PRUNED CELLULAR FREE RESOLUTIONS OF MONOMIAL IDEALS

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Abstract. Using discrete Morse theory, we give an algorithm that prunes the excess of information in the Taylor resolution and constructs a new cellular free resolution for an arbitrary monomial ideal. The pruned resolution is not simplicial in general, but we can slightly modify our algorithm in order to obtain a simplicial resolution. We also show that the Lyubeznik resolution fits into our pruning strategy. The pruned resolution is not always minimal but it is a lot closer to the minimal resolution than the Taylor and the Lyubeznik resolutions as we will see in some examples. We finally use our methods to give a different approach to the theory of splitting of monomial ideals. We deduce from this splitting strategy that the pruned resolution is always minimal in the case of cycles and paths.

1. Introduction

Let $R = \mathbb{k}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{k}$ and $I \subseteq R$ a monomial ideal. The study of minimal free resolutions of these ideals has been a very active area of research during the last decades. There are topological and combinatorial formulae, as those of Hochster [15] or Gasharov, Peeva and Welker [12], to describe their multigraded Betti numbers but, except for some specific classes of monomial ideals (see, e.g., [8], [16] or [9]), the problem of describing a minimal multigraded free resolution explicitly was shown to be difficult.

Another strategy is to study non-minimal free resolutions. These reveal less information than minimal free resolutions do but are often much easier to describe. The most significant ones are the Taylor resolution [21] and the Lyubeznik resolution [18]. An interesting feature of these two resolutions is that they fit in the theory of simplicial resolutions introduced by Bayer, Peeva and Sturmfels in [5] and further extended to regular cellular resolutions and CW-resolutions in [6] and [17] respectively. The idea behind these three concepts is to associate to a free resolution of a monomial ideal a simplicial complex (respectively a regular cell complex, a CW-complex) that carries in its structure the algebraic structure of the free resolution. It is worth pointing out that Velasco proved in [23] that there exist monomial ideals whose minimal free resolutions cannot be described by a CW-complex.

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By adapting the discrete Morse theory developed by Forman [10] and Chari [7], Batzies and Welker provided in [4] a method to reduce a given regular cellular resolution. In particular, they proved that the Lyubeznik resolution can be obtained in this way from the Taylor resolution. Let's point out that discrete Morse theory has the inconvenient that it can't be used iteratively. To overcome this issue, one can use the algebraic discrete Morse theory developed independently by Sköldberg [20] and Jöllenbeck and Welker [17]. In this work, we use a similar strategy to reduce the Taylor resolution and obtain cellular and simplicial free resolutions that are closer to the minimal one than the Lyubeznik resolution. Essentially, the information given by the Taylor resolution can be encoded in a directed graph and the obstruction to its minimality can be observed in some of the edges of this graph. What we will do is to remove, in a convenient order, some of these edges to provide a smaller resolution. In some sense, we are pruning the excess of information given by the Taylor resolution in a simple and efficient way.

The organization of this paper is as follows. In Section 2, we review the notion of cellular resolution and introduce the basics on discrete Morse theory that will be needed throughout this work. In Section 3, we present our main results. We first provide an algorithm (Algorithm 3.1) that, starting from the Taylor resolution of a monomial ideal, allows to construct a smaller cellular free resolution (Theorem 3.3 and Corollary 3.4). The resolution that we obtain is not simplicial in general, but we can adapt our pruning algorithm to produce a simplicial free resolution (Algorithm 3.7). Indeed, the Lyubeznik resolution fits into this pruning strategy as shown in Algorithm 3.9. Other variants of our method are also mentioned.

In Section 4, we illustrate our results with several examples. We implemented our algorithms using CoCoALib [1] for constructing pruned resolutions in the non-trivial examples contained in this section. Finally, in Section 5 we present a connection between our method and the theory of Betti splittings introduced by Eliahou and Kervaire [8] and later developed by Francisco, Hà and Van Tuyl [11]. We provide a sufficient condition for having a Betti splitting by checking some prunings in our algorithm. We use this approach to prove that the pruned resolution is minimal for edge ideals associated to paths and cycles.

2. Cellular resolutions using discrete Morse theory

Let $R = \mathbb{k}[x_1, \dots, x_n]$ be the polynomial ring in n variables with coefficients in a field $\mathbb{k}$. An ideal $I \subseteq R$ is monomial if it may be generated by monomials $\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha \in \mathbb{Z}^n_{\geq 0}$. As usual, we denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\varepsilon_1, \dots, \varepsilon_n$ will be the natural basis of $\mathbb{Z}^n$. Moreover, given a set of generators of a monomial ideal I, $\{m_1, \dots, m_r\}$, we will consider the monomials $m_\sigma := \operatorname{lcm}(m_i \mid \sigma_i = 1)$ for any $\sigma \in \{0,1\}^r$.

A $\mathbb{Z}^n$-graded free resolution of R/I is an exact sequence of free $\mathbb{Z}^n$-graded modules:

$$\mathbb{F}_\bullet : \quad 0 \longrightarrow F_p \xrightarrow{d_n} \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow R/I \longrightarrow 0 , \tag{2.1}$$

where the i-th term is of the form

$$F_i = \bigoplus_{\alpha \in \mathbb{Z}^n} R(-\alpha)^{\beta_{i,\alpha}} .$$

We say that $\mathbb{F}_\bullet$ is minimal if the matrices of the homogeneous morphisms $d_i : F_i \longrightarrow F_{i-1}$ do not contain invertible elements. In this case, the exponents $\beta_{i,\alpha}$ form a set of invariants of R/I known as its *multigraded Betti numbers*. Throughout this work, we will mainly consider the coarser $\mathbb{Z}$-graded free resolution. In this case, we will encode the $\mathbb{Z}$-graded Betti numbers in the so-called *Betti diagram* of R/I where the entry on the ith row and jth column of the table is $\beta_{i,i+j}$:

$$\begin{array}{lcccc} & 0 & 1 & 2 & \cdots \\ \text{total:} & . & . & . & \\ 0: & \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \cdots \\ 1: & \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

2.1. **Cellular resolutions.** A CW-complex X is a topological space obtained by attaching cells of increasing dimensions to a discrete set of points $X^{(0)}$. Let $X^{(i)}$ denote the set of i-cells of X and consider the set of all cells $X^{(*)} := \bigcup_{i\geq 0} X^{(i)}$. Then, we can view $X^{(*)}$ as a poset with the partial order given by $\sigma' \leq \sigma$ if and only if σ' is contained in the closure of σ. We can also give a $\mathbb{Z}^n$-graded structure to X by means of an order preserving map $gr : X^{(*)} \longrightarrow \mathbb{Z}^n_{\geq 0}$.

We say that the free resolution (2.1) is *cellular* (or is a *CW-resolution*) if there exists a $\mathbb{Z}^n$-graded CW-complex (X, gr) such that, for all $i \geq 1$:

- there exists a basis $\{e_\sigma\}$ of F_i indexed by the $(i-1)$-cells of X, such that if $e_\sigma \in R(-\alpha)^{\beta_{i,\alpha}}$ then $gr(\sigma) = \alpha$, and
- the differential $d_i : F_i \longrightarrow F_{i-1}$ is given by

$$e_\sigma \ \mapsto \sum_{\sigma \geq \sigma' \in X^{(i-1)}} [\sigma : \sigma'] \, \mathbf{x}^{gr(\sigma) - gr(\sigma')} \, e_{\sigma'} \ , \quad \forall \sigma \in X^{(i)}$$

  where $[\sigma : \sigma']$ denotes the coefficient of σ' in the image of σ by the differential map in the cellular homology of X.

