SOME NUMERICAL INVARIANTS OF LOCAL RINGS

JOSEP ÀLVAREZ MONTANER

Abstract. Let R be a formal power series ring over a field of characteristic zero and $I \subseteq R$ be any ideal. The aim of this work is to introduce some numerical invariants of the local rings R/I by using the theory of algebraic $\mathcal{D}$-modules. More precisely, we will prove that the multiplicities of the characteristic cycle of the local cohomology modules $H_I^{n-i}(R)$ and $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$, where $\mathfrak{p} \subseteq R$ is any prime ideal that contains I, are invariants of R/I.

1. Introduction

Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension n containing the field k, and A a local ring which admits a surjective ring homomorphism $\pi : R \longrightarrow A$. Set $I = \operatorname{Ker} \pi$. G. Lyubeznik [10] defines a new set of numerical invariants of A by means of the Bass numbers $\lambda_{p,i}(A) := \mu_p(\mathfrak{m}, H_I^{n-i}(R)) := \dim_k \operatorname{Ext}_R^p(k, H_I^{n-i}(R))$. This invariant depends only on A, i and p, but neither on R nor on π. Completion does not change $\lambda_{p,i}(A)$ so one can assume $R = k[[x_1, \dots, x_n]]$, with $x_1, \dots, x_n$ independent variables.

Lyubeznik numbers can be described as the multiplicities of the characteristic cycle of the local cohomology modules $H_{\mathfrak{m}}^p(H_I^{n-i}(R))$. The aim of this work is to prove that the multiplicities of the characteristic cycle of the local cohomology modules $H_I^{n-i}(R)$ and $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$, where $\mathfrak{p} \subseteq R$ is any prime ideal that contains I, are also invariants of R/I. Among these invariants we may find the Bass numbers $\mu_p(\mathfrak{p}, H_I^{n-i}(R)) := \dim_{k(\mathfrak{p})} \operatorname{Ext}_{R_{\mathfrak{p}}}^p(k(\mathfrak{p}), H_{IR_{\mathfrak{p}}}^{n-i}(R_{\mathfrak{p}}))$.

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2. The characteristic cycle

In the sequel, $\mathcal{D}$ will denote the ring of differential operators corresponding to the formal power series ring $R = k[[x_1, \dots, x_n]]$, where k is a field of characteristic zero and $x_1, \dots, x_n$ are independent variables. For details we refer to [5], [6]. The ring $\mathcal{D}$ has a natural increasing filtration given by the order such that the corresponding associated graded ring $gr(\mathcal{D})$ is isomorphic to the polynomial ring $R[\xi_1, \dots, \xi_n]$.

Let M be a finitely generated $\mathcal{D}$-module equipped with a good filtration, i.e. an increasing sequence of finitely generated R-submodules such that the associated graded module $gr(M)$ is a finitely generated $gr(\mathcal{D})$-module. The characteristic ideal of M is the ideal in $gr(\mathcal{D}) = R[\xi_1, \dots, \xi_n]$ given by $J(M) := \operatorname{rad}(\operatorname{Ann}_{gr(\mathcal{D})}(gr(M)))$. One may prove that $J(M)$ is independent of the good

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filtration on M. The characteristic variety of M is the closed algebraic set given by:

$$C(M) := V(J(M)) \subseteq \mathrm{Spec}\,(gr(\mathcal{D})) = \mathrm{Spec}\,(R[\xi_1, \dots, \xi_n]).$$

The characteristic variety allows us to describe the support of a finitely generated $\mathcal{D}$-module as R-module. Let $\pi : \mathrm{Spec}\,(R[\xi_1, \dots, \xi_n]) \longrightarrow \mathrm{Spec}\,(R)$ be the map defined by $\pi(x, \xi) = x$. Then $\mathrm{Supp}_R(M) = \pi(C(M))$.

