ and we set $R_{fg}(s) = R_{fg}[s] \otimes_{\mathbb{K}[s]} \mathbb{K}(s)$, and $D_{fg}(s) = R_{fg} \otimes_R D(s)$. We have a chain of isomorphisms

$$\begin{aligned}
\operatorname{Ext}^i_{D_f(s)}(\mathscr{M}(s), D_{fg}(s)) &\cong \operatorname{Ext}^i_{D_{fg}(s)}(R_{fg}(s) \otimes_{R_{fg}(s)} \mathscr{M}(s), D_{fg}(s)) \\
&\cong \operatorname{Ext}^i_{D_{fg}(s)}(R_{fg}(s), \operatorname{Hom}_{R_{fg}(s)}(\mathscr{M}(s), D_{fg}(s))) \\
&\cong \operatorname{Ext}^i_{D_{fg}(s)}(R_{fg}(s), D_{fg}(s)) \otimes_{D_{fg}(s)} \operatorname{Hom}_{R_{fg}(s)}(\mathscr{M}(s), D_{fg}(s))) \\
&\cong \operatorname{Ext}^i_{D_{fg}(s)}(R_{fg}, D_{fg}) \otimes_{D_{fg}} \operatorname{Hom}_{R_{fg}(s)}(\mathscr{M}(s), D_{fg}(s))) \\
&\cong (\operatorname{Ext}^i_D(R, D) \otimes_R R_{fg}) \otimes_{D_{fg}} \operatorname{Hom}_{R_{fg}(s)}(\mathscr{M}(s), D_{fg}(s))).
\end{aligned}$$

Since $\operatorname{Ext}^i_D(R, D) \otimes_R R_{fg} = 0$ for $i \neq d$, we conclude that $\operatorname{Ext}^i_{D_{fg}(s)}(\mathscr{M}(s), D_{fg}(s)) \neq 0$ for $i \neq d$. Then, $\mathscr{M}(s)$ has a $D(s)$-submodule N in the Bernstein class of $D(s)$ such that

$N_{fg} = \mathscr{M}(s)$ [MNM91, Proposition 1.2.7 and Proof of Theorem 3.1.1]. Then, there exists $\ell \in \mathbb{N}$ such that $f^\ell g^\ell \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}} \in N$. Since N has finite length as $D_{A(s)|\mathbb{K}(s)}$-module the chain

$$D(s) f^\ell g^\ell \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}} \supseteq D(s) f^{\ell+1} g^\ell \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}} \supseteq D(s) f^{\ell+1} g^\ell \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}} \supseteq \dots$$

stabilizes. Then, there exists $m \in \mathbb{N}$ and a differential operator $\delta(s) \in D_{A(s)|\mathbb{K}(s)}$ such that

$$\delta(s) f^{\ell+m+1} \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}} = f^{\ell+m} \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}}.$$

After clearing denominators and applying a shifting, there exists $\widetilde{\delta}(s) \in D_{A|\mathbb{K}}[s]$ such that

$$\widetilde{\delta}(s) f \mathbf{f^s} = \mathbf{f^s}.$$

□

Definition 4.3. Let R be a differentiably admissible $\mathbb{K}$-algebra. Let $f, g \in R$ be nonzero elements and $\alpha \in \mathbb{R}_{\geq 0}$. The Bernstein-Sato polynomial $b^\alpha_{f/g}(s) \in \mathbb{K}[s]$ of order α of the meromorphic function f/g is the monic polynomial of smallest degree satisfying the functional equation in Proposition 4.2.

Proposition 4.4. *Let R be a differentiably admissible $\mathbb{K}$-algebra. Let $f, g \in R$ be nonzero elements and $\alpha \in \mathbb{R}_{\geq 0}$. The following are equivalent:*

(1) The Bernstein-Sato polynomial of f/g of order α;

(2) The minimal polynomial of the action of s on $\dfrac{D_{R|\mathbb{K}}[s] \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}}}{D_{R|\mathbb{K}}[s] f \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}}}$;

(3) The monic element of smallest degree in $\mathbb{K}[s] \cap (\mathrm{Ann}_{D_{R|\mathbb{K}}[s]}(\frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}}) + D_{R|\mathbb{K}}[s] f)$.

Proof. The equivalence between the first two follows from the definition. For the equivalence between the second and the third, we observe that

$$\frac{D_{R|\mathbb{K}}[s] \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}}}{D_{R|\mathbb{K}}[s] f \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}}} \cong \operatorname{coker} \left(\frac{D_{R|\mathbb{K}}[s]}{\mathrm{Ann}_{D_{R|\mathbb{K}}[s]}(\mathbf{f^s})} \xrightarrow{\cdot f} \frac{D_{R|\mathbb{K}}[s]}{\mathrm{Ann}_{D_{R|\mathbb{K}}[s]}(\frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}})} \right)$$
$$\cong \frac{D_{R|\mathbb{K}}[s]}{\mathrm{Ann}_{D_{R|\mathbb{K}}[s]}(\frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}}) + D_{R|\mathbb{K}}[s] f}.$$

□

We give a $D_{R[t]|\mathbb{K}}$-module structure on $\mathscr{M}^\alpha_{f/g}[s]$ where the new variable t acts as multiplication by f. We define

$$t \cdot \frac{h(s)}{f^a g^b} \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}} = \frac{h(s+1)}{f^a g^b} f \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}}$$
$$\partial_t \cdot \frac{h(s)}{f^a g^b} \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}} = \frac{h(s-1)}{f^a g^b} \frac{1}{f} \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}}$$

A simple computation shows that $\partial_t t - t\partial_t = 1$ and that $-\partial_t t$ acts as multiplication by s. Moreover $ts - st = t$ and we may consider the ring $D_{R|\mathbb{K}}\langle s, t\rangle \subseteq D_{R[t]|\mathbb{K}}$. Denote for simplicity

$$\mathscr{N}^\alpha_{f/g} = D_{R|\mathbb{K}}[s] \frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}} \subseteq \mathscr{M}^\alpha_{f/g}[s]$$

We have that $\mathscr{N}^{\alpha}_{f/g}$ is indeed a $D_{R|\mathbb{K}}\langle s,t\rangle$-module. As in the classical case, we may view the Bernstein-Sato polynomial of f/g of order α as the minimal polynomial of the action of $-\partial_t t = s$ on

$$\frac{\mathscr{N}^{\alpha}_{f/g}}{t\,\mathscr{N}^{\alpha}_{f/g}}$$

Building upon the previous construction we deduce a Bernstein-Sato polynomial of a meromorphic function that mimics the classical case.

Theorem 4.5. *Let R be a differentiably admissible $\mathbb{K}$-algebra. Let $f, g \in R$ be nonzero elements. There exists $b(s) \in \mathbb{K}[s] \setminus \{0\}$ and $\delta[s] \in D_{R|\mathbb{K}}[s]$ such that*

$$\delta(s)\frac{f}{g}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} = b(s)\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}}$$

which is a functional equation in $\mathscr{M}^{1}_{f/g}[s]$.

Proof. By Proposition 4.2, we have the functional equation

$$\delta(s)f\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s+1}}} = b(s)\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s+1}}}$$

in $\mathscr{M}^{1}_{f/g}[s]$. Let $\phi : \mathscr{M}^{1}_{f/g}[s] \to \mathscr{M}^{0}_{f/g}[s]$ be the map defined by

$$\frac{p(s)h}{f^a g^b}\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s+1}}} \mapsto \frac{p(s)h}{f^a g^{b+1}}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}}.$$

We note that ϕ is an isomorphism of $D_{R|\mathbb{K}}[s]$-modules. Then, after appliying ϕ, we get that

$$\delta(s)\frac{f}{g}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}} = b(s)\frac{1}{g}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)^{\mathbf{s}}$$

The result follows after multiplying by g. $\square$

Definition 4.6. Let R be a differentiably admissible $\mathbb{K}$-algebra and $f, g \in R$ be nonzero elements. The Bernstein-Sato polynomial $b_{f/g}(s) \in \mathbb{K}[s]$ of the meromorphic function f/g is the monic polynomial of smallest degree satisfying the functional equation in Theorem 4.5.

Remark 4.7. In this situation we may interpret the Bernstein-Sato polynomial of f/g as the minimal polynomial of the action of s on

$$\frac{D_{R|\mathbb{K}}[s]\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}}}{D_{R|\mathbb{K}}[s]\frac{f}{g}\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}}}.$$

We may also give a $D_{R[t]|\mathbb{K}}$-module structure where the new variable t acts now as multiplication by f/g.