In the sequel, whenever we want to emphasize such a cellular structure, we will denote the free resolution as $\mathbb{F}_\bullet = \mathbb{F}_\bullet^{(X,gr)}$. If X is a simplicial complex, we say that the free resolution is *simplicial*. This is the case for the following two well-known examples.

• **The Taylor resolution**: The most recurrent example of simplicial free resolution is the Taylor resolution discovered in [21]. Using the above terminology, we can describe it as follows. Let $I = \langle m_1, \ldots, m_r \rangle \subseteq R$ be a monomial ideal. Consider the full simplicial complex on r vertices, $X_{\texttt{Taylor}}$, whose faces are labelled by $\sigma \in \{0,1\}^r$ or, equivalently, by the corresponding monomials m_σ. We have a natural $\mathbb{Z}^n$-grading on $X_{\texttt{Taylor}}$ by assigning

$gr(\sigma) = \alpha \in \mathbb{Z}^n$ where $\mathbf{x}^\alpha = m_\sigma$. The *Taylor resolution* is the simplicial resolution $\mathbb{F}_\bullet^{(X_{\mathtt{Taylor}},gr)}$.

• **The Lyubeznik resolution**: Another important example of simplicial resolution is the one considered by Lyubeznik in [18]. Let's s start fixing an order $m_1 \leq \cdots \leq m_r$ on a generating set of a monomial ideal $I \subseteq R$. Consider the simplicial subcomplex $X_{\mathtt{Lyub}} \subseteq X_{\mathtt{Taylor}}$ whose faces of dimension s are labelled by those $\sigma = \varepsilon_{i_0} + \cdots + \varepsilon_{i_s} \in \{0,1\}^r$ such that, for all $t < s$ and all $j < i_t$

$$m_j \nmid \operatorname{lcm}(m_{i_t}, \ldots, m_{i_s}).$$

The *Lyubeznik resolution* is the simplicial resolution $\mathbb{F}_\bullet^{(X_{\mathtt{Lyub}},gr)}$.

2.2. **Discrete Morse theory.** Forman introduced in [10] the discrete Morse theory as a method to reduce the number of cells in a CW-complex without changing its homotopy type. Batzies and Welker adapted this technique in [4] to the study of cellular resolutions; see also [24]. Indeed, they used the reformulation of discrete Morse theory in terms of acyclic matchings given by Chari in [7] in order to obtain, given a regular cellular resolution (most notably the Taylor resolution), a reduced cellular resolution.

This is also our approach in this work. Let's start recalling from [4] the preliminaries on discrete Morse theory. Consider the directed graph G_X on the set of cells of a regular $\mathbb{Z}^n$-graded CW-complex (X, gr) which edges are given by

$$E_X = \{\sigma \longrightarrow \sigma' \mid \sigma' \leq \sigma,\ \dim \sigma' = \dim \sigma - 1\}.$$

For a given set of edges $\mathcal{A} \subseteq E_X$, denote by $G_X^{\mathcal{A}}$ the graph obtained by reversing the direction of the edges in $\mathcal{A}$, i.e., the directed graph with edges[1]

$$E_X^{\mathcal{A}} = (E_X \setminus \mathcal{A}) \cup \{\sigma' \Longrightarrow \sigma \mid \sigma \longrightarrow \sigma' \in \mathcal{A}\}.$$

When each cell of X occurs in at most one edge of $\mathcal{A}$, we say that $\mathcal{A}$ is a *matching* on X. A matching $\mathcal{A}$ is *acyclic* if the associated graph $G_X^{\mathcal{A}}$ is acyclic, i.e., does not contain any directed cycle. Given an acyclic matching $\mathcal{A}$ on X, the $\mathcal{A}$-*critical cells* of X are the cells of X that are not contained in any edge of $\mathcal{A}$. Finally, an acyclic matching $\mathcal{A}$ is *homogeneous* whenever $gr(\sigma) = gr(\sigma')$ for any edge $\sigma \longrightarrow \sigma' \in \mathcal{A}$.

Proposition 2.1 ([4, Proposition 1.2]). *Let (X, gr) be a regular $\mathbb{Z}^n$-graded CW-complex and $\mathcal{A}$ a homogeneous acyclic matching. Then, there is a (not necessarily regular) CW-complex $X_{\mathcal{A}}$ whose i-cells are in one-to-one correspondence with the $\mathcal{A}$-critical i-cells of X, such that $X_{\mathcal{A}}$ is homotopically equivalent to X, and that inherits the $\mathbb{Z}^n$-graded structure.*

In the theory of cellular resolutions, we have the following consequence.

Theorem 2.2 ([4, Theorem 1.3]). *Let $I \subseteq R = \mathbb{k}[x_1, \ldots, x_n]$ be a monomial ideal. Assume that (X, gr) is a regular $\mathbb{Z}^n$-graded CW-complex that defines a cellular resolution $\mathbb{F}_\bullet^{(X,gr)}$ of R/I. Then, for a homogeneous acyclic matching $\mathcal{A}$ on G_X, the $\mathbb{Z}^n$-graded CW-complex $(X_{\mathcal{A}}, gr)$ supports a cellular resolution $\mathbb{F}_\bullet^{(X_{\mathcal{A}},gr)}$ of R/I.*

[1]For the sake of clarity, the arrows that we reverse will be denoted by $\Longrightarrow$.

The differentials of the cellular resolution $\mathbb{F}_\bullet^{(X_\mathcal{A},gr)}$ can be explicitely described in terms of the differentials of $\mathbb{F}_\bullet^{(X,gr)}$ (see [4, Lemma 7.7]).

2.3. **Algebraic discrete Morse theory.** One of the main inconvenients of discrete Morse theory is that the CW-complex $(X_\mathcal{A}, gr)$ that we obtain for a given homogeneous acyclic matching $\mathcal{A}$ is not necessarily regular. Therefore, we cannot always iterate the procedure. To overcome such an obstacle, one may use *algebraic discrete Morse theory* developed independently by Sköldberg [20] and Jöllenbeck and Welker [17].

This approach works directly with an initial free resolution $\mathbb{F}_\bullet$ without paying attention whether it has or not a cellular structure. Given a basis $X = \bigcup_{i\geq 0} X^{(i)}$ of the corresponding free modules F_i, we may consider the directed graph G_X on the set of basis elements with the corresponding set of edges E_X. Then, we may follow the same constructions as in the previous subsection. Namely, we may define an acyclic matching $\mathcal{A} \subseteq E_X$ (see [17, Definition 2.1]) but, in this case, we have to make sure that the coefficient $[\sigma : \sigma']$ in the differential corresponding to an edge $\sigma \longrightarrow \sigma' \in \mathcal{A}$ is a unit. We consider the $\mathcal{A}$-critical basis elements $X_\mathcal{A}$ and construct a free resolution $\mathbb{F}_\bullet^{X_\mathcal{A}}$ that is homotopically equivalent to $\mathbb{F}_\bullet$ (see [17, Theorem 2.2]).

3. Pruning the Taylor resolution

In [4], the Lyubeznik resolution is obtained from the Taylor resolution through discrete Morse theory by detecting a suitable homogeneous acyclic matching $\mathcal{A}$ on the simplicial complex $X_{\texttt{Taylor}}$ such that $X_{\texttt{Lyub}} = X_\mathcal{A}$. In this section, we use a similar approach to provide some new cellular free resolutions for monomial ideals. We should point out that the framework considered in [4] is slightly more general. To keep notations as simple as possible, we decided to stick to the case of monomial ideals in a polynomial ring.