The characteristic cycle of M is defined as:

$$CC(M) = \sum m_i V_i$$

where the sum is taken over all the irreducible components $V_i = V(\mathfrak{q}_i)$ of the characteristic variety $C(M)$, where $\mathfrak{q}_i \in \mathrm{Spec}\,(gr(\mathcal{D}))$ and m_i is the multiplicity of the module $gr(M)_{\mathfrak{q}_i}$. Notice that the contraction of $\mathfrak{q}_i$ to R is a prime ideal so the variety $\pi(V_i)$ is irreducible.

2.1. **Bass numbers and characteristic cycle.** Let $\mathfrak{p} \in \mathrm{Spec}\,(R)$ be a prime ideal. The Bass numbers $\mu_p(\mathfrak{p}, H_I^{n-i}(R))$ of the local cohomology modules $H_I^{n-i}(R)$, where $I \subseteq R$ is any ideal, can be described as the multiplicities of the characteristic cycle of $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$. Namely we have:

Proposition 2.1. *Let $I \subseteq R$ be an ideal, $\mathfrak{p} \subseteq R$ a prime ideal and*

$$CC(H_{\mathfrak{p}}^p(H_I^{n-i}(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha} \;\; V_\alpha$$

be the characteristic cycle of the local cohomology module $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$. Then, the Bass numbers with respect to $\mathfrak{p}$ of $H_I^{n-i}(R)$ are

$$\mu_p(\mathfrak{p}, H_I^{n-i}(R)) = \lambda_{\mathfrak{p},p,i,\alpha_{\mathfrak{p}}},$$

where $\pi(V_{\alpha_{\mathfrak{p}}})$ is the subvariety of $X = \mathrm{Spec}\,(R)$ defined by $\mathfrak{p}$.

Proof. Let $\widehat{R_{\mathfrak{p}}}$ be the completion with respect to the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ of the localization $R_{\mathfrak{p}}$. Notice that $\widehat{R_{\mathfrak{p}}}$ is a formal power series ring of dimension $\mathrm{ht}\,\mathfrak{p}$. Since Bass numbers are invariant by completion we have:

$$\mu_p(\mathfrak{p}, H_I^{n-i}(R)) = \mu_p(\mathfrak{p}\widehat{R_{\mathfrak{p}}}, H_{I\widehat{R_{\mathfrak{p}}}}^{n-i}(\widehat{R_{\mathfrak{p}}})) = \mu_0(\mathfrak{p}\widehat{R_{\mathfrak{p}}}, H_{\mathfrak{p}\widehat{R_{\mathfrak{p}}}}^p(H_{I\widehat{R_{\mathfrak{p}}}}^{n-i}(\widehat{R_{\mathfrak{p}}}))),$$

where the last assertion follows from [10, Lemma 1.4]. By using [10, Theorem 3.4] we have:

$$H_{\mathfrak{p}\widehat{R_{\mathfrak{p}}}}^p(H_{I\widehat{R_{\mathfrak{p}}}}^{n-i}(\widehat{R_{\mathfrak{p}}})) = E_{\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}}/\mathfrak{p}\widehat{R_{\mathfrak{p}}})^{\mu_0(\mathfrak{p}\widehat{R_{\mathfrak{p}}}, H_{\mathfrak{p}\widehat{R_{\mathfrak{p}}}}^p(H_{I\widehat{R_{\mathfrak{p}}}}^{n-i}(\widehat{R_{\mathfrak{p}}})))}.$$

So, its characteristic cycle is:

$$CC(H_{\mathfrak{p}\widehat{R_{\mathfrak{p}}}}^p(H_{I\widehat{R_{\mathfrak{p}}}}^{n-i}(\widehat{R_{\mathfrak{p}}}))) = \mu_p(\mathfrak{p}, H_I^{n-i}(R)) \;\; V'_{\alpha_{\mathfrak{p}}},$$

where $\pi(V'_{\alpha_{\mathfrak{p}}})$ is the subvariety of $X' = \mathrm{Spec}\,\widehat{R_{\mathfrak{p}}}$ defined by the ideal $\mathfrak{p}\widehat{R_{\mathfrak{p}}}$. Notice that we have used the following fact (see [10] and [1] for details):

$$CC(H_{\mathfrak{p}\widehat{R_{\mathfrak{p}}}}^{\mathrm{ht}\,\mathfrak{p}}(\widehat{R_{\mathfrak{p}}})) = CC(E_{\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}}/\mathfrak{p}\widehat{R_{\mathfrak{p}}})) = V'_{\alpha_{\mathfrak{p}}}.$$