Remark 4.8. Takeuchi [Tak21] has independently introduced a theory of Bernstein-Sato polynomials which slightly differs from the one in Definition 4.3. He defines the Bernstein-Sato polynomial $b^{\mathrm{mero}}_{f/g,\alpha}(s)$ of order α of the meromorphic function f/g as the monic polynomial of smallest degree satisfying the functional equation

$$\delta_1[s]\frac{f}{g}\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}} + \cdots + \delta_\ell(s)\frac{f^\ell}{g^\ell}\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}} = b(s)\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}} \tag{4.0.1}$$

for some $\delta_i[s] \in D_{R|\mathbb{K}}[s]$ and $\ell \geq 1$ large enough. He points out that one may also consider the functional equation

$$\delta(s)\frac{f}{g}\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}} = b(s)\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}}, \tag{4.0.2}$$

which would lead to a Bernstein-Sato polynomial that we temporaly denote by $b_{f/g,\alpha}(s)$. They satisfy the following relation

$$b^{\mathrm{mero}}_{f/g,\alpha}(s) \mid b_{f/g,\alpha}(s).$$

When comparing to our family of Bernstein-Sato polynomials of order α, which satisfy the functional equation

$$\delta(s)f\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}} = b(s)\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}}, \tag{4.0.3}$$

we have

$$b^{\alpha}_{f/g}(s) \mid b_{f/g,\alpha}(s).$$

We point out that Takeuchi's $b_{f/g,0}(s)$ coincides with our $b_{f/g}(s)$ and, following the proof of Theorem 4.5, we have

$$b_{f/g}(s) \mid b^{1}_{f/g}(s).$$

Example 4.9 (Separated variables)**.** Take two sets of independent variables $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_l)$, i.e. $x_i \neq y_j$ for every $i = 1, \dots, k$ and $j = 1, \dots, l$. Let $b_f(s)$ be the Bernstein- Sato polynomial of $f(x)$, and let $P(s, x, \partial_x)$ be a differential operator such that

$$P(s, x, \partial_x)f^{s+1} = b_f(s)f^s.$$

Then for any $\alpha \in \mathbb{R}_{\geq 0}$ and $g(y)$ we have

$$P(s, x, \partial_x)f\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}} = b_f(s)\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}},$$

and thus $b_f(s) \mid b^{\alpha}_{f/g}(s)$.

Example 4.10. As a particular case of the preceding example, consider the function

$$h(x_1, \dots, x_n) = \frac{x_1^{m_1} \cdots x_k^{m_k}}{x_{k+1}^{m_{k+1}} \cdots x_n^{m_n}}.$$

Then for every $\alpha \in \mathbb{R}_{\geq 0}$ the polynomial $b^{\alpha}_h(s)$ equals

$$\prod_{i=1}^{k}\left(\prod_{j=1}^{m_i}\left(s + \frac{j}{m_i}\right)\right).$$

Remark 4.11. Since the operator $\delta(s) = 1$ verifies the functional equation

$$\delta(s)\frac{1}{g^{s+\alpha}} = 1\frac{1}{g^{s+\alpha}},$$

we have that $b^{\alpha}_{1/g}(s) = 1$ for every $\alpha \in \mathbb{R}_{\geq 0}$, and thus the Bernstein-Sato polynomial of $1/g$ has no roots. However, there exist non constant functions g for which the topological zeta function of $1/g$ has non trivial poles [GVL14]. This phenomenon shows that a direct generalization of the strong monodromy conjecture in the case of meromorphic functions is not possible and deserves further investigation.

We now show that -1 is always a root of $b^{\alpha}_{f/g}$ in most cases (see also Remark 4.11).

Proposition 4.12. *Let R be a differentiably admissible $\mathbb{K}$-algebra and $f, g \in R$ be nonzero elements such that f is a nonzero divisor in R/gR nor a unit. Then, $b^{\alpha}_{f/g}(-1) = 0$ for every $\alpha \in \mathbb{R}$.*

Proof. We fix $\alpha \in \mathbb{R}$. We set $M^{\alpha}_{f/g} = R_{fg}\frac{\mathbf{1}}{\mathbf{g}^{\alpha}}$. This free module has an structure of $D_{R|\mathbb{K}}$-module given by

$$\partial \cdot \frac{h}{f^a g^b}\frac{\mathbf{1}}{\mathbf{g}^{\alpha}} = \partial\left(\frac{h}{f^a g^b}\right)\frac{\mathbf{1}}{\mathbf{g}^{\alpha}} - \alpha\frac{\delta(g)}{f^a g^{b+1}}\frac{\mathbf{1}}{\mathbf{g}^{\alpha}}$$

where ∂ is a $\mathbb{K}$-linear derivative on R. There is a specialization map of $D_{R|\mathbb{K}}$-modules from $\mathscr{M}^{\alpha}_{f/g}[s] \to M^{\alpha}_{f/g}$ by sending $s \mapsto -1$.

There exists $\delta(s) \in D_{R|\mathbb{K}}[s]$ such that

$$\delta(s) f \frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}} = b(s)\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}}. \tag{4.0.4}$$

After specializing, we have

$$\delta(-1)\frac{\mathbf{1}}{\mathbf{g}^{\alpha}} = b(-1)\frac{1}{f}\frac{\mathbf{1}}{\mathbf{g}^{\alpha}}. \tag{4.0.5}$$

We note that $\delta(-1)\frac{\mathbf{1}}{\mathbf{g}^{\alpha}} \in R_g\frac{\mathbf{1}}{\mathbf{g}^{\alpha}}$. Then, there exists $u \in R$ and $\theta \in \mathbb{N}$ such that $\delta(-1)\frac{\mathbf{1}}{\mathbf{g}^{\alpha}} = \frac{u}{g^{\theta}}\frac{\mathbf{1}}{\mathbf{g}^{\alpha}}$. Then Equation 4.0.5 implies $\frac{u}{g^{\theta}} = b(-1)\frac{1}{f}$, which is equivalent to $fu = b(-1)g^{\theta}$. If $\theta = 0$, we have that $b(-1) = 0$, because f is not a unit. Since f is a nonzero divisor in R/gR, it is also a nonzero divisor in $R/g^{\theta}R$ if $\theta \neq 0$. Thus $b^{\alpha}_{f/g}(-1) = 0$. □

Corollary 4.13. *Let $R = \mathbb{C}[x_1, \ldots, x_d]$ and $f, g \in R$ be nonzero elements such that f does not divide g. Then, $b^{\alpha}_{f/g}(-1) = 0$ for every $\alpha \in \mathbb{R}$.*

Proof. We can assume that f and g are relatively prime. If g is a unit, this is a known fact of the classical Bernstein-Sato polynomial. Then, the claim follows from Proposition 4.12, because f and g form a regular sequence. We note that f is not a unit because it does not divide g. □

4.1. **Rationality of the roots of the meromorphic Bernstein-Sato polynomial.** A classical result of Kashiwara [Kas77] asserts that the roots of the Bernstein-Sato polynomial are negative rational numbers. This result was later refined by Lichtin [Lic89] by giving a set of candidate roots described in terms of the numerical data of the log resolution of the singularity. A similar result also holds in the meromorphic case.

Let $\mathcal{O}_{\mathbb{C}^n,0}$ be the ring of germs of holomorphic functions around a point $0 \in \mathbb{C}^n$, which we identify with $R = \mathbb{C}\{x_1, \ldots, x_n\}$ by taking local coordinates. We take a small neighborhood of the origin $U \subseteq \mathbb{C}^n$ where f and g are holomorphic. Let $J_f = (\frac{df}{dx_1}, \ldots, \frac{df}{dx_1})$ and $J_g = (\frac{dg}{dx_1}, \ldots, \frac{dg}{dx_1})$ be the Jacobian ideals of f and g respectively. We may assume that the zero locus of J_f is contained in the zero locus of f and the same condition holds for g.

Let $\pi : X \to U$ be a log resolution of the meromorphic germ f/g, which is in particular a log resolution of $f^{-1}(0) \cup g^{-1}(0)$. In the sequel we will use the notations in Section 2.4

but just recall that we associated to f/g the divisor $\tilde{F} = \tilde{F}_0 + \tilde{F}_\infty$ where

$$\tilde{F}_0 = \sum_{i\in I_0} N_{f/g,i} E_i \quad \text{and} \quad \tilde{F}_\infty = \sum_{i\in I_\infty} N_{f/g,i} E_i.$$

Theorem 4.14. *Let $f, g \in R$ be nonzero holomorphic functions and $\alpha \in \mathbb{R}_{\geq 0}$. The roots of $b^\alpha_{f/g}(s)$ are negative rational numbers. Moreover, they are contained in the set*

$$\left\{ \frac{k_i + 1 + \ell}{N_{f/g,i}} \ ; \ \ell \geq 0, \ i \in I_0 \right\}.$$

Proof. Let $\pi : X \to U$ be a log resolution of the meromorphic germ f/g and let D_X and D denote the corresponding rings of differential operators. Assume that in local coordinates around a point $p \in X$ we have either

$$\frac{\tilde{f}}{\tilde{g}} = z_1^{N_1} \cdots z_m^{N_m} \quad \text{or} \quad \frac{\tilde{f}}{\tilde{g}} = \frac{1}{z_1^{N_1} \cdots z_m^{N_m}}$$