3.1. **A cellular free resolution.** Let $I = \langle m_1, \dots, m_r\rangle \subseteq R$ be a monomial ideal. Our starting point is the Taylor resolution $\mathbb{F}_\bullet^{(X_{\texttt{Taylor}},gr)}$. This resolution is, in general, far from being minimal. In other words, the directed graph $G_{X_{\texttt{Taylor}}}$ associated to $X_{\texttt{Taylor}}$ contains a lot of unnecessary information. Our goal is to prune this excess of information in a very simple way. More precisely, we give an algorithm that produces a homogeneous acyclic matching $\mathcal{A}_P$ on $X_{\texttt{Taylor}}$. Using discrete Morse theory, this will provide a cellular free resolution of R/I. It will not be minimal in general, but it will be smaller than the Lyubeznik resolution.

Algorithm 3.1. *(Pruned resolution)*

Input: The set of edges $E_{X_{\texttt{Taylor}}}$.

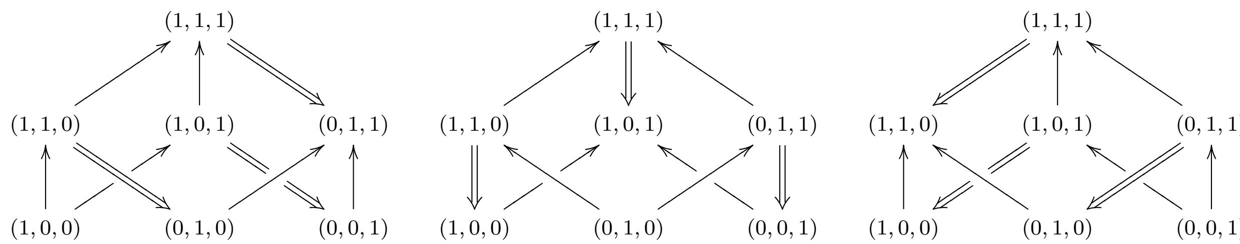
For j from 1 to r, incrementing by 1

(j) *Prune* the edge $\sigma \longrightarrow \sigma + \varepsilon_j$ for all $\sigma \in \{0,1\}^r$ such that $\sigma_j = 0$, where 'prune' means remove the edge[2] if it survived after step $(j-1)$ and $gr(\sigma) = gr(\sigma + \varepsilon_j)$.

[2] When we remove an edge, we also remove its two vertices and all the edges passing through these two vertices.

RETURN: The set $\mathcal{A}_P$ of edges that have been pruned.

Example 3.2. For the case $r = 3$ we can visualize the steps of the algorithm over the directed graph as follows:



The double arrows indicate the direction of the pruning step, that is, the arrows that will be pruned at each step if the degree of their two vertices coincide (and if they have not been pruned at a previous step).

The main result in this section is the following:

Theorem 3.3. *Let $\mathcal{A}_P \subseteq E_{X_{\texttt{Taylor}}}$ be the set of pruned edges obtained using Algorithm 3.1. Then $\mathcal{A}_P$ is a homogeneous acyclic matching on $X_{\texttt{Taylor}}$.*

As a consequence, we get our desired cellular free resolution.

Corollary 3.4. *Let $I \subseteq R = \Bbbk[x_1, \dots, x_n]$ be a monomial ideal and $\mathcal{A}_P \subseteq E_{X_{\texttt{Taylor}}}$ be the set of pruned edges obtained using Algorithm 3.1. Then, the $\mathbb{Z}^n$-graded CW-complex $(X_{\mathcal{A}_P}, gr)$ supports a cellular free resolution $\mathbb{F}_\bullet^{(X_{\mathcal{A}_P}, gr)}$ of R/I.*

Remark 3.5. The free resolution $\mathbb{F}_\bullet^{(X_{\mathcal{A}_P}, gr)}$, like the Lyubeznik resolution, strongly depends on the order of the generators of the monomial ideal I. In general, it is neither simplicial (while the Lyubeznik resolution is always simplicial) nor minimal.

The proof of Theorem 3.3 follows closely the one given in [4, §3] to show that the Lyubeznik resolution can be obtained using discrete Morse theory. However, we will need some reformulation of their preliminary results.

Let's start with some notations. Given $\sigma \in \{0,1\}^r$, we define:

- $t(\sigma) := \min\{\sigma' \leq \sigma \mid \exists\ 1 \leq i \leq r \text{ s.t. } \sigma'_i = 0 \text{ and } gr(\sigma') = gr(\sigma' + \varepsilon_i)\}$;
- $j(\sigma) := \min\{i,\ 1 \leq i \leq r \mid t(\sigma)_i = 0 \text{ and } gr(t(\sigma)) = gr(t(\sigma) + \varepsilon_i)\}$.

Of course, $gr(t(\sigma)) = gr(t(\sigma) + \varepsilon_i)$ implies $gr(\sigma) = gr(\sigma + \varepsilon_i)$. It follows that $j(\sigma)$ is the step in the algorithm where the edge containing σ, namely $\sigma \longrightarrow \sigma + \varepsilon_{j(\sigma)}$, is pruned. Note that we may have $t(\sigma) = \sigma$. For simplicity, we will set $t(\sigma) = 0$ when no edge containing σ has been pruned in the algorithm.

The following properties will be useful in the proof of Theorem 3.3:

i) If $gr(\sigma) = gr(\sigma + \varepsilon_k)$ for some k s.t. $\sigma_k = 0$, then $t(\sigma + \varepsilon_k) \geq t(\sigma)$.

Proof. The result obviously holds if $t(\sigma) = 0$. When $t(\sigma) \neq 0$, the result follows from the fact that $gr(t(\sigma)) = gr(t(\sigma) + \varepsilon_{j(\sigma)})$ implies $gr(t(\sigma) + \varepsilon_k) = gr(t(\sigma) + \varepsilon_k + \varepsilon_{j(\sigma)})$. □

ii) If $gr(\sigma) = gr(\sigma + \varepsilon_k)$ and $t(\sigma + \varepsilon_k) = t(\sigma) \neq 0$ for some k s.t. $\sigma_k = 0$, then $j(\sigma + \varepsilon_k) = j(\sigma)$.

Proof. We have $gr(t(\sigma)) = gr(t(\sigma) + \varepsilon_{j(\sigma)})$ so it follows that

$$gr(t(\sigma + \varepsilon_k)) = gr(t(\sigma + \varepsilon_k) + \varepsilon_{j(\sigma)})$$

and therefore $j(\sigma + \varepsilon_k) \geq j(\sigma)$. On the other hand, we also have the equality $gr(t(\sigma + \varepsilon_k)) = gr(t(\sigma) + \varepsilon_{j(\sigma+\varepsilon_k)})$ so

$$gr(t(\sigma)) = gr(t(\sigma) + \varepsilon_{j(\sigma+\varepsilon_k)})$$

and the reverse inequality $j(\sigma + \varepsilon_k) \leq j(\sigma)$ follows. □

iii) If $t(\sigma) \neq 0$, then $t(\sigma - \varepsilon_{j(\sigma)}) = t(\sigma) = t(\sigma + \varepsilon_{j(\sigma)})$.