Finally, by using the flatness of the morphism $R \longrightarrow \widehat{R_{\mathfrak{p}}}$, this characteristic cycle can be obtained from the characteristic cycle of $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$. Namely, if

$$CC(H_{\mathfrak{p}}^p(H_I^{n-i}(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha} \;\; V_\alpha$$

is the characteristic cycle of the module $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$, then we have

$$CC(H^p_{\mathfrak{p}\widehat{R_{\mathfrak{p}}}}(H^{n-i}_{I\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}}))) = \lambda_{\mathfrak{p},p,i,\alpha_{\mathfrak{p}}} \;\; V'_{\alpha_{\mathfrak{p}}}.$$

□

2.2. **Inverse and direct image.** Some geometrical operations as the direct image have a key role in the theory of $\mathcal{D}$-modules. Our aim in this section is to give a brief survey of this operations in the particular case of the injection of $\mathbb{A}^n_k$ in $\mathbb{A}^{n+1}_k$. The main references we will use in this section are [6] and [11].

Let $\mathcal{D}_{n+1}$ and $\mathcal{D}_n$ be the rings of differential operators corresponding to $R' = k[[x_1, \dots, x_n, t]]$ and $R = k[[x_1, \dots, x_n]]$ respectively. Let M be a $\mathcal{D}_n$-module. The direct image corresponding to the injection is the $\mathcal{D}_{n+1}$-module $i_+(M)$ defined as

$$i_+(M) = k[\partial_t]\widehat{\otimes}_k M = M[\partial_t].$$

The characteristic variety of $i_+(M)$ can be computed from the characteristic variety of M. Namely, we have:

$$C(i_+(M)) = \{(\mathbf{x}, 0, \xi, \tau) \;\mid\; (\mathbf{x}, \xi) \in C(M)\} \subseteq \operatorname{Spec}(R'[\xi_1, \dots, \xi_n, \tau]),$$

where we have considered $C(M) \subseteq \operatorname{Spec}(R[\xi_1, \dots, \xi_n])$.

The direct image of local cohomology modules can be easily described. The following result is stated in the way we will use in our work.

Lemma 2.2. *Let $\mathfrak{p} \subseteq R$ be a prime ideal that contains an ideal $I \subseteq R$. The direct image of the local cohomology module $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$ is:*

$$i_+(H^p_{\mathfrak{p}}(H^{n-i}_I(R))) = H^1_{(t)}(H^p_{\mathfrak{p}R'}(H^{n-i}_{IR'}(R'))).$$

Proof. Let $\mathcal{D}_t$ be the ring of differential operators corresponding to the formal power series ring $k[[t]]$. For simplicity we will denote the local cohomology modules $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$ and $H^p_{\mathfrak{p}R'}(H^{n-i}_{IR'}(R'))$ by N and N' respectively. Then we have:

$$H^1_{(t)}(N') = H^1_{(t)}(N\hat{\otimes}_k k[[t]]) = H^1_{(t)}(k[[t]])\hat{\otimes}_k N = (\mathcal{D}_t/\mathcal{D}_t \cdot (t))\hat{\otimes}_k N = i_+(N)$$

□

Remark 2.3. In general, let $I_1, \dots, I_s$ be a set of ideals of R. Then, the direct image of the local cohomology module $H^{i_1}_{I_1}(\cdots(H^{i_s}_{I_s}(R))\cdots)$ is:

$$i_+(H^{i_1}_{I_1}(\cdots(H^{i_s}_{I_s}(R))\cdots)) = H^1_{(t)}(H^{i_1}_{I_1R'}(\cdots(H^{i_s}_{I_sR'}(R'))\cdots)).$$

3. Multiplicities of the characteristic cycle

Let A be a ring that admits a presentation $A \cong R/I$ for a given ideal $I \subseteq R = k[[x_1, \dots, x_n]]$. Recall that we have $\operatorname{Spec}(A) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}$. Throughout this section, a prime ideal of A will also mean the corresponding prime ideal of R that contains I.