Therefore, locally at the point p, we have that the Bernstein-Sato polynomial of order α is either

$$b^\alpha_{\tilde{f}/\tilde{g}}(s) = \prod_{i=1}^{m} \left(\prod_{j=1}^{N_i} \left(s + \frac{j}{N_i} \right) \right) \quad \text{or} \quad b^\alpha_{\tilde{f}/\tilde{g}}(s) = 1.$$

By considering a cover of X by affine open subsets where we can use local coordinates as above, we obtain the global Bernstein-Sato polynomial, that we also denote by $b^\alpha_{\tilde{f}/\tilde{g}}(s)$ if no confusion arises, as the least common multiple of the local Bernstein-Sato polynomials. Then the rationality result boils down to proving the existence of an integer $\ell \geq 0$ such that

$$b^\alpha_{f/g}(s) \mid b^\alpha_{\tilde{f}/\tilde{g}}(s) b^\alpha_{\tilde{f}/\tilde{g}}(s+1) \cdots b^\alpha_{\tilde{f}/\tilde{g}}(s+\ell). \tag{4.1.1}$$

The proof of this fact in the meromorphic case is analogous to the classical case considered by Kashiwara. We only have to be careful when dealing with the module

$$\mathscr{N}^\alpha_{\tilde{f}/\tilde{g}} = D_X[s] \frac{\tilde{\mathbf{f}}^{\mathbf{s}}}{\tilde{\mathbf{g}}^{\mathbf{s}+\alpha}}$$

If $\frac{\tilde{f}}{\tilde{g}} = z_1^{N_1} \cdots z_m^{N_m}$ we have that $\mathscr{N}^\alpha_{\tilde{f}/\tilde{g}}$ is subholonomic as D_X-module as in the classical case. Specifically, its characteristic variety is the closure of $\{(x, sdf(x)) \ ; \ f(x) \neq 0, s \in \mathbb{C}\}$ in the cotangent bundle T^*X and thus it has dimension $n+1$. $\mathscr{N}^\alpha_{\tilde{f}/\tilde{g}}$ has the same characteristic variety when $\frac{\tilde{f}}{\tilde{g}} = \frac{1}{z_1^{N_1} \cdots z_m^{N_m}}$ so it is subholonomic as well.

The rest of the proof follows the same lines of reasoning of Kashiwara so we will just sketch the key points and refer to [Kas77] for details. Let

$$\mathscr{N} = \mathbb{R}^0 \pi_+ \mathscr{N}^\alpha_{\tilde{f}/\tilde{g}} = \mathbb{R}^0 \pi_* (D_{U\leftarrow X} \otimes^{\mathbf{L}}_D \mathscr{N}^\alpha_{\tilde{f}/\tilde{g}})$$

be the degree zero direct image of $\mathscr{N}^\alpha_{\tilde{f}/\tilde{g}}$, where $D_{U\leftarrow X}$ denotes the transfer bimodule. There is a canonical section $u \in \mathscr{N}$ associated to $\mathbf{1}_{U\leftarrow X} \otimes \frac{\tilde{\mathbf{f}}^{\mathbf{s}}}{\tilde{\mathbf{g}}^{\mathbf{s}+\alpha}} \in D_{U\leftarrow X} \otimes^{\mathbf{L}}_D \mathscr{N}^\alpha_{\tilde{f}/\tilde{g}}$ that allows us to describe a D-submodule

$$\mathscr{N}' = D[s]u \subseteq \mathscr{N}$$

This inclusion is indeed a morphism of $D\langle s,t\rangle$-modules. We have that $\mathscr{N}$ is subholonomic as D-module [Kas77, Lemma 5.7] and moreover $\mathscr{N}/t\mathscr{N}$ is holonomic so the action of s has a minimal polynomial which we denote $b_{\mathscr{N}}(s)$. We have that $\mathscr{N}'/t\mathscr{N}'$ is holonomic as well which leads to a polynomial $b_{\mathscr{N}'}(s)$. Using that $\mathscr{N}/\mathscr{N}'$ is also holonomic we get by [Kas77, Proposition 5.11] that there exist $\ell \geq 0$ large enough such that $t^\ell \mathscr{N} \subseteq \mathscr{N}'$. From the relation

$$b_{\mathscr{N}}(s+j)t^j\mathscr{N} = t^j b_{\mathscr{N}}(s)\mathscr{N} \subseteq t^{j+1}\mathscr{N},$$

$j \geq 0$, we obtain

$$b_{\mathscr{N}}(s+\ell)\cdots b_{\mathscr{N}}(s)\mathscr{N}' \subseteq b_{\mathscr{N}}(s+\ell)\cdots b_{\mathscr{N}}(s)\mathscr{N} \subseteq t^{\ell+1}\mathscr{N} \subseteq t\mathscr{N}'$$

and thus $b_{\mathscr{N}'}(s) \mid b_{\mathscr{N}}(s+\ell)\cdots b_{\mathscr{N}}(s)$. Finally, we have to relate this equation to (4.1.1). On the one hand, since $b^{\alpha}_{\tilde{f}/\tilde{g}}(s)\mathscr{N}^{\alpha}_{\tilde{f}/\tilde{g}} \subseteq t\mathscr{N}^{\alpha}_{\tilde{f}/\tilde{g}}$ there exist a D_X-module endomorphism $h : \mathscr{N}^{\alpha}_{\tilde{f}/\tilde{g}} \to \mathscr{N}^{\alpha}_{\tilde{f}/\tilde{g}}$ such that $b^{\alpha}_{\tilde{f}/\tilde{g}}(s) = t\circ h$. Applying the functor $\mathbb{R}^0\pi_+$ we get

$$b^{\alpha}_{\tilde{f}/\tilde{g}}(s)\mathscr{N} = t\circ \mathbb{R}^0\pi_+(h)\mathscr{N} \subseteq t\mathscr{N}$$

and thus $b_{\mathscr{N}}(s) \mid b^{\alpha}_{\tilde{f}/\tilde{g}}(s)$. On the other hand we have a surjection of $D\langle s,t\rangle$-modules $\mathscr{N}' \twoheadrightarrow \mathscr{N}^{\alpha}_{f/g}$ and thus $b_{\mathscr{N}'}(s)\mathscr{N}^{\alpha}_{f/g} \subseteq t\mathscr{N}^{\alpha}_{f/g}$ which gives $b^{\alpha}_{f/g}(s)) \mid b_{\mathscr{N}'}(s)$ and Equation (4.1.1) follows.

The refinement given by Lichtin [Lic89] requires to pass from left to right D-modules in order to plug in all the information given by the relative canonical divisor. Namely, for a left D-module $\mathscr{M}$ we consider the right D-module $\mathscr{M}^{(r)} := \omega_R \otimes_R \mathscr{M}$. In local coordinates, this operation is given by the involution on D that sends a differential operator δ to its adjoint δ^* described uniquely by the properties $(\delta\delta')^* = \delta'^*\delta^*$, $f^* = f$ for any $f \in R$ and $\partial_i^* = -\partial_i$. This gives an equivalence between left and right D-modules. We can extend this to an equivalence between left and right $D\langle s,t\rangle$-modules by taking $t^* = t$, $\partial_t^* = -\partial_t$ and $s^* = (-\partial_t t)^* = t\partial_t = \partial_t t - 1 = -s-1$. For an element u in a $D\langle s,t\rangle$-module $\mathscr{M}$ we set $u^* = dx_1 \wedge \cdots \wedge dx_n \otimes u$. Then we have $u^*\delta^* = (\delta u)^*$ for any $\delta \in D\langle s,t\rangle$.

Now, for $u = \frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}} \in \mathscr{N}^{\alpha}_{f/g}$ we have that the functional equation

$$\delta(s) f \frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}} = b(s)\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}} \quad \text{becomes} \quad f\left(\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}}\right)^*(\delta(s))^* = (b(s))^*\left(\frac{\mathbf{f}^{\mathbf{s}}}{\mathbf{g}^{\mathbf{s}+\alpha}}\right)^*$$

and the minimal polynomials satisfying these equations are related by

$$b^{\alpha}_{f/g}(s) = b_u(s) = b_{u^*}(-s-1) = b^{\alpha}_{(f/g)^*}(-s-1). \tag{4.1.2}$$

In particular $b_{u^*}(s)$ is the minimal polynomial of the action of s on $(\mathscr{N}^{\alpha}_{f/g})^{(r)}/(\mathscr{N}^{\alpha}_{f/g})^{(r)}t$.