Proof. Assume that $\varepsilon_{j(\sigma)} \leq \sigma$. Then we have $t(\sigma - \varepsilon_{j(\sigma)}) \leq t(\sigma)$ by definition. To prove the reverse inequality we only have to notice that $\varepsilon_{j(\sigma)} \not\leq t(\sigma)$ and equivalently $t(\sigma) \leq \sigma - \varepsilon_{j(\sigma)} < \sigma$. Using similar arguments we also obtain $t(\sigma) = t(\sigma + \varepsilon_{j(\sigma)})$ when $\varepsilon_{j(\sigma)} \not\leq \sigma$. □

Proof. of Theorem 3.3. It is clear from its construction that the set $\mathcal{A}_P$ is a matching since we are removing edges (or pairs of cells) at each step of the algorithm. To check that it is acyclic, assume that we have a cycle

$$\sigma_0 \Longrightarrow \tau_0 \longrightarrow \sigma_1 \Longrightarrow \tau_1 \longrightarrow \cdots \Longrightarrow \tau_{k-1} \longrightarrow \sigma_k = \sigma_0 \,.$$

Then $gr(\sigma_0) = gr(\tau_0) \leq gr(\sigma_1) = \cdots = gr(\tau_{k-1}) \leq gr(\sigma_0)$, and hence equality must hold everywhere. Let's see that we also have $j(\sigma_0) = j(\tau_0) = \cdots = j(\tau_{k-1}) = j(\sigma_0)$.

We start considering the case of a reversed arrow $\sigma \Longrightarrow \tau$, where $\sigma = \tau + \varepsilon_{j(\tau)}$. We have $0 \neq t(\tau) = t(\tau + \varepsilon_{j(\tau)})$ by iii) so, by ii), we also have $j(\tau) = j(\tau + \varepsilon_{j(\tau)}) = j(\sigma)$.

For a direct arrow $\tau \longrightarrow \sigma$ we have $\tau = \sigma - \varepsilon_k$ for some k. We have

$$0 \neq t(\sigma - \varepsilon_k) \leq t(\sigma) \leq \sigma$$

were the first inequality is due to the fact that the vertex τ must be pruned in the next link of the chain. The second inequality comes from i) and it can not be strict because of the definition of $t(\sigma)$. Therefore $t(\sigma - \varepsilon_k) = t(\sigma)$ and we have $j(\sigma - \varepsilon_k) = j(\sigma)$ by ii), that is, $j(\tau) = j(\sigma)$.

Since all the elements in the cycle have the same $j = j(\sigma_0) = \cdots = j(\tau_{k-1})$, we deduce that such cycle cannot exist due to the fact that we can only go in the direction of ε_j when constructing the edges of the cycle. At the end, we would have an edge with both directions and we get a contradiction.

This shows that $\mathcal{A}_P$ is an acyclic matching on $X_{\texttt{Taylor}}$, and it is homogeneous by construction of $\mathcal{A}_P$ in our pruning algorithm. □

A nice feature about the pruning algorithm is that we do not need to care if the original system of generators of the monomial ideal is minimal or not. Roughly speaking, the pruning algorithm will always remove the excess of information given by the extra generators.

Lemma 3.6. *Let $I = \langle m_1, \ldots, m_s \rangle \subseteq R$ be a monomial ideal, and let $\{m_{i_1}, \ldots, m_{i_r}\}$ be its minimal set of generators with $1 \leq i_1 < \cdots < i_r \leq s$. Let X_s and X_r be the Taylor simplicial complexes associated to these two sets of generators*[3] *and $\mathcal{A}_P^s \subseteq E_{X_s}$, $\mathcal{A}_P^r \subseteq E_{X_r}$ be the sets of pruned edges obtained by applying Algorithm 3.1 to each case. Then, there is an isomorphism of $\mathbb{Z}^n$-graded CW-complexes $(X_{\mathcal{A}_P^r}, gr) \cong (X_{\mathcal{A}_P^s}, gr)$. In particular, both sets of generators lead to the same cellular free resolution of R/I in Corollary 3.4.*

Proof. Assume that m_j is a generator of I that does not belong to the minimal generating set $\{m_{i_1}, \ldots, m_{i_r}\}$. We want to check that all the vertices $\sigma = (\sigma_1, \ldots, \sigma_s)$ with $\sigma_j = 1$ in the graph associated to the Taylor complex X_s are pruned using Algorithm 3.1.

There exists a minimal generator m_{i_k} such that $m_{i_k} | m_j$. Therefore, at the step (i_k) of Algorithm 3.1, we prune all the edges $\sigma \longrightarrow \sigma + \varepsilon_{i_k}$ with $\sigma_j = 1$ and $\sigma_{i_k} = 0$ that have survived in the previous steps. But if we have pruned, at a previous step (ℓ) of the algorithm, the edge $\sigma + \varepsilon_{i_k} \longrightarrow \sigma + \varepsilon_\ell + \varepsilon_{i_k}$, then we have also pruned the edge $\sigma \longrightarrow \sigma + \varepsilon_\ell$ at the same step (ℓ) and hence neither the vertex σ nor the vertex $\sigma + \varepsilon_{i_k}$ survived at the step (ℓ) of the algorithm. □

3.2. **A simplicial free resolution.** The cellular complex $X_{\mathcal{A}_P}$ in Theorem 3.3 may not be simplicial, that is, it may not satisfy the property that given $\tau \in X_{\mathcal{A}_P}$ then $\sigma \in X_{\mathcal{A}_P}$ for any $\sigma \leq \tau$. In other words, the pruned free resolution $\mathbb{F}_\bullet^{(X_{\mathcal{A}_P}, gr)}$ obtained in Corollary 3.4 is cellular but it may not be simplicial. If we choose carefully the edges that we prune in Algorithm 3.1 in order to preserve this property, we will obtain a simplicial free resolution of R/I that, in general, will be bigger than the one in Corollary 3.4.

Algorithm 3.7. *(Simplicial pruned resolution)*

INPUT: The set of edges $E_{X_{\texttt{Taylor}}}$.

For j from 1 to r, incrementing by 1

- **(j)** *Prune* the edge $\sigma \longrightarrow \sigma + \varepsilon_j$ for all $\sigma \in \{0,1\}^r$ such that $\sigma_i = 0$, where 'prune' means remove the edge if it survived after step $(j-1)$, $gr(\sigma) = gr(\sigma + \varepsilon_j)$ and no face $\tau > \sigma$ survives at this step (j).

RETURN: *The set $\mathcal{A}_S$ of edges that have been pruned.*

We have that $\mathcal{A}_S$ is an acyclic matching on $X_{\texttt{Taylor}}$ and the corresponding free resolution $\mathbb{F}_\bullet^{(X_{\mathcal{A}_S}, gr)}$ is a simplicial free resolution of the monomial ideal I.

[3] We have that X_r is a subcomplex of X_s.

Remark 3.8. The following situation could happen: an edge $\sigma \longrightarrow \sigma + \varepsilon_j$, that would be pruned at step (j) of Algorithm 3.1, may not be pruned at step (j) of Algorithm 3.7 because one face $\tau > \sigma$ survives at this step. But τ may be pruned in a posterior step. In this case, due to the fact that $X_{\mathcal{A}_S}$ is simplicial, we may apply Algorithm 3.7 once again, with the set of edges $E_{X_{\mathcal{A}_S}}$ as input intead of $E_{X_{\mathtt{Taylor}}}$. The edge $\sigma \longrightarrow \sigma + \varepsilon_j$ will be pruned during this second iteration and we will get a smaller simplicial resolution. We will observe this phenomenon later in Example 4.1 contructing the simplicial pruned resolution of the 5-cycle.

3.3. **The Lyubeznik resolution revisited.** The Lyubeznik resolution can be also obtained from the Taylor resolution using our pruning algorithm. In this case, the edges of the Taylor complex $X_{\mathtt{Taylor}}$ that we prune are obtained using the following:

Algorithm 3.9. *(The Lyubeznik resolution via pruning)*

INPUT: The set of edges $E_{X_{\mathtt{Taylor}}}$.