Let R/I and R'/I' be two different presentations of the local ring A. Then, for any prime ideal of A, we will denote $\mathfrak{p}' \in \operatorname{Spec}(R')$ the prime ideal that corresponds to $\mathfrak{p} \in \operatorname{Spec}(R)$ by the isomorphism $\operatorname{Spec}(R/I) \cong \operatorname{Spec}(R'/I')$

Theorem 3.1. *Let A be a local ring which admits a surjective ring homomorphism $\pi : R \longrightarrow A$, where $R = k[[x_1, \dots, x_n]]$ is the formal power series ring. Set $I = \ker \pi$, let $\mathfrak{p} \subseteq A$ be a prime ideal and let*

$$CC(H^p_{\mathfrak{p}}(H^{n-i}_I(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha} \ \ V_\alpha,$$

be the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$. Then the multiplicities $\lambda_{\mathfrak{p},p,i,\alpha}$ depend only on A, $\mathfrak{p}$, p, i and α but neither on R nor on π.

The proof of the theorem is inspired in the proof of [10, Theorem 4.1], but here we must be careful with the behavior of the characteristic cycle so instead of [10, Lemma 4.3] we will use the following:

Lemma 3.2. *Let $g : R' \longrightarrow R$ be a surjective ring homomorphism, where R' is a formal power series ring of dimension n'. Set $I' = \ker \pi g$ and let*

$$CC(H^p_{\mathfrak{p}}(H^{n-i}_I(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha} \ \ V_\alpha,$$

be the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$. Then, the characteristic cycle of $H^p_{\mathfrak{p}'}(H^{n'-i}_{I'}(R'))$ is

$$CC(H^p_{\mathfrak{p}'}(H^{n'-i}_{I'}(R'))) = \sum \lambda_{\mathfrak{p},p,i,\alpha} \ \ V'_\alpha,$$

where $\pi(V'_\alpha)$ is the subvariety of $X' = \operatorname{Spec} R'$ defined by the defining ideal of $\pi(V_\alpha)$ contracted to R'.

Proof. R is regular so $\operatorname{Ker} g$ is generated by $n' - n$ elements that form part of a minimal system of generators of the maximal ideal $\mathfrak{m}' \subseteq R'$. By induction on $n' - n$ we are reduced to the case $n' - n = 1$, so $\operatorname{Ker} g$ is generated by one element $f \in \mathfrak{m}' \setminus {\mathfrak{m}'}^2$. By Cohen's structure theorem $R' = k[[x_1, \dots, x_n, t]]$ where we assume $f = t$ by a change of variables. We identify R with the subring $k[[x_1, \dots, x_n]]$ of R'. In particular we have to consider $I' = IR' + (t)$ and $\mathfrak{p}' = \mathfrak{p}R' + (t)$.

By using Lemma 2.2 and the degeneration of the Grothendieck's spectral sequence $E_2^{p,q} = H^p_{(t)}(H^q_J(M)) \Longrightarrow H^{p+q}_{J+(t)}(M)$ we have:

$$\begin{aligned} i_+(H^p_{\mathfrak{p}}(H^{n-i}_I(R))) &= H^1_{(t)}(H^p_{\mathfrak{p}R'}(H^{n-i}_{IR'}(R'))) = H^p_{\mathfrak{p}R'}(H^1_{(t)}(H^{n-i}_{IR'}(R'))) = \\ &= H^p_{\mathfrak{p}R'}(H^{n+1-i}_{IR'+(t)}(R')) = H^p_{\mathfrak{p}R'}(H^{n'-i}_{I'}(R')) = \\ &= H^p_{\mathfrak{p}R'+(t)}(H^{n'-i}_{I'}(R')) = H^p_{\mathfrak{p}'}(H^{n'-i}_{I'}(R')), \end{aligned}$$

where the second last assertion comes from the fact that $H^{n'-i}_{I'}(R')$ is a (t)-torsion module. Then we are done by the results in Section 2.2.