Let $\pi : X \to U$ be a log resolution of f/g. We may construct $(\mathscr{N}^{\alpha}_{\tilde{f}/\tilde{g}})^{(r)} = \omega_X \otimes_{\mathcal{O}_X} \mathscr{N}^{\alpha}_{\tilde{f}/\tilde{g}}$ as before but, following Lichtin, we consider the $D_X\langle s,t\rangle$-submodule

$$\mathscr{N}_v := vD_X[s]$$

with $v = \pi^*(dx_1 \wedge \cdots \wedge dx_n) \otimes \frac{\tilde{\mathbf{f}}^{\mathbf{s}}}{\tilde{\mathbf{g}}^{\mathbf{s}+\alpha}} \in \left(\mathscr{N}^{\alpha}_{\tilde{f}/\tilde{g}}\right)^{(r)}$. In local coordinates around a point $p \in X$ we have either

$$v = wz_1^{N_1 s+k_1}\cdots z_m^{N_m s+k_m} \quad \text{or} \quad v = wz_1^{-N_1 s+k_1}\cdots z_m^{-N_m s+k_m},$$

where w is a unit. The Bernstein-Sato polynomial associated to these elements is either

$$b_v(s) = \prod_{i=1}^{m} \left(\prod_{j=1}^{N_i} \left(s + \frac{k_i + j}{N_i} \right) \right) \quad \text{or} \quad b_v(s) = 1,$$

and we have a relation with the Bernstein-Sato polynomial associated to $\mathscr{N}_v$, i.e. the minimal polynomial of the action of s on $\mathscr{N}_v/\mathscr{N}_v t$, given by $b_{\mathscr{N}_v}(s) = b_v(-s-1)$. By considering a cover of X we get the global Bernstein-Sato polynomial that we also denote by $b_{\mathscr{N}_v}(s)$ if no confusion arises.

Then the refined version of Lichtin result follows from proving the existence of an integer $\ell \geq 0$ such that the polynomial $b_{u^*}(s)$ in Equation (4.1.2) satisfies

$$b_{u^*}(s) \mid b_{\mathscr{N}_v}(s) b_{\mathscr{N}_v}(s-1) \cdots b_{\mathscr{N}_v}(s-\ell). \tag{4.1.3}$$

The rest of the proof is analogous to Kashiwara's proof of rationality but working in the category of right D-modules.

□

Corollary 4.15. *Let $f, g \in R$ be nonzero holomorphic functions. The roots of $b_{f/g}(s)$ are negative rational numbers. Moreover, they are contained in the set*

$$\left\{ \frac{k_i + 1 + \ell}{N_{f/g,i}} \; ; \; \ell \geq 0, \; i \in I_0 \right\}.$$

Proof. It follows from the fact that $b_{f/g}(s)$ divides $b^1_{f/g}(s)$. □

5. Archimedean Local Zeta Functions

In this section we use the Bernstein-Sato polynomial of f/g to study Archimedean local zeta functions. Let $\mathcal{O}_{\mathbb{C}^n,0}$ be the ring of germs of holomorphic functions around a point $0 \in \mathbb{C}^n$, which we identify with $R = \mathbb{C}\{x_1, \ldots, x_n\}$ by taking local coordinates. For definiteness we take a small neighborhood of the origin $U \subseteq \mathbb{C}^n$ where f and g are both holomorphic. By a test function we mean a smooth function with compact support on $\mathbb{C}^n$. Then the local zeta function attached to a test function ϕ and f/g is defined as

$$Z_\phi(s, f/g) = \int_{U \setminus H} \phi(x) \left| \frac{f(x)}{g(x)} \right|^{2s} \, \mathrm{d}x,$$

where s is a complex number and $H = f^{-1}(0) \cup g^{-1}(0)$. When $g = 1$ (the classical case), local zeta functions were introduced by Gel'fand and Shilov in the 50's [GS16]. In that case is not difficult to show that $Z_\phi(s, f/1) = Z_\phi(s, f)$ converges on the half plane $\{s \in \mathbb{C} \; ; \mathrm{Re}(s) > 0\}$ and defines a holomorphic function there. In contrast, the convergence of the parametric integral $Z_\phi(s, f/g)$ is a delicate matter as Veys and Zúñiga-Galindo noted [VZG17, Remark 2.1 (1)]. In particular they explain that the convergence of the integral does not follow from the fact that ϕ has compact support and they use an embedded resolution of H to show that the integral has a region of convergence [VZG17, Theorem 3.5 (1)]. As in the classical case, the meromorphic continuation of $Z_\phi(s, f/g)$ does not depend on the set U.

Building on the properties of the log canonical threshold of f and g we describe below a simple region of convergence for $Z_\phi(s, f/g)$. First, we recall the definition of the log canonical threshold of f and some of its properties regarding the local zeta function of f and ϕ.

Definition 5.1. Let f be a holomorphic function in an open set $U \subseteq \mathbb{C}^n$. The log-canonical threshold of f (at the origin) is the number

$$\operatorname{lct}_0(f) = \sup\left\{\lambda \in \mathbb{R}_{>0} \ ; \ \int_{B_\varepsilon(0)} |f|^{-2\lambda} < \infty, \text{ for some } \varepsilon > 0\right\}.$$

It is known that $\operatorname{lct}_0(f)$ is a positive rational number. Furthermore, we have that $Z_\phi(s, f)$ is an holomorphic function on $\operatorname{Re}(s) > -\operatorname{lct}_0(f)$. Using this fact one may show that $Z_\phi(s, g^{-1}) = \int_{U\setminus g^{-1}(0)} \phi(x)\, |g(x)|^{-2s}\ \mathrm{d}x$, is holomorphic on $\operatorname{Re}(s) < \operatorname{lct}_0(g)$.

Lemma 5.2. *Let f and g be nonzero holomorphic functions. Then, the integral $Z_\phi(s, f/g)$ converges for $-\operatorname{lct}_0(f) < \operatorname{Re}(s) < \operatorname{lct}_0(g)$. Furthermore, it defines a holomorphic function there.*

Proof. First we show that $Z_\phi(s, f/g)$ is finite on the interval $-\operatorname{lct}_0(f) < \operatorname{Re}(s) \leq 0$. Note that in this region the function $|g(x)|^{-2s}$ is well defined and continuous over $U \setminus H$. Then

$$\left|\int_{U\setminus H} \phi(x) \left|\frac{f(x)}{g(x)}\right|^{2s} \ \mathrm{d}x\right| \leq \int_{U\setminus H} |\phi(x)| \left|\frac{f(x)}{g(x)}\right|^{2\operatorname{Re}(s)} \ \mathrm{d}x < \infty,$$

since the the support of ϕ is compact and $\left|\frac{f(x)}{g(x)}\right|^{2\operatorname{Re}(s)}$ is continuous. A symmetric argument shows that $Z_\phi(s, f/g) < \infty$ for $0 \leq \operatorname{Re}(s) < \operatorname{lct}_0(g)$. The fact that $Z_\phi(s, f/g)$ defines a holomorphic function on $-\operatorname{lct}_0(f) < \operatorname{Re}(s) < \operatorname{lct}_0(g)$ can be proved as in the classical case by showing that $\partial/\partial\bar{s}(Z_\phi(s, f/g)) = 0$, which follows in particular from Lebesgue's dominated convergence theorem [Igu78, Theorem 3.1]. ☐

Remark 5.3. The region of convergence of Lemma 5.2 is in general smaller that the region of convergence described in the work of Veys and Zuñiga-Galindo [VZG17]. According to their Example 3.13 (4), the integral $Z_\phi(s, y^2 + x^4/x^2 + y^4)$ converges in $-3/2 < \operatorname{Re}(s) < 3/2$, whereas Lemma 5.2 shows that the integral converges in $-3/4 < \operatorname{Re}(s) < 3/4$. We point out that the region of convergence of Veys and Zuñiga-Galindo is optimal in the sense that the endpoints give actual poles for $Z_\phi(s, f/g)$, i.e. $-3/2$ and $3/2$ are poles of $Z_\phi(s, y^2 + x^4/x^2 + y^4)$.

Another property of $Z_\phi(s, f)$ that one may also study is the existence of a meromorphic continuation to the whole complex plane. This fact was conjectured by Gel'fand in the 50's and proved by Bernstein and Gel'fand [BG69], Atiyah [Ati70] using resolution of singularities, and Bernstein [Ber72] using the Bernstein-Sato polynomial. In the case of meromorphic functions, Veys and Zúñiga-Galindo [VZG17, Theorem 3.5 (2)] showed that $Z_\phi(s, f/g)$ has a meromorphic continuation to the whole complex plane and its poles are described by means of the numerical data of an embedded resolution of singularities of H. By using the theory of Bernstein-Sato polynomials developed in Sections 3 and 4 we present a proof of the meromorphic continuation of $Z_\phi(s, f/g)$, resembling the original proof of Bernstein [Ber72], giving in addition a shorter list of candidate poles.