For j from 1 to r, incrementing by 1

(j) *Prune* the edge $\sigma \longrightarrow \sigma + \varepsilon_j$ for all $\sigma \in \{0,1\}^r$ such that $\sigma_i = 0$ for all $i \leq j$, where 'prune' means remove the edge if it survived after step $(j-1)$ and $gr(\sigma) = gr(\sigma + \varepsilon_j)$.

RETURN: *The set $\mathcal{A}_L$ of edges that have been pruned.*

We have that $\mathcal{A}_L$ is a homogeneous acyclic matching on $X_{\mathtt{Taylor}}$ and that $X_{\mathcal{A}_L} = X_{\mathtt{Lyub}}$. Moreover, the Lyubeznik resolution is simplicial because the pruning that we consider in this case preserves the property of being simplicial. In this sense, we may understand the pruned resolution given in Corollary 3.4 and the simplicial pruned resolution given in Subsection 3.2 as a refinement of the Lyubeznik resolution. Finally, we mention that generalizations of Lyubeznik resolutions have been studied by Novik in [19] using a completely different approach.

3.4. **Some variants of the pruning algorithm.** The methods developed in this work can be extended in several different directions. The aim of this subsection is to present some of them.

· *General setup:* Batzies and Welker [4] use a slightly more general framework for discrete Morse theory than the one considered in this paper. The interested reader should be able to adapt our pruning algorithm to their framework.

· *Partial pruning:* Notice that the acyclic matchings considered in Algorithms 3.1, 3.7 and 3.9 satisfy $\mathcal{A}_P \supseteq \mathcal{A}_S \supseteq \mathcal{A}_L$. One may also consider any convenient subset $\mathcal{A}_P \supseteq \mathcal{A}'$ of pruning edges. Indeed, we may consider many different variants using algebraic discrete Morse theory iteratively. We only have to pick a convenient edge at each iteration.

· *Pruning other resolutions:* The method that we present here always starts with the Taylor resolution but we may start with other non-minimal free resolutions, the Lyubeznik resolution for example. The advantage of the Taylor resolution is that it does not depend on the order of the generators, and the simplicity of its construction makes our algorithm very easy to present and implement.

$\cdot\nu$*-invariants:* A new set of invariants that measure the acyclicity of the linear strands of a minimal free resolution of a graded ideal was introduced in [2]. We may obtain an approximation to these invariants by applying the following iteration of the pruning algorithm. We first apply Algorithm 3.1 to obtain the CW-complex $X_{\mathcal{A}_P}$ and its corresponding free resolution. Then, we apply the pruning algorithm to $X_{\mathcal{A}_P}$ with the following variant:

- **(j)** *Prune* the edge $\sigma \longrightarrow \sigma + \varepsilon_j$ for all $\sigma \in \{0,1\}^r$ such that $\sigma_j = 0$, where 'prune' means remove the edge if it survived after step $(j-1)$ and $gr(\sigma) = gr(\sigma+\varepsilon_j) - 1$. Here $gr(\sigma) = |\alpha| \in \mathbb{Z}$ where $\mathbf{x}^\alpha = m_\sigma$.

4. Examples

In this section, we will illustrate that the pruned free resolution described in Section 3.1 is fairly close to the minimal free resolution in some examples. Indeed, we will compare the pruned resolution to the simplicial free resolution obtained in Section 3.2 and to the Lyubeznik resolution by means of their corresponding Betti diagram.

4.1. **First examples.**

Already for some simple examples we can appreciate the better behavior of the pruned resolutions with respect to the Lyubeznik resolution.

Example 4.1. We are going to describe the steps of the pruning Algorithms described in Section 3 for the edge ideals associated to a 5-path and a 5-cycle:

• Let $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5)$ be the edge ideal of a 5-path. The steps performed with Algorithm 3.1 are the following:

· **Step (1):** No edge is pruned.
· **Step (2):** We prune the edges $(1,0,1,0) \Leftarrow (1,1,1,0)$ and $(1,0,1,1) \Leftarrow (1,1,1,1)$.
· **Step (3):** We prune the edge $(0,1,0,1) \Leftarrow (0,1,1,1)$.
· **Step (4):** No edge is pruned.

The two edges pruned in Step (2) are also pruned using Algorithm 3.7 but are not pruned using Algorithm 3.9. The edge pruned in Step (3) is not be pruned neither using Algorithm 3.7 nor Algorithm 3.9. Therefore, the Betti diagrams of the pruned, the simplicial pruned, and the Lyubeznik resolutions are respectively:

	0	1	2	3
total:	1	4	4	1
0:	1	.	.	.
1:	.	4	3	.
2:	.	.	1	1

	0	1	2	3
total:	1	4	5	2
0:	1	.	.	.
1:	.	4	3	1
2:	.	.	2	1

	0	1	2	3	4
total:	1	4	6	4	1
0:	1	.	.	.	.
1:	.	4	3	2	1
2:	.	.	3	2	.

The pruned resolution is minimal. The simplicial pruned resolution is not minimal but it is as small as possible since the ideal I has no minimal simplicial resolution as one can check using an argument similar to the one used in [16, Section 3.2] for the 4-cycle. The

Lyubeznik resolution coincides with the Taylor resolution since no edge has been pruned through Algorithm 3.9.

• Let $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$ be the edge ideal of a 5-cycle. The steps performed with Algorithm 3.1 are the following:

· **Step (1):** We prune four edges $(0,1,0,0,1) \Leftarrow (1,1,0,0,1)$, $(0,1,1,0,1) \Leftarrow (1,1,1,0,1)$, $(0,1,0,1,1) \Leftarrow (1,1,0,1,1)$ and $(0,1,1,1,1) \Leftarrow (1,1,1,1,1)$.
· **Step (2):** We prune $(1,0,1,0,0) \Leftarrow (1,1,1,0,0)$ and $(1,0,1,1,0) \Leftarrow (1,1,1,1,0)$.
· **Step (3):** We prune the edge $(0,1,0,1,0) \Leftarrow (0,1,1,1,0)$.
· **Step (4):** We prune $(0,0,1,0,1) \Leftarrow (0,0,1,1,1)$ and $(1,0,1,0,1) \Leftarrow (1,0,1,1,1)$.
· **Step (5):** We prune the edge $(1,0,0,1,0) \Leftarrow (1,0,0,1,1)$.

The edges pruned at Step (1) are also pruned using Algorithm 3.7 and Algorithm 3.9. The two edges pruned at Step (2) can not be pruned using Algorithm 3.7 because the vertices $(1,0,1,0,1)$ and $(1,0,1,1,1)$ remain at this step of the algorithm. The edges pruned at Step (3) and (4) are also pruned in Algorithm 3.7. The edge pruned at Step (5) is not pruned in Algorithm 3.7. Notice that the two edges that prohibited the pruning at Step (3) of Algorithm 3.7 were pruned at Step (4). This means that we can perform another round of Algorithm 3.7 that will prune the two edges that could not be pruned before at step (2) as observed in Remark 3.8.

On the other hand, note that after the first step, no more edge can be pruned through Algorithm 3.9. We thus obtain that the Betti diagrams of the pruned, the simplicial pruned, and the Lyubeznik resolutions are the following respectively:

	0	1	2	3
total:	1	5	5	1
0:	1	.	.	.
1:	.	5	5	.
2:	.	.	.	1

	0	1	2	3
total:	1	5	6	2
0:	1	.	.	.
1:	.	5	5	1
2:	.	.	1	1

	0	1	2	3	4
total:	1	5	9	7	2
0:	1	.	.	.	.
1:	.	5	5	4	2
2:	.	.	4	3	.

As in the previous example, the pruned resolution is minimal and the simplicial pruned resolution is not minimal but it is as small as possible.