□

Now we continue the proof of Theorem 3.1.

Proof. Let $\pi' : R' \longrightarrow A$ and $\pi'' : R'' \longrightarrow A$ be surjections with $R' = k[[y_1, \dots, y_{n'}]]$ and $R'' = k[[z_1, \dots, z_{n''}]]$. Let $I' = \ker \pi'$ and let $I'' = \ker \pi''$. Let $R''' = R' \widehat{\otimes}_k R''$ be the external tensor product, $\pi''' = \pi' \widehat{\otimes}_k \pi'' : R' \widehat{\otimes}_k R'' \longrightarrow A$ and $I''' = \ker \pi'''$.

By Lemma 3.2, if the characteristic cycle of $H^p_{\mathfrak{p}'}(H^{n'-i}_{I'}(R'))$ is

$$CC(H^p_{\mathfrak{p}'}(H^{n'-i}_{I'}(R'))) = \sum \lambda'_{\mathfrak{p},p,i,\alpha} \ \ V'_\alpha,$$

then the characteristic cycle of $H^p_{\mathfrak{p}'''}(H^{n'+n''-i}_{I'''}(R'''))$ is

$$CC(H^p_{\mathfrak{p}'''}(H^{n'+n''-i}_{I'''}(R'))) = \sum \lambda'_{\mathfrak{p},p,i,\alpha} \ \ V'''_\alpha,$$

where $\pi(V'''_\alpha)$ is the subvariety of $X''' = \operatorname{Spec} R'''$ defined by the defining ideal of $\pi(V'_\alpha)$ contracted to R'''.

By Lemma 3.2, if the characteristic cycle of $H^p_{\mathfrak{p}''}(H^{n''-i}_{I''}(R''))$ is

$$CC(H^p_{\mathfrak{p}''}(H^{n''-i}_{I''}(R''))) = \sum \lambda''_{\mathfrak{p},p,i,\alpha} \ \ V''_\alpha,$$

then the characteristic cycle of $H^p_{\mathfrak{p}'''}(H^{n'+n''-i}_{I'''}(R'''))$ is

$$CC(H^p_{\mathfrak{p}'''}(H^{n'+n''-i}_{I'''}(R'))) = \sum \lambda''_{\mathfrak{p},p,i,\alpha} \ \ V'''_\alpha,$$

where $\pi(V'''_\alpha)$ is the subvariety of $X''' = \operatorname{Spec} R'''$ defined by the defining ideal of $\pi(V''_\alpha)$ contracted to R'''.

In particular we have $\lambda'_{\mathfrak{p},p,i,\alpha} = \lambda''_{\mathfrak{p},p,i,\alpha}$ for all $\mathfrak{p}$, p, i and α.

□

Remark 3.3. With the same arguments one may prove that the multiplicities of the characteristic cycle of the local cohomology modules $H^{i_1}_{I_1}(\cdots(H^{i_s}_{I_s}(R))\cdots)$, where $I_1,\dots,I_s$ is a set of ideals of R containing the ideal $I = I_s$, are also invariants of R/I.