For a differential operator $\delta(s) \in D_{R|\mathbb{C}}[s]$ we denote the *conjugate operator* by $\overline{\delta}(s)$. This is the operator obtained from δ by replacing x_i with $\overline{x_i}$ and ∂_{x_i} with $\partial_{\overline{x_i}}$. A simple calculation shows that the functional equation in Lemma 4.2 implies

$$\delta(s)\overline{\delta}(s)\frac{|f|^{2(s+1)}}{|g|^{2(s+\alpha)}} = b^{\alpha}_{f/g}(s)^2\frac{|f|^{2s}}{|g|^{2(s+\alpha)}} \text{ in } \mathscr{M}^{\alpha}_{f/g}[s]. \tag{5.0.1}$$

Theorem 5.4. *Let f and g be nonzero holomorphic functions and take $\alpha = \operatorname{lct}_0 g$. The local zeta function $Z_\phi(s, f/g)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$, and its poles are contained in the set*

$$\Big\{\zeta - k\alpha \ ; \ k \in \mathbb{Z}_{\geq 0}\Big\}_{\zeta \text{ root of } b^{\alpha}_{f/g}(s)} \bigcup \Big\{k\alpha - \xi \ ; \ k \in \mathbb{Z}_{\geq 0}\Big\}_{\xi \text{ root of } b^{\alpha}_{g/f}(s)}.$$

In particular, the poles of $Z_\phi(s, f/g)$ are rational numbers.

Proof. Let $b^{\alpha}_{f/g}(s) \in \mathbb{C}[s]$ be the nonzero polynomial of Lemma 4.2, note that there exists a map of $D_{R|\mathbb{C}}$-modules

$$\psi_\lambda : \mathscr{M}^{\alpha}_{f/g}[s] \to \operatorname{Frac}(\mathbb{C}\{x_1, \dots, x_d\})$$

that maps $s \mapsto \lambda$ and $\frac{\mathbf{f^s}}{\mathbf{g^{s+\alpha}}} \mapsto \frac{f^\lambda}{g^{\lambda+\alpha}}$ for every $\lambda \in \mathbb{C}$. We also notice that after possible shrinking U, (5.0.1) remains valid for $f, g \in R$, and then multiplication of $Z_\phi(s, f/g)$ by $b^{\alpha}_{f/g}(s)^2$ gives

$$\begin{aligned}
b^{\alpha}_{f/g}(s)^2\, Z_\phi(s, f/g) &= \int_{U\setminus H} \phi(x) b^{\alpha}_{f/g}(s)^2 \left|\frac{f(x)}{g(x)}\right|^{2s} \mathrm{d}x \\
&= \int_{U\setminus H} \phi(x)|g(x)|^{2\alpha}\, b^{\alpha}_{f/g}(s)^2 \frac{1}{|g(x)|^{2\alpha}} \left|\frac{f(x)}{g(x)}\right|^{2s} \mathrm{d}x \\
&= \int_{U\setminus H} \phi(x)|g(x)|^{2\alpha} \left(\delta(s)\overline{\delta}(s)\frac{|f(x)|^{2(s+1)}}{|g(x)|^{2(s+\alpha)}}\right) \mathrm{d}x \\
&= \int_{U\setminus H} \phi(x)|g(x)|^{2\alpha} \left(\delta(s)\overline{\delta}(s)|f(x)|^{2-2\alpha}\left|\frac{f(x)}{g(x)}\right|^{2(s+\alpha)}\right) \mathrm{d}x \\
&= \int_{U\setminus H} \left(|f(x)|^{2-2\alpha}\delta^*(s)\overline{\delta}^*(s)\cdot\phi(x)|g(x)|^{2\alpha}\right) \left|\frac{f(x)}{g(x)}\right|^{2(s+\alpha)} \mathrm{d}x \\
&= Z_\psi(s+\alpha, f/g).
\end{aligned}$$

Here δ^* denotes the adjoint operator of δ and $\psi = |f(x)|^{2-2\alpha}\delta^*(s)\overline{\delta}^*(s)\cdot\phi(x)|g(x)|^{2\alpha}$ is a complex test function. By Lemma 5.2, $Z_\psi(s+\alpha, f/g)$ converges for $\beta - \alpha < \operatorname{Re}(s) < 0$ and thus

$$Z_\phi(s, f/g) = \frac{Z_\psi(s+\alpha, f/g)}{b^{\alpha}_{f/g}(s)^2}$$

converges in $\beta - \alpha < \operatorname{Re}(s) < \alpha$ outside the possible roots of $b_{f/g}(s)$. If we now multiply $Z_\psi(s+\alpha, f/g)$ by $b^{\alpha}_{f/g}(s+\alpha)^2$ and repeat the reasoning above, we may conclude that $Z_\phi(s, f/g)$ can be extended meromorphically to $\beta - 2\alpha < \operatorname{Re}(s) < \alpha$ with possible poles in the roots of $b^{\alpha}_{f/g}(s) \cdot b^{\alpha}_{f/g}(s+\alpha)$. Iterating this process, we obtain a meromorphic extension to $\mathrm{Re}(s) < \alpha$ for $Z_\phi(s, f/g)$, and the possible poles of the zeta function are contained in the set

$$\Big\{\zeta - k\alpha \ ; \ k \in \mathbb{Z}_{\geq 0}\Big\}_{\zeta \text{ root of } b^{\alpha}_{f/g}(s)}.$$

For the continuation 'to the right' of $Z_\phi(s, f/g)$ we use Lemma 4.2 in the following version

$$\gamma(-s)\frac{f^{\alpha-s}}{g^{1-s}} = b^{\alpha}_{g/f}(-s)\frac{f^{s-\alpha}}{g^{s}},$$

which implies the analogue of 5.0.1: $\gamma(-s)\overline{\gamma}(-s)\frac{|f|^{2(\alpha-s)}}{|g|^{2(1-s)}} = b^{\alpha}_{g/f}(-s)^2\frac{|f|^{2(s-\alpha)}}{|g|^{2s}}$. If we multiply $Z_\phi(s, f/g)$ by $b^{\alpha}_{g/f}(-s)^2$ we obtain

$$Z_\phi(s, f/g) = \frac{Z_\varphi(\alpha - s, f/g)}{b^{\alpha}_{g/f}(-s)^2},$$

showing that $Z_\phi(s, f/g)$ converges in $\beta < \mathrm{Re}(s) < \alpha - \beta$ away from the possible roots of $b^{\alpha}_{g/f}(s)$. Further iteration of this process provides the desired conclusion. The rationality of the poles of $Z_\phi(s, f/g)$ follows from Theorem 4.14. □

Remark 5.5. Veys and Zúñiga-Galindo study local zeta functions for meromorphic functions over local fields of characteristic zero [VZG17]. The statement and proof of Lemma 5.2 are also valid in this generality, providing also a simple region of convergence for the non Archimedean $Z_\phi(s, f/g)$, cf. [VZG17, Theorem 3.2 (1)]. We do not known if the theory developed in sections 3 and 4 is also valid in this generality, but it is certainly true over Archimedean local fields of characteristic zero, that is, $\mathbb{R}$ or $\mathbb{C}$. The proof of Theorem 5.4 for the real case can be given by adapting the ideas of our proof and following the lines of Igusa's work [Igu00, Theorem 5.3.1].

6. Multiplier ideals: analytic construction

In this section we describe the theory of analytic multiplier ideals for meromorphic functions and relate their jumping numbers to roots of the meromorphic Bernstein-Sato polynomial. To this end, we use the previous notation, identifying $\mathcal{O}_{\mathbb{C}^n,0}$ with the ring $R = \mathbb{C}\{x_1, \dots, x_n\}$ by taking local coordinates.

Definition 6.1. Let $f, g \in R$ be nonzero elements such that f/g is not constant and $\lambda \in \mathbb{R}_{\geq 0}$. We define the multiplier ideal (at the origin) of f/g at λ by

$$\mathcal{J}\left(\left(\frac{f}{g}\right)^{\lambda}\right) = \left\{h \in R \ ; \ \int_{B_r(0)} \frac{|h|^2|g|^{2\lambda}}{|f|^{2\lambda}} < \infty \text{ for some } r > 0\right\},$$

where $B_r(0)$ denotes a closed ball of radius $r > 0$ around the origin 0.

The following useful remark shows that we can describe the meromorphic multiplier ideal from the theory of mixed multiplier ideals $\mathcal{J}(f^{\lambda_1}g^{\lambda_2})$ associated to the pair of functions.