Remark 4.2. It is not surprising that in the previous examples we could find a cellular minimal resolution since both the 5-path and the 5-cycle fall into [6, Example 1.7].

For larger examples the difference between the pruned and the Luybeznik resolutions becomes more apparent.

Example 4.3. Consider the ideal $I = (x_1^4, x_2^4, x_2^2x_3^2, x_3^4, x_4^4, x_1x_4^2x_5, x_5^4, x_2^2x_6^2, x_6^4, x_4^2x_7^2, x_7^4)$. The pruned resolution is minimal while the Lyubeznik resolution is not. The Betti diagrams are:

	0	1	2	3	4	5	6	7
total:	1	11	49	114	148	107	40	6
0:	1	.	.	.	.	.	.	.
1:	.	.	.	.	.	.	.	.
2:	.	.	.	.	.	.	.	.
3:	.	11	.	.	.	.	.	.
4:	.	.	9	.	.	.	.	.
5:	.	.	2	2	.	.	.	.
6:	.	.	38	4	.	.	.	.
7:	.	.	.	56	2	.	.	.
8:	.	.	.	10	31	.	.	.
9:	.	.	.	42	30	9	.	.
10:	.	.	.	.	71	32	1	.
11:	.	.	.	.	2	37	14	.
12:	.	.	.	.	12	6	6	2
13:	.	.	.	.	.	22	6	.
14:	.	.	.	.	.	.	11	2
15:	.	.	.	.	.	1	.	1
16:	.	.	.	.	.	.	2	.
17:	.	.	.	.	.	.	.	1

	0	1	2	3	4	5	6	7	8	9	10
total:	1	11	54	156	294	378	336	204	81	19	2
0:	1	.	.	.	.	.	.	.	.	.	.
1:	.	.	.	.	.	.	.	.	.	.	.
2:	.	.	.	.	.	.	.	.	.	.	.
3:	.	11	.	.	.	.	.	.	.	.	.
4:	.	.	9	.	.	.	.	.	.	.	.
5:	.	.	2	7	.	.	.	.	.	.	.
6:	.	.	43	4	2	.	.	.	.	.	.
7:	.	.	.	58	4	.	.	.	.	.	.
8:	.	.	.	12	64	2	.	.	.	.	.
9:	.	.	.	75	32	44	.	.	.	.	.
10:	.	.	.	.	106	48	22	.	.	.	.
11:	.	.	.	.	18	114	48	7	.	.	.
12:	.	.	.	.	68	40	77	30	1	.	.
13:	.	.	.	.	.	86	44	42	12	.	.
14:	.	.	.	.	.	10	85	30	18	2	.
15:	.	.	.	.	.	34	16	50	10	6	.
16:	.	.	.	.	.	.	33	10	23	2	1
17:	.	.	.	.	.	.	2	27	4	6	.
18:	.	.	.	.	.	.	9	2	10	.	1
19:	.	.	.	.	.	.	.	5	.	3	.
20:	.	.	.	.	.	.	.	.	3	.	.
21:	.	.	.	.	.	.	.	1	.	.	.

4.2. **Independence on the characteristic of the base field.** It is well-known that the Betti numbers of a monomial ideal depend on the characteristic of the base field. This phenomenon can be understood through Hochster's formula. Namely, given a monomial ideal $I \subseteq R = \Bbbk[x_1, \ldots, x_n]$, we may assume that it is squarefree since its Betti numbers can always be obtained from a squarefree monomial ideal using polarization. Now, a squarefree monomial ideal I is always the Stanley-Reisner ideal of a simplicial complex Δ. For any squarefree degree $\alpha \in \{0,1\}^n$, we consider $F_\alpha := \{x_i \mid \alpha_i \neq 0\} \subseteq \{x_1, \ldots, x_n\}$. Let $\Delta_{|F_\alpha}$ be the simplicial complex obtained by taking all the faces of Δ whose vertices belong to F_α. Hochster's formula states that the Betti numbers of R/I can be computed as the dimensions of reduced simplicial homology groups:

$$\beta_{i,\alpha}(R/I) = \dim_{\Bbbk} \widetilde{H}_{|\alpha|-i-1}(\Delta_{|F_\alpha}; \Bbbk).$$

By the universal coefficient theorem, we may just compute the homology modules over the integer numbers $\widetilde{H}_{|\alpha|-i-1}(\Delta_{|F_\alpha}; \mathbb{Z})$. It follows that Betti numbers depend on the characteristic of the base field whenever these homology groups have torsion.

Example 4.4. The most recurrent example that illustrates the behavior of Betti numbers with respect to the characteristic of the base field is the Stanley-Reisner ideal associated to a minimal triangulation of $\mathbb{P}^2_{\mathbb{R}}$. Namely, consider the following ideal in $R = \Bbbk[x_1, \ldots, x_6]$:

$$I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_2x_4x_5, x_3x_4x_5, x_2x_3x_6, x_1x_4x_6, x_3x_4x_6, x_1x_5x_6, x_2x_5x_6).$$

The Betti diagrams in characteristic zero and two respectively are

	0	1	2	3
total:	1	10	15	6
0:	1	.	.	.
1:	.	.	.	.
2:	.	10	15	6

	0	1	2	3	4
total:	1	10	15	7	1
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	.	10	15	6	1
3:	.	.	.	1	.

On the other hand, the results that we obtain with the pruned and the Lyubeznik resolution respectively are:

	0	1	2	3	4
total:	1	10	15	7	1
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	.	10	15	6	1
3:	.	.	.	1	.

	0	1	2	3	4
total:	1	10	27	27	9
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	.	10	15	18	9
3:	.	.	12	9	.

Remark 4.5. All the free resolutions that we consider in this work are obtained from the Taylor resolution using discrete Morse theory as in [4] or [17]. If one follows closely the construction of the CW-complex $X_{\mathcal{A}}$ that is homotopically equivalent to $X_{\texttt{Taylor}}$, one may check that it is independent of the characteristic of the base field. In this sense, the result obtained with the pruned resolution in Example 4.4 is the best that we can achieve using discrete Morse theory.

4.3. **Minimality.** A necessary and sufficient condition for the minimality of a cellular free resolution $\mathbb{F}_\bullet^{(X_{\mathcal{A}},gr)}$ is described in [4, Lemma 7.5]. The pruned resolution obtained in Corollary 3.4 is minimal if and only if the following condition is satisfied:

Recall that the complex $X_{\texttt{Taylor}}$ associated to a monomial ideal $I = \langle m_1, \ldots, m_r \rangle$ has faces labelled by $\sigma \in \{0,1\}^r$. Then, the cells $\sigma \in X_{\mathcal{A}}$ that survive to any of the pruning processes described in Section §3 must satisfy $gr(\sigma) \neq gr(\sigma + \varepsilon_j)$ whenever $\sigma_j = 0$.

Remark 4.6. Barile [3] noticed that in the case of the Lyubeznik resolution, it is enough checking out the aforementioned condition for maximal faces of $X_{\texttt{Lyub}}$.

The Taylor resolution is rarely minimal. Some examples of minimal Lyubeznik resolutions include shellable ideals (see [4]) and the matroid ideal of a finite projective space (see [19]). Finally, let's point out that Torrente and Varbaro [22] provided a fast algorithm for computing Betti diagrams of a monomial ideal taking as a starting point the Lyubeznik resolution of the ideal. Using the pruned resolution as starting point of their algorithm would speed up their computation of the Betti diagram.

5. Betti splittable monomial ideals

A technique that has been successfully applied to describe Betti numbers of monomial ideals is based on the concept of splitting introduced by Eliahou and Kervaire in [8]; see [14] for a nice survey. A refinement of this concept was coined by Francisco, Hà and Van Tuyl in [11].