Since Bass numbers $\mu_p(\mathfrak{p}, H^{n-i}_I(R))$ are multiplicities of the characteristic cycle of $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$, we recover Lyubeznik's result:

Corollary 3.4. *Let A be a ring which admits a surjective ring homomorphism $\pi : R \longrightarrow A$, where $R = k[[x_1,\dots,x_n]]$ is the formal power series ring. Set $I = \ker\pi$ and let $\mathfrak{p} \subseteq A$ be a prime ideal. The Bass numbers $\mu_p(\mathfrak{p}, H^{n-i}_I(R))$ depend only on A, $\mathfrak{p}$, p and i but neither on R nor on π.*

When $\mathfrak{p}$ is the zero ideal, we obtain the invariance with respect to R/I of the multiplicities of the characteristic cycle of $H^{n-i}_I(R)$.

Corollary 3.5. *Let A be a local ring which admits a surjective ring homomorphism $\pi : R \longrightarrow A$, where $R = k[[x_1,\dots,x_n]]$ is the formal power series ring. Set $I = \ker\pi$ and let*

$$CC(H^{n-i}_I(R)) = \sum m_{i,\alpha} \ \ V_\alpha,$$

be the characteristic cycle of the local cohomology modules $H^{n-i}_I(R)$. Then the multiplicities $m_{i,\alpha}$ depend only on A, i and α but neither on R nor on π.

Collecting these multiplicities by the dimension of the corresponding irreducible varieties we define the following invariants:

Definition 3.6. Let $I \subseteq R$ be an ideal. If $CC(H^{n-i}_I(R)) = \sum m_{i,\alpha} \ V_\alpha$ is the characteristic cycle of the local cohomology modules $H^{n-i}_I(R)$ then we define:

$$\gamma_{p,i}(R/I) := \{\sum m_{i,\alpha} \ \mid \ \dim(\pi(V_\alpha)) = p\}.$$

One may prove that these invariants have the same properties as Lyubeznik numbers (see [10, Section 4]). Namely, let $d = \dim(R/I)$ then $\gamma_{p,i}(R/I) = 0$ if $i > d$, $\gamma_{p,i}(R/I) = 0$ if $p > i$ and $\gamma_{d,d}(R/I) \neq 0$. In particular we can collect them in a triangular matrix that we will denote by $\Gamma(R/I)$. We point out that these invariants are finer than the Lyubeznik numbers.

Example 3.7. Let $R = k[[x_1, x_2, x_3, x_4, x_5]]$. Consider the ideals:

- $I_1 = (x_1, x_2, x_5) \cap (x_3, x_4, x_5)$.
- $I_2 = (x_1, x_2, x_5) \cap (x_3, x_4, x_5) \cap (x_1, x_2, x_3, x_4)$.

The characteristic cycle of the corresponding local cohomology modules can be computed by means of [1, Theorem 3.8]. Collecting the multiplicities we obtain the triangular matrices:

$$\Gamma(R/I_1) = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 2 \end{pmatrix} \qquad \Gamma(R/I_2) = \begin{pmatrix} 0 & 2 & 0 \\ & 1 & 0 \\ & & 2 \end{pmatrix}$$

Computing the Lyubeznik numbers (see [1, Theorem 4.4]), we obtain the triangular matrix:

$$\Lambda(R/I_1) = \Lambda(R/I_2) = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 2 \end{pmatrix}$$

We have to point out that the quotient ring R/I_1 is Buchsbaum but R/I_2 is not.

Remark 3.8. In order to compute the Lyubeznik numbers $\lambda_{p,i}(R/I)$ for a given ideal $I \subseteq R$ and arbitrary i, p we have to refer to U. Walther's algorithm [12]. When I is a squarefree monomial ideal, a description of these invariants is given in [1] and [14]. Some other particular computations may also be found in [7], [8],[9] and [13]. The multiplicities of the characteristic cycle of $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$, where I is a squarefree monomial ideal and $\mathfrak{p}$ is any homogeneous prime ideal, have been computed in [2].

When I is a squarefree monomial ideal (resp. the defining ideal of an arrangement of linear varieties), the multiplicities of the characteristic cycle of $H^{n-i}_I(R)$ have been computed in [1] (resp. [3]).

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Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Avinguda Diagonal 647, Barcelona 08028, SPAIN
E-mail address: `Josep.Alvarez@upc.es`