Remark 6.2. Let $f, g \in R$ be nonzero elements and $\lambda \in \mathbb{R}$. Pick $t \in \mathbb{N}$ such that $t \geq \lambda$. Then,

$$\begin{aligned}
\mathcal{J}\left(f^{\lambda} g^{t-\lambda}\right) : g^t &= \left\{h \in R\ ;\ hg^t \in \mathcal{J}\left(f^{\lambda} g^{t-\lambda}\right)\right\} \\
&= \left\{h \in R\ ;\ \int_{B_r(0)} \frac{|h|^2 |g|^{2t}}{|f|^{2\lambda}|g|^{2t-2\lambda}} < \infty \text{ for some } r > 0\right\} \\
&= \left\{h \in R\ ;\ \int_{B_r(0)} \frac{|h|^2 |g|^{2\lambda}}{|f|^{2\lambda}} < \infty \text{ for some } r > 0\right\} \\
&= \mathcal{J}\left(\left(\frac{f}{g}\right)^{\lambda}\right).
\end{aligned}$$

Remark 6.3. Let $f, g \in R$ be nonzero elements and $\lambda \in \mathbb{R}$. Then,

$$\mathcal{J}\left(\left(\frac{f}{g}\right)^{\lambda}\right) = \mathcal{J}\left(\left(\frac{g}{f}\right)^{-\lambda}\right).$$

Definition 6.4. Let $f, g \in R$ be nonzero elements and $\lambda \in \mathbb{R}$. We say that λ is a jumping number for f/g if for every $\varepsilon > 0$ we have either

$$\mathcal{J}\left(\left(\frac{f}{g}\right)^{\lambda}\right) \neq \mathcal{J}\left(\left(\frac{f}{g}\right)^{\lambda-\varepsilon}\right)$$

or

$$\mathcal{J}\left(\left(\frac{f}{g}\right)^{\lambda}\right) \neq \mathcal{J}\left(\left(\frac{f}{g}\right)^{\lambda+\varepsilon}\right).$$

We stress that multiplier ideals of meromorphic functions are not necessarily continuous on neither side. For instance, if $g = 1$, it is known that

$$\mathcal{J}\left(f^{\lambda}\right) = \mathcal{J}\left(f^{\lambda+\varepsilon}\right)$$

for small enough $\varepsilon > 0$. In contrast, if $f = 1$,

$$\mathcal{J}\left(\frac{1}{g^{\lambda}}\right) \neq \mathcal{J}\left(\frac{1}{g^{\lambda-\varepsilon}}\right)$$

for small enough $\varepsilon > 0$.

Proposition 6.5. *Let $f, g \in R$ be a nonzero elements and $\lambda \in \mathbb{R}_{\geq 0}$. Then,*

$$\mathcal{J}\left(\left(\frac{f}{g}\right)^{\lambda+\ell}\right) = \left(\frac{f^{\ell}}{g^{\ell}}\left(\mathcal{J}\left(\frac{f}{g}\right)^{\lambda}\right)\right) \bigcap R$$

for every $\ell \in \mathbb{N}$. In particular, if $\lambda + 1$ is a jumping number, then λ is a jumping number.

Proof. We fix $t \geq \lambda + 1$. The result follows from the equality

$$\mathcal{J}\left(f^{\lambda+\ell} g^{t-\lambda-1}\right) : g^t = \left(\left(\frac{f^{\ell}}{g^{\ell}} \mathcal{J}(f^{\lambda} g^{t-\lambda})\right) : g^t\right) \bigcap R = \left(\frac{f^{\ell}}{g^{\ell}}\left(\mathcal{J}(f^{\lambda} g^{t-\lambda}) : g^t\right)\right) \bigcap R.$$

given by Remark 6.2. ∎

We note that Proposition 6.5 gives a weaker version of Skoda's Theorem [Laz04, 9.6.21].

Theorem 6.6. *Let $f, g \in R$ be nonzero elements and $\lambda \in \mathbb{R}$. Then, the set of jumping numbers of f/g is a set of rational numbers with no accumulation points.*

Proof. It suffices to show that the set of jumping numbers of f/g in $[0,t]$ is a finite set of rational numbers for any $t \in \mathbb{Z}_{\geq 0}$. By Remark 6.2, the jumping numbers of $\mathcal{J}((\frac{f}{g})^{\lambda})$ is a subset of the jumping numbers of $\mathcal{J}\left(f^{\lambda}g^{t-\lambda}\right)$. The last set is the intersection of the jumping walls of $\mathcal{J}(f^{\lambda_1}g^{\lambda_2})$ with the line $\lambda_1 + \lambda_2 = t$. We conclude that the set of jumping numbers of f/g is a set of rational numbers with no accumulation points, because the jumping walls are the boundaries of the constancy regions of $\mathcal{J}(f^{\lambda_1}g^{\lambda_2})$ and, by construction, they are defined by linear equations with rational coefficients and there are only finitely many in any bounded part of the positive orthant $\mathbb{R}^2_{\geq 0}$. $\square$

We state the following extension in our setting of the classical result of Ein, Lazarsfeld, Smith and Varolin [ELSV04, Theorem 2.1].

Theorem 6.7. *Let $f, g \in R$ be nonzero elements and λ be a jumping number of f/g such that $\lambda \in (1 - \mathrm{lct}_0(g), 1]$. Then, $-\lambda$ is a root of the Bernstein-Sato polynomial of f/g.*

Proof. Suppose that $\lambda > 1 - \mathrm{lct}_0(g)$ is a jumping number of f/g and take $1 - \mathrm{lct}_0(g) < \lambda' < \lambda$. If we fix an arbitrary $c \in [\lambda', \lambda)$ then by the definition of jumping number there exist a function $h \in \mathcal{J}\left(\left(\frac{f}{g}\right)^{c}\right)$ such that $h \notin \mathcal{J}\left(\left(\frac{f}{g}\right)^{\lambda-\varepsilon}\right)$, (we may assume without lost of generality that the sign of the definition is negative). Equivalently, there exist r, r' (with $0 < r' < r$) such that

$$\int_{B_r(0)} \frac{|h|^2|g|^{2c}}{|f|^{2c}} < \infty \quad \text{and} \quad \int_{B_{r'}(0)} \frac{|h|^2|g|^{2(\lambda-\varepsilon)}}{|f|^{2(\lambda-\varepsilon)}} = \infty. \tag{6.0.1}$$

We shall show that the first integral in Equation (6.0.1) becomes unbounded when c approaches λ. To do so, note first that Theorem 4.5 implies the following analogue of Equation (5.0.1)

$$\delta(s)\overline{\delta}(s)\frac{|f|^{2(s+1)}}{|g|^{2(s+1)}} = b_{f/g}(s)^2\frac{|f|^{2s}}{|g|^{2s}}.$$

In particular, for $s = -c$ and for any positive test function ϕ supported on $B_r(0)$ we have

$$\begin{aligned}\int_{B_r(0)} |h|^2\phi\ b_{f/g}(-c)^2\frac{|g|^{2c}}{|f|^{2c}} &= \int_{B_r(0)} |h|^2\phi\ \delta(-c)\overline{\delta}(-c)\frac{|f|^{2(-c+1)}}{|g|^{2(-c+1)}} \\ &= \int_{B_r(0)} \frac{|f|^{2(-c+1)}}{|g|^{2(-c+1)}}\delta^*(-c)\overline{\delta}^*(-c)(|h|^2\phi),\end{aligned} \tag{6.0.2}$$

where $\delta^*(s)$ and $\overline{\delta}^*(s)$ denote the respective adjoint operators. Since $c > \lambda' > 1 - \mathrm{lct}_0(g)$, the proof of Lemma 5.2 implies that the right-hand side of Equation (6.0.2) is uniformly bounded by a positive number depending only on h and φ. If we now take ϕ as the characteristic function of the ball $B_{r'}(0)$, then

$$\int_{B_r(0)} |h|^2\phi\ b_{f/g}(-c)^2\frac{|g|^{2c}}{|f|^{2c}} \geq \int_{B'_r(0)} |h|^2\phi\ b_{f/g}(-c)^2\frac{|g|^{2c}}{|f|^{2c}} = b_{f/g}(-c)^2\int_{B_{r'}(0)} \frac{|h|^2|g|^{2c}}{|f|^{2c}}.$$

Our previous discussion implies that $b_{f/g}(-c)^2 \int_{B_{r'}(0)} \frac{|h|^2|g|^{2c}}{|f|^{2c}}$ is bounded for any $c \in [\lambda', \lambda)$, but Equation (6.0.1) shows that the last integral tends to infinity as c tends to λ, implying that $b_{f/g}(-\lambda) = 0$.

□

7. Multiplier ideals: algebraic construction

In this section we define the algebraic version of multiplier ideals for meromorphic functions using log resolutions. Let $\mathcal{O}_{\mathbb{C}^n,0}$ be the ring of germs of holomorphic functions around a point $0 \in \mathbb{C}^n$, which we identify with $R = \mathbb{C}\{x_1, \ldots, x_n\}$ by taking local coordinates. Throughout this section we use the notation introduced in Section 2.4 concerning the numerical data associated to a log resolution $\pi : X \to U$ of the meromorphic germ f/g.

Definition 7.1. Let $f, g \in R$ be nonzero elements such that f/g is not constant and $\lambda \in \mathbb{R}_{\geq 0}$. We define the 0-multiplier ideal of f/g at λ as the stalk at the origin of

$$\mathcal{J}\left(\left(\frac{f}{g}\right)^{\lambda}\right) = \pi_* \mathcal{O}_X(-[\lambda \cdot \tilde{F}] + K_\pi),$$

where $[\cdot]$ denotes the integer part of a real number or $\mathbb{R}$-divisor. If no confusion arises we denote the stalk at the origin in the same way and thus $\mathcal{J}\left(\left(\frac{f}{g}\right)^{\lambda}\right) \subseteq R$.

