An Overview of the Degree/Diameter Problem for Directed, Undirected and Mixed Graphs

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Abstract

A well-known fundamental problem in extremal graph theory is the *degree/diameter problem*, which is to determine the largest (in terms of the number of vertices) graphs or digraphs or mixed graphs of given maximum degree, respectively, maximum out-degree, respectively, mixed degree; and given diameter. General upper bounds, called Moore bounds, exist for the largest possible order of such graphs, digraphs and mixed graphs of given maximum degree d (respectively, maximum out-degree d, respectively, maximum mixed degree) and diameter k.

In recent years, there have been many interesting new results in all these three versions of the problem, resulting in improvements in both the lower bounds and the upper bounds on the largest possible number of vertices. However, quite a number of questions regarding the degree/diameter problem are still wide open. In this paper we present an overview of the current state of the degree/diameter problem, for undirected, directed and mixed graphs, and we outline several related open problems.

1 Introduction

We are interested in relationships among three graph parameters, namely, maximun degree (respectively, maximum out-degree, respectively, maximum mixed degree), diameter and order (i.e., the number of vertices) of a graph (respectively, digraph, respectively, mixed graph). Fixing the values of two of the parameters, we then wish to maximise or minimise the value of the third parameter. Then there are six possible problems, depending on which parameter we maximise or minimise; however, three of these problems are trivial and so below we formulate only the three non-trivial problems. For undirected graphs the problem statements are then as follows.

- *Degree/diameter problem*: Given natural numbers d and k, find the largest possible number of vertices $n_{d,k}$ in a graph of maximum degree d and diameter $\leq k$.

- *Order/degree problem*: Given natural numbers n and d, find the smallest possible diameter $k_{n,d}$ in a graph of order n and maximum degree d.

- *Order/diameter problem*: Given natural numbers n and k, find the smallest possible maximum degree $d_{n,k}$ in a graph of order n and diameter k.

The statements of the directed version of the problems differ only in that 'degree' is replaced by 'out-degree'. The corresponding statements for the mixed version of the problems use both the (undirected) maximum degree and the maximum out-degree.

The three problems are related but as far as we know they are not equivalent. For both undirected and directed cases, most of the attention has been given to the first problem, some attention has been received by the second problem but the third problem has been largely overlooked so far. The mixed version of all three problems was the last to be formulated and has received only very limitted attention until recently.

In this paper we will consider mainly the degree/diameter problem. For most fixed values of d and k, this problem is still wide open. In the next section we give an overview of the undirected version of the degree/diameter problem. In Section 3 we consider the degree/diameter problem for directed graphs. In Section 4 we present the status of the degree/diameter problem for mixed graphs. The paper concludes with some interesting open problems.

2 Undirected graphs

There is a natural straightforward upper bound on the largest possible order $n_{d,k}$ of a graph G of maximum degree d and diameter k. Trivially, if $d = 1$ then $k = 1$ and $n_{1,1} = 2$; in what follows we therefore assume that $d \geq 2$. Let v be a vertex of the graph G and let n_i, for $0 \leq i \leq k$, be the number of vertices at distance i from v. Then $n_i \leq d(d-1)^{i-1}$, for $1 \leq i \leq k$, and so

$$\begin{aligned} n_{d,k} = \sum_{i=0}^{k} n_i &\leq 1 + d + d(d-1) + \cdots + d(d-1)^{k-1} \\ &= 1 + d(1 + (d-1) + \cdots + (d-1)^{k-1}) \\ &= \begin{cases} 1 + d\,\frac{(d-1)^k - 1}{d-2} & \text{if } d > 2 \\ 2k+1 & \text{if } d = 2 \end{cases} \end{aligned} \qquad (1)$$

The right-hand side of (1) is called the *Moore bound* and is denoted by $M_{d,k}$. A graph whose maximum degree is d, diameter k, and order equal to the Moore bound $M_{d,k}$ is called a *Moore graph*; such a graph is necessarily regular of degree d.