Definition 5.1. We say that $I = J + K$ is a *Betti splitting* for the monomial ideal I if the following formula for the $\mathbb{Z}^n$-graded Betti numbers is satisfied:

$$\beta_{i,\alpha}(I) = \beta_{i,\alpha}(J) + \beta_{i,\alpha}(K) + \beta_{i-1,\alpha}(J \cap K)\,.$$

We can mimic the same concept introducing the *pruned Betti numbers* as the Betti numbers that we obtain in the pruned free resolution of Section 3.1, and that we will denote by $\overline{\beta}_{i,\alpha}(I)$ to distinguish them from the usual Betti numbers.

Definition 5.2. We say that $I = J + K$ is a *pruned Betti splitting* for the monomial ideal I if the following formula for the $\mathbb{Z}^n$-graded Betti numbers is satisfied:

$$\overline{\beta}_{i,\alpha}(I) = \overline{\beta}_{i,\alpha}(J) + \overline{\beta}_{i,\alpha}(K) + \overline{\beta}_{i-1,\alpha}(J \cap K)\,.$$

Remark 5.3. Obviously, a pruned Betti splitting provides a Betti splitting whenever the pruning algorithm gives a minimal free resolution of I.

Necessary and sufficient conditions describing Betti splittings have been given in [11]. Using our main Algorithm 3.1, we will give some conditions that are easy to check and that provide a pruned Betti splitting.

5.1. **A partial pruning of $J \cap K$.** Let $I = \langle m_1, \dots, m_r \rangle$ be a monomial ideal, and consider the Taylor simplicial complex $X_{\texttt{Taylor}}$ on r vertices which faces are labeled by $\sigma \in \{0,1\}^r$. Consider a decomposition $I = J + K$ with $J = \langle m_1, \dots, m_s \rangle$ and $K = \langle m_{s+1}, \dots, m_r \rangle$. In order to compute the pruned resolution of J and K we have to consider the subcomplexes:

- $X_J \subseteq X_{\texttt{Taylor}}$ with faces labelled by $\sigma = (\sigma_1, \dots, \sigma_s, 0, \dots, 0) \in \{0,1\}^r$, and
- $X_K \subseteq X_{\texttt{Taylor}}$ with faces labelled by $\sigma = (0, \dots, 0, \sigma_{s+1}, \dots, \sigma_r) \in \{0,1\}^r$.

Denote by X' the set obtained by removing the faces of X_J and X_K from $X_{\texttt{Taylor}}$. Notice that X' is not a simplicial subcomplex of $X_{\texttt{Taylor}}$ and, in particular, it is not the Taylor simplicial complex associated to the intersection. However, applying a partial pruning algorithm to $X_{J\cap K}$ we obtain X'. The proof of this fact is quite tedious but straightforward. We will simply sketch it and leave the details for the interested reader.

Consider the Taylor complex $X_{J\cap K}$ associated to the (possibly non-minimal) set of generators of $J \cap K$ given by the monomials

$$\{m_{1,s+1}, \dots, m_{s,s+1}, m_{1,s+2}, \dots, m_{s,s+2}, \dots, m_{1,r}, \dots, m_{s,r}\},$$

where $m_{i,k} = \operatorname{lcm}(m_i, m_k)$. In the sequel we will also denote $m_{i_1,\dots,i_\ell} = \operatorname{lcm}(m_{i_1}, \dots, m_{i_\ell})$. The Taylor complex $X_{J\cap K}$ is the full simplicial complex on sr vertices with faces labelled by $\sigma \in \{0,1\}^{sr}$. Notice that the vertex ε_j corresponds to the monomial $m_{\varepsilon_j} := m_{i,k}$ if we have $j = ks + i$ for $k \in \{0, \dots, r-1\}$ and $i \in \{1, \dots, s\}$. In general, a face σ corresponds to $m_\sigma := \operatorname{lcm}(m_{\varepsilon_j} \mid \sigma_j = 1)$.

Now we want to apply our Algorithm 3.1 but we only want to prune the edges $\sigma \longrightarrow \sigma + \varepsilon_j$ whenever the corresponding lcm's involve the same monomials $m_i \in I$. For example, we have that $\operatorname{lcm}(m_{i,b}, m_{a,k}) = \operatorname{lcm}(m_{i,k}, m_{i,b}, m_{a,k}) = m_{i,k,a,b}$ so both lcm 's involve the same monomials $m_i, m_k, m_a, m_b \in I$ for all a, b. This means that the corresponding edge would be pruned at step $j = ks + i$.

The partial pruning algorithm that we apply is the following:

Algorithm 5.4. *(Partial pruning)*

INPUT: The set of edges $E_{X_{J \cap K}}$.

For j from 1 to sr, incrementing by 1

(j) *Prune* the edge $\sigma \longrightarrow \sigma + \varepsilon_j$ for all $\sigma \in \{0,1\}^{sr}$ such that $\sigma_j = 0$, where 'prune' means remove the edge if it survived after step $(j-1)$ and m_σ involves the monomials m_i and m_k, where $j = ks + i$.

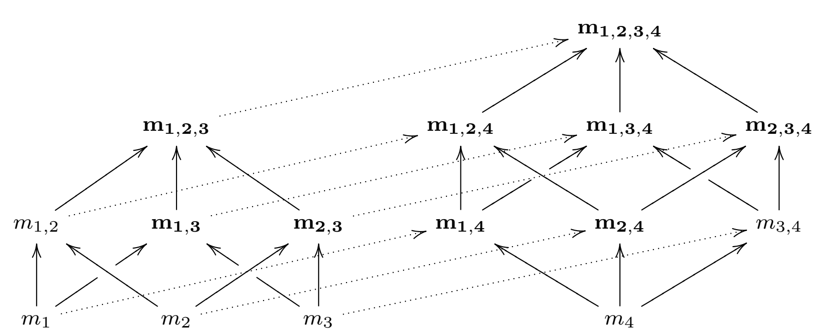
RETURN: The set $\mathcal{A}'$ of edges that have been pruned.

The tedious part is to check that using this algorithm we obtain X', that is, $X_{\mathcal{A}'} = X'$. To illustrate this fact we present the following:

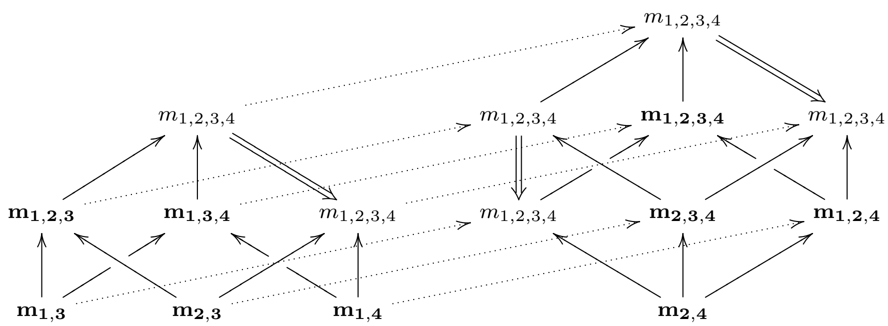
Example 5.5. Let $I = \langle m_1, m_2, m_3, m_4 \rangle$ be a monomial ideal generated by four monomials. We are going to consider the following decompositions:

· Consider $I = J + K$ with $J = \langle m_1, m_2, m_3 \rangle$ and $K = \langle m_4 \rangle$. In this case we have $J \cap K = \langle m_{1,4}, m_{2,4}, m_{3,4} \rangle$ so X' coincides with $X_{J \cap K}$.