Remark 7.2.

(1) The above definition is independent of the choice of the log resolution π. This follows from analogous considerations to those in the classical case when $g = 1$ [Laz04, Example 9.1.16, Theorem 9.2.18 & Lemma 9.2.19].

(2) Given $h \in R$, the condition $h \in \mathcal{J}((\frac{f}{g})^{\lambda})$ is equivalent to
$$\mathrm{ord}_{E_i} \pi^* h \geq [\lambda \cdot N_{f/g,i}] - k_i \quad \text{for every } \ i \in I.$$

(3) Since $\mathrm{ord}_{E_i} \pi^* h g^t = \mathrm{ord}_{E_i} \pi^* h + t N_{g,i}$, and $[\lambda \cdot N_{f/g,i}] + t N_{g,i} = [\lambda \cdot N_{f,i} + (t-\lambda) \cdot N_{g,i}]$ for all $t \in \mathbb{N}$ such that $t \geq \lambda$, we have that the condition $h \in \mathcal{J}((\frac{f}{g})^{\lambda})$ is equivalent to the conditions $\mathrm{ord}_{E_i} \pi^* h g^t \geq [\lambda \cdot N_{f,i} + (t-\lambda) \cdot N_{g,i}] + k_i$ for all $t \in \mathbb{N}$ with $t \geq \lambda$ and all $i \in I$. This gives a different proof of Remark 6.2.

(4) Given $h \in R$, we have $\mathrm{ord}_{E_i} \pi^* h \geq 0$ for all $i \in I$. Since $\mathcal{J}((\frac{f}{g})^{\lambda}) \subseteq R$, it follows that
$$\mathcal{J}\left(\left(\frac{f}{g}\right)^{\lambda}\right) = \pi_* \mathcal{O}_X(-[\lambda \cdot \tilde{F}_0] + K_\pi).$$

(5) One can associate ∞-multiplier ideals $\mathcal{J}^{\infty}((\frac{f}{g})^{\lambda})$ to the meromorphic germ f/g and the parameter λ as follows $\mathcal{J}^{\infty}((\frac{f}{g})^{\lambda}) = \mathcal{J}((\frac{g}{f})^{\lambda})$.

The following properties of multiplier ideals $\mathcal{J}((\frac{f}{g})^{\lambda})$ are analogous to the case of multiplier ideals associated to holomorphic germs f, that is, when $g = 1$.

Lemma 7.3. *There exists a discrete strictly increasing sequence of rational numbers*
$$\lambda_1 < \lambda_2 < \cdots$$

such that

$$\mathcal{J}\left(\left(\frac{f}{g}\right)^{\lambda_{i+1}}\right) \subsetneq \mathcal{J}\left(\left(\frac{f}{g}\right)^{c}\right) = \mathcal{J}\left(\left(\frac{f}{g}\right)^{\lambda_i}\right)$$

for $c \in [\lambda_i, \lambda_{i+1})$*, and all* i*. These rational numbers are called the* 0*-jumping numbers of* f/g*.*

Remark 7.4.

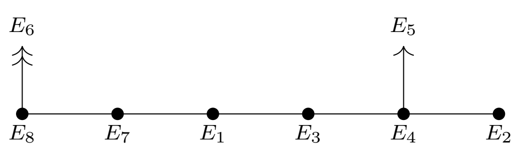
- The candidate 0-jumping numbers of f/g have the form $\frac{k_i+1+\ell}{N_{f/g,i}}$ with $\ell \in \mathbb{Z}_{\geq 0}$.
- Let $\lambda \in \mathbb{Q}$ be a candidate jumping number of f associated to the divisor E_i. Then $\lambda \cdot \frac{N_{f,i}}{N_{f/g,i}}$ is a candidate 0-jumping number of f/g. Notice that $\frac{N_{f,i}}{N_{f/g,i}} \geq 1$.

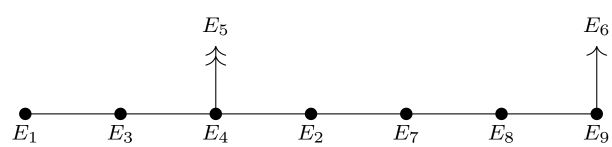
Lemma 7.5. *For any* $\lambda \in \mathbb{R}_{>0}$ *we have* $\mathcal{J}(f^\lambda) \subseteq \mathcal{J}((\frac{f}{g})^\lambda)$*. In addition, we have* $\mathcal{J}((\frac{f}{g})^n) = (f^n)$ *for any* $n \in \mathbb{Z}_{>0}$*.*

Proof. The inclusion $\mathcal{J}(f^\lambda) \subseteq \mathcal{J}((\frac{f}{g})^\lambda)$ follows from the fact that $N_{f,i}/N_{f/g,i} \geq 1$. Therefore, we have that $(f^n) = \mathcal{J}(f^n) \subseteq \mathcal{J}((\frac{f}{g})^n)$. Moreover, for any E_i in the strict transform of f we have that $N_{f,i}/N_{f/g,i} = 1$, and $k_i = 0$. Hence, if $h \in \mathcal{J}((\frac{f}{g})^n)$, we have that $\mathrm{ord}_{E_i} \pi^* h \geq n \cdot N_{f,i}$. It follows that f divides h and hence $\mathcal{J}((\frac{f}{g})^n) \subseteq (f^n)$. □

Remark 7.6. In general $\mathcal{J}((\frac{f}{g})^{\lambda+n}) \neq (f^n) \cdot \mathcal{J}((\frac{f}{g})^\lambda)$, as Example 7.7 shows. As a consequence, the periodicity of jumping numbers fails. Alternatively, Skoda's Theorem for multiplier ideals of meromorphic germs is weaker than the classic one.

Example 7.7. Let us consider $f = y^3 + x^5$, $g_1 = x$, $g_2 = y$, and f/g_i with $i = 1, 2$. The minimal resolution of fg_i is obtained after 4 point blow-ups. Denote E_i with $i = 1, 2, 3, 4$ the exceptional divisors, E_5 (resp. E_6) the strict transform of f (resp. of g_i). Notice that E_5 intersects E_4, and E_6 intersects E_1 if $i = 1$ or E_2 if $i = 2$. Two additional point blow-ups, with exceptional divisors E_7 and E_8 are needed to construct the resolution of f/g_i. The following figures and tables give the corresponding dual resolution graphs and resolution data





	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
$N_{f,i}$	3	5	9	15	1	0	3	3
$N_{g_1,i}$	1	1	2	3	0	1	2	3
k_i	1	2	4	7	0	0	2	3

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9
$N_{f,i}$	3	5	9	15	1	0	5	5	5
$N_{g_2,i}$	1	2	3	5	0	1	3	4	5
k_i	1	2	4	7	0	0	3	4	5

The set of jumping numbers of f/g_1 is $\left\{\frac{8}{12}, \frac{11}{12}, 1, \frac{23}{12}\right\} \cup \mathbb{Z}_{\geq 2}$, the set of jumping numbers of f/g_2 is $\left\{\frac{8}{10}\right\} \cup \mathbb{Z}_{>0}$ and the multiplier ideals of f/g_1 and f/g_2 are

$$\mathcal{J}\left(\left(\frac{f}{g_1}\right)^{\lambda}\right) = \begin{cases} 1 & 0 < \lambda < 8/12, \\ (x,y) & 8/12 \leq \lambda < 11/12, \\ (x^2,y) & 11/12 \leq \lambda < 1, \\ f & 1 \leq \lambda < 23/12, \\ (x,y)f & 23/12 \leq \lambda < 2 \\ f^n & n \leq \lambda < n+1, \\ & \quad \text{and } n \in \mathbb{Z}_{\geq 2}. \end{cases} \quad \text{and} \quad \mathcal{J}\left(\left(\frac{f}{g_2}\right)^{\lambda}\right) = \begin{cases} 1 & 0 < \lambda < 8/10, \\ (x,y) & 8/10 \leq \lambda < 1, \\ f^n & n \leq \lambda < n+1, \\ & \quad \text{and } n \in \mathbb{Z}_{\geq 1}. \end{cases}$$

For comparison's sake, recall that the set of jumping numbers of f is $\left\{\frac{8}{15}, \frac{11}{15}, \frac{13}{15}, \frac{14}{15}\right\} + \mathbb{Z}_{\geq 0}$, and the multiplier ideals of f are

$$\mathcal{J}(f^{\lambda}) = \begin{cases} f^n & n < \lambda < 8/15 + n, \text{ and } \lambda \neq 0, \\ (x,y)f^n & 8/15 + n \leq \lambda < 11/15 + n, \\ (x^2,y)f^n & 11/15 + n \leq \lambda < 13/15 + n, \\ (x,y)^2 f^n & 13/15 + n \leq \lambda < 14/15 + n, \\ (x^3, xy, y^2)f^n & 14/15 + n \leq \lambda < 1 + n. \end{cases}$$

Acknowledgments

We thank Nero Budur for pointing out the phenomenon described in Remark 4.11.