Moore graphs do exist: For diameter $k = 1$ and degree $d \geq 1$, they are the complete graphs K_{d+1}. For diameter $k = 2$, Hoffman and Singleton [13] proved that Moore graphs can exist only for $d = 2, 3, 7$ and possibly 57; they are the cycle C_5 for degree $d = 2$, the Petersen graph for degree $k = 3$, and the Hoffman-Singleton graph for degree $k = 7$. The existence of a Moore graph of degree 57 is still an open problem. Damerell [9] proved that there are no Moore graphs (other than cycles K_{2k+2}) of diameter $k \geq 3$. An independent proof of this result was also given by Bannai and Ito [1].

Since Moore graphs exist only in a small number of cases, the study of the existence of large graphs focuses on graphs whose order is 'close' to the Moore bound, that is, graphs of order $M_{d,k} - \delta$, for δ small. The parameter δ is called the *defect*, and the most usual understanding of 'small defect' is that $\delta \leq d$. For convenience, by a (d,k)*-graph* we will understand any graph of maximum degree d and of diameter at most k; if such a graph has order $M^*_{d,k} - \delta$ then it will be referred to as a (d,k)*-graph of defect* δ.

Erdös, Fajtlowitcz and Hoffman [10] proved that, apart from the cycle C_4, there are no graphs of degree d, diameter 2 and defect 1, that is, of order one less than the Moore bound. This was subsequently generalized by Bannai and Ito [2] and also by Kurosawa and Tsujii [15] to all diameters. Hence, for all $d \geq 3$ there are no (d,k)-graphs of defect 1, and for $d = 2$ the only such graphs are the cycles C_{2k}. It follows that for $d \geq 3$ we have $n_{d,k} \leq M_{d,k} - 2$. Only a few values of $n_{d,k}$ are known. Apart from those already mentioned, we have also $n_{4,2} = 15$, $n_{5,2} = 24$, $n_{3,3} = 20$ and $n_{3,4} = 38$. The general frontier in the study of the upper bound of $n_{d,k}$ is defect 2.

Miller, Nguyen and Pineda-Villavicencio [17] found several structural properties of $(d,2)$-graphs with defect 2, and showed the nonexistence of such graphs for infinitely many values of d. Conde and Gimbert [7] used factorisation of certain polynomials related to the characteristic polynomial of a graph of diameter 2 and defect 2 to prove the nonexistence of $(d,2)$-graphs with defect 2 for other values of d. Combining these results we obtain that for degree d, $6 \leq d \leq 50$, there are no $(d,2)$-graphs with defect 2. Moreover, we believe that the following conjecture holds.

Conjecture 1 *For degree $d \geq 6$, there are no $(d,2)$-graphs with defect 2.*

Little is known about defects larger than two. Jorgensen [14] proved that a graph with maximum degree 3 and diameter $k \geq 4$ cannot have defect two. Taking into account the handshaking lemma when defect is odd, this shows that $n_{3,k} \leq M_{3,k} - 4$ if $k \geq 4$. In 2008, this was improved by Pineda-Villavicencio and Miller [18] to $n_{3,k} \leq M_{3,k} - 6$ if $k \geq 5$. Miller and Simanjuntak [19] proved that for $k \geq 3$, a $(4,k)$-graph cannot have defect 2, showing that $n_{4,k} \leq M_{4,k} - 3$ if $k \geq 3$. Currently, for most values of d and k, the existence or otherwise of (d,k)-graphs with defect 2 remains an open problem.

The lower bounds on $n_{d,k}$ and $n^*_{d,k}$ are obtained from constructions of the corresponding graphs and digraphs. There are many interesting techniques used in these constructions, including algebraic specifications (used to produce de Bruijn and Kautz graphs and digraphs), star product, compounding, and graph lifting - the last three methods all producing large graphs from suitable smaller "seed" or "base" graphs. Additionally, many new largest known graphs have been obtained with the assistance of computers.

In the case of undirected graphs, the gap between the lower bound and the upper bound on $n_{d,k}$ is in most cases wide, providing a good motivation for researchers to race each other for ever larger graphs. Further stimulation is provided by the current table of largest graphs (for degree up to 16 and diameter up to 10), kept up to date by Francesc Comellas on the website

`http://maite71.upc.es/grup_de_grafs/grafs/taula_delta_d.html`

A larger table (for degree up to 20 and diameter up to 10) is kept by Eyal Loz, Hebert Perez-Roses and Guillermo Pineda-