· If $J = \langle m_1, m_2 \rangle$ and $K = \langle m_3, m_4 \rangle$ then we have that $J \cap K = \langle m_{1,3}, m_{2,3}, m_{1,4}, m_{2,4} \rangle$. The directed graph associated to the Taylor simplicial complex of I is:



and the monomials corresponding to the set X' are indicated with bold letters. On the other hand, applying the partial pruning algorithm to the Taylor complex $X_{J \cap K}$ we also get the set X' as shown in the following graph:



5.2. **A sufficient condition for Betti splittings.** Let $I = J + K$ be a decomposition of a monomial ideal. In order to have a Betti splitting, it is enough to check that applying Algorithm 3.1 to I, the pruning steps are realized independently within the edges associated to X_J, X_K and X'. This gives the following result that provides a sufficient condition to have a Betti splitting:

Proposition 5.6. *Let $I = \langle m_1, \dots, m_r \rangle \subseteq \Bbbk[x_1, \dots, x_n]$ be a monomial ideal and consider the ideals $J = \langle m_1, \dots, m_s \rangle$ and $K = \langle m_{s+1}, \dots, m_r \rangle$. If we only perform pruning steps within the edges of X_J, X_K or X' separatedly in Algorithm 3.1, then $I = J + K$ is a pruned Betti splitting for I.*

In the particular case that we remove just one generator from our initial ideal, we obtain the following:

Corollary 5.7. *Let $I = \langle m_1, \dots, m_r \rangle \subseteq \Bbbk[x_1, \dots, x_n]$ be a monomial ideal and consider the ideals $J = \langle m_1, \dots, m_{r-1} \rangle$ and $K = \langle m_r \rangle$. If no pruning has been made at the step r of Algorithm 3.1, then $I = J + K$ is a pruned Betti splitting for I.*

Betti splitting techniques can be used to provide a recursive method to compute Betti numbers of monomial ideals. For example, the case of edge ideals of graphs has been successfully studied in [13] and [11] where they considered *splitting edges* and *splitting vertices*. Splitting edges are easy to describe (see [11, Theorem 3.4]) but, in general they are difficult to find. On the other hand, every vertex is a splitting vertex except for some limit cases where the vertex is isolated or its complement consists of isolated vertices.

One can also check that any vertex also provides a pruned Betti splitting. Indeed, let $I = I(G) \subseteq \Bbbk[x_1, \dots, x_n]$ be the edge ideal of a graph G on n vertices. Consider a vertex, say x_n, and the decomposition $I = J + K$ where $J = I(G \setminus \{x_n\})$ is the edge ideal of the graph obtained from G by removing the vertex x_n and all the edges passing through this vertex. Notice that K is the edge ideal of a bipartite graph $\mathcal{K}_{1,d}$ where d is the number of vertices adjacent to x_n in G. We have a pruned Betti splitting because the ideal J does not involve the variable x_n and the pruning steps of the algorithm are performed within X_K and X' separately. Moreover, it is not difficult to see that the pruning algorithm provides a minimal free resolution for K, and hence

$$\overline{\beta}_{i,\alpha}(I) = \overline{\beta}_{i,\alpha}(J) + \beta_{i,\alpha}(K) + \overline{\beta}_{i-1,\alpha}(J \cap K).$$

In particular, if Algorithm 3.1 provides a minimal free resolution for J and $J \cap K$, then it is also provides a minimal free resolution for I.

For paths and cycles, we can use these splitting techniques to prove that the pruning algorithm will always provide a minimal free resolution as we already observed in Example 4.1 for the 5-path and the 5-cycle. The Betti numbers of these ideals have already been computed by Jacques in [16].

Example 5.8 (n-paths)**.** Let $I = \langle x_1x_2, x_2x_3, \dots, x_{n-1}x_n\rangle$ be the edge ideal of an n-path. A decomposition $I = J + K$ with $J = \langle x_1x_2, x_2x_3, \dots, x_{n-2}x_{n-1}\rangle$ and $K = \langle x_{n-1}x_n\rangle$ is a pruned Betti splitting simply because the ideal J does not involve the vertex x_n. Therefore we have

$$\overline{\beta}_{i,\alpha}(I) = \overline{\beta}_{i,\alpha}(J) + \overline{\beta}_{i,\alpha}(K) + \overline{\beta}_{i-1,\alpha}(J \cap K).$$

Indeed, this is a Betti splitting and the pruning algorithm provides a minimal free resolution, thus

$$\beta_{i,\alpha}(I) = \beta_{i,\alpha}(J) + \beta_{i,\alpha}(K) + \beta_{i-1,\alpha}(J \cap K).$$

By induction, we get a minimal free resolution for the ideals J and K so we only have to control the intersection in order to get the desired result. Recall that, using Lemma 3.6, we only have to consider a minimal set of generators. In our case we have

$$J \cap K = \langle \underbrace{x_1x_2x_{n-1}x_n, \dots, x_{n-4}x_{n-3}x_{n-1}x_n}_{J'}, \underbrace{x_{n-2}x_{n-1}x_n}_{K'} \rangle.$$

If we consider the decomposition $J \cap K = J' + K'$, we have a Betti splitting. Namely, the pruning algorithm applied on the ideals J' and $J' \cap K'$ is equivalent to the one given for the path $\langle x_1x_2, \dots, x_{n-4}x_{n-3}\rangle$, and we are done by induction.

Example 5.9 (n-cycles)**.** Let $I = \langle x_1x_2, x_2x_3, \dots, x_{n-1}x_n, x_nx_1\rangle$ be the edge ideal of an n-cycle. The decomposition $I = J+K$ with $J = \langle x_1x_2, x_2x_3, \dots, x_{n-1}x_n\rangle$ and $K = \langle x_nx_1\rangle$ is not a Betti splitting by [11, Theorem 3.4]. Indeed, there is a pruning in the last step of Algorithm 3.1.

It is more convenient to consider a splitting vertex . Namely, consider $I = J + K$ with $J = \langle x_1x_2, x_2x_3, \dots, x_{n-2}x_{n-1}\rangle$ and $K = \langle x_{n-1}x_n, x_nx_1\rangle$. Since J and K are the ideal of an $(n-1)$-path and a 3-path respectively, the pruning algorithm provides a minimal free resolution as shown in Example 5.8. Therefore we have:

$$\overline{\beta}_{i,\alpha}(I) = \beta_{i,\alpha}(J) + \beta_{i,\alpha}(K) + \overline{\beta}_{i-1,\alpha}(J \cap K).$$

A minimal set of generators of $J \cap K$ is

$$\langle \underbrace{x_2x_3x_{n-1}x_n, \dots, x_{n-4}x_{n-3}x_{n-1}x_n, x_{n-2}x_{n-1}x_n}_{J'}, \underbrace{x_1x_2x_n, x_1x_3x_4x_n, \dots, x_1x_{n-3}x_{n-2}x_n}_{K'} \rangle.$$

We have a pruned Betti splitting given by $J \cap K = J' + K'$. The pruning algorithm gives a minimal free resolution for the ideals J' and K' as we have seen when dealing with the case of paths. A minimal set of generators for the intersection $J' \cap K'$ is

$$\langle x_1x_2x_3x_{n-1}x_n, x_1x_3x_4x_{n-1}x_n, \ldots, x_1x_{n-3}x_{n-2}x_{n-1}x_n, x_1x_2x_{n-2}x_{n-1}x_n\rangle$$

but the pruning algorithm applied to this ideal is equivalent to the one for the cycle $\langle x_2x_3, x_3x_4, \ldots, x_{n-3}x_{n-2}, x_2x_{n-2}\rangle$, and we are done by induction.

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