References

[Arn98] V. I. Arnol'd. Singularities of fractions and the behavior of polynomials at infinity. *Tr. Mat. Inst. Steklova*, 221:48–68, 1998. 2

[Ati70] M. F. Atiyah. Resolution of singularities and division of distributions. *Comm. Pure Appl. Math.*, 23:145–150, 1970. 3, 17

[Ber72] I. N. Bernšteĭn. Analytic continuation of generalized functions with respect to a parameter. *Funkcional. Anal. i Priložen.*, 6(4):26–40, 1972. 1, 2, 17

[BG69] I. N. Bernšteĭn and S. I. Gel'fand. Meromorphy of the function P^{λ}. *Funkcional. Anal. i Priložen.*, 3(1):84–85, 1969. 3, 17

[BG20] M. Bocardo-Gaspar. Poles of non-archimedean zeta functions for non-degenerate rational functions, 2020. 2

[BGZG18] Miriam Bocardo-Gaspar and W. A. Zúñiga-Galindo. Local zeta functions for rational functions and Newton polyhedra. In *Singularities and foliations. geometry, topology and applications*, volume 222 of *Springer Proc. Math. Stat.*, pages 363–385. Springer, Cham, 2018. 2

[Bjö79] J.-E. Björk. *Rings of differential operators*, volume 21 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1979. 6

[BP07] Arnaud Bodin and Anne Pichon. Meromorphic functions, bifurcation sets and fibred links. *Math. Res. Lett.*, 14(3):413–422, 2007. 2

[BPS09] Arnaud Bodin, Anne Pichon, and José Seade. Milnor fibrations of meromorphic functions. *J. Lond. Math. Soc. (2)*, 80(2):311–325, 2009. 2

[ELSV04] Lawrence Ein, Robert Lazarsfeld, Karen E. Smith, and Dror Varolin. Jumping coefficients of multiplier ideals. *Duke Math. J.*, 123(3):469–506, 2004. 21

[Gro67] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967. 3

[GS16] I. M. Gel'fand and G. E. Shilov. *Generalized functions. Vol. 1.* AMS Chelsea Publishing, Providence, RI, 2016. Properties and operations, Translated from the 1958 Russian original

[MR0097715] by Eugene Saletan, Reprint of the 1964 English translation [MR0166596]. 16

[GVL14] Manuel González Villa and Ann Lemahieu. The monodromy conjecture for plane meromorphic germs. *Bull. Lond. Math. Soc.*, 46(3):441–453, 2014. 2, 12

[GVLM16] Manuel González Villa, Anatoly Libgober, and Laurenţiu Maxim. Motivic infinite cyclic covers. *Adv. Math.*, 298:413–447, 2016. 2

[GVLM18] Manuel González Villa, Anatoly Libgober, and Laurenţiu Maxim. Motivic zeta functions and infinite cyclic covers. In *Local and global methods in algebraic geometry*, volume 712 of *Contemp. Math.*, pages 117–141. Amer. Math. Soc., Providence, RI, 2018. 2

[GZLMH98] Sabir Guseĭn-Zade, Ignacio Luengo, and Alejandro Melle Hernández. Zeta functions of germs of meromorphic functions, and the Newton diagram. *Funktsional. Anal. i Prilozhen.*, 32(2):26–35, 95, 1998. 2

[GZLMH99a] Sabir Guseĭn-Zade, Ignacio Luengo, and Alejandro Melle Hernández. On atypical values and local monodromies of meromorphic functions. *Tr. Mat. Inst. Steklova*, 225(Solitony Geom. Topol. na Perekrest.):168–176, 1999. 2

[GZLMH99b] Sabir Guseĭn-Zade, Ignacio Luengo, and Alejandro Melle Hernández. On the topology of germs of meromorphic functions and its applications. *Algebra i Analiz*, 11(5):92–99, 1999. 2

[GZLMH01] Sabir Guseĭn-Zade, Ignacio Luengo, and Alejandro Melle Hernández. Bifurcations and topology of meromorphic germs. In *New developments in singularity theory (Cambridge, 2000)*, volume 21 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 279–304. Kluwer Acad. Publ., Dordrecht, 2001. 2

[Igu78] Jun-ichi Igusa. *Forms of higher degree*, volume 59 of *Lectures on Mathematics and Physics*. Tata Institute of Fundamental Research. The Narosa Publishing House, New Delhi, 1978. 17

[Igu00] Jun-ichi Igusa. *An introduction to the theory of local zeta functions*, volume 14 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Cambridge, MA, 2000. 19

[Kas77] Masaki Kashiwara. B-functions and holonomic systems. Rationality of roots of B-functions. *Invent. Math.*, 38(1):33–53, 1976/77. 2, 13, 14, 15

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals. 20, 22

[LC17] E. León-Cardenal. An explicit formula for the local zeta function of a Laurent polynomial. *J. Number Theory*, 173:230–242, 2017. 2

[LCZG13] E. León-Cardenal and W. A. Zúñiga-Galindo. Local zeta functions for non-degenerate Laurent polynomials over p-adic fields. *J. Math. Sci. Univ. Tokyo*, 20(4):569–595, 2013. 2

[Lic89] B. Lichtin. Poles of $|f(z,w)|^{2s}$ and roots of the b-function. *Ark. Mat.*, 27(2):283–304, 1989. 2, 13, 15

[Lyu00] Gennady Lyubeznik. Finiteness properties of local cohomology modules for regular local rings of mixed characteristic: the unramified case. volume 28, pages 5867–5882. 2000. Special issue in honor of Robin Hartshorne. 5

[MNM91] Z. Mebkhout and L. Narváez-Macarro. La théorie du polynôme de Bernstein-Sato pour les algèbres de Tate et de Dwork-Monsky-Washnitzer. *Ann. Sci. École Norm. Sup. (4)*, 24(2):227–256, 1991. 5, 6, 9, 10

[NB13] Luis Núñez-Betancourt. On certain rings of differentiable type and finiteness properties of local cohomology. *J. Algebra*, 379:1–10, 2013. 5

[NT19] Tat Thang Nguyen and Kiyoshi Takeuchi. Meromorphic nearby cycle functors and monodromies of meromorphic functions (with appendix by t. saito), 2019. 2

[PS08] Anne Pichon and José Seade. Fibred multilinks and singularities $f\overline{g}$. *Math. Ann.*, 342(3):487–514, 2008. 2

[Put18] Tony J. Puthenpurakal. On the ring of differential operators of certain regular domains. *Proc. Amer. Math. Soc.*, 146(8):3333–3343, 2018. 5

[Rai12] Michel Raibaut. Motivic Milnor fibers of a rational function. *C. R. Math. Acad. Sci. Paris*, 350(9-10):519–524, 2012. 2

[Rai13] Michel Raibaut. Motivic Milnor fibers of a rational function. *Rev. Mat. Complut.*, 26(2):705–734, 2013. 2

[SS72] Mikio Sato and Takuro Shintani. On zeta functions associated with prehomogeneous vector spaces. *Proc. Nat. Acad. Sci. U.S.A.*, 69:1081–1082, 1972. 1

[ST04] Dirk Siersma and Mihai Tibăr. On the vanishing cycles of a meromorphic function on the complement of its poles. In *Real and complex singularities*, volume 354 of *Contemp. Math.*, pages 277–289. Amer. Math. Soc., Providence, RI, 2004. 2

[Tak21] Kiyoshi Takeuchi. On a berstein-sato polynomial of a meromorphic function, 2021. 3, 11

[Tib02] Mihai Tibăr. Singularities and topology of meromorphic functions. In *Trends in singularities*, Trends Math., pages 223–246. Birkhäuser, Basel, 2002. 2

[VZG17] Willem Veys and W. A. Zúñiga-Galindo. Zeta functions and oscillatory integrals for meromorphic functions. *Adv. Math.*, 311:295–337, 2017. 2, 3, 16, 17, 19

Departament de Matemàtiques and Institut de Matemàtiques de la UPC-BarcelonaTech (IMTech), Universitat Politècnica de Catalunya, Spain
E-mail address: josep.alvarez@upc.edu

Centro de investigación en Matemáticas, Apartado Postal 402, C.P. 36000, Guanajuato, GTO, México.
E-mail address: manuel.gonzalez@cimat.mx

CONACYT–Centro de investigación en Matemáticas, Unidad Zacatecas, Av. Lasec Andador Galileo Galilei, Manzana 3 Lote 7, C.P. 98160, Zacatecas, ZAC, México.
E-mail address: edwin.leon@cimat.mx

Centro de investigación en Matemáticas, Apartado Postal 402, C.P. 36000, Guanajuato, GTO, México.
E-mail address: luisnub@cimat.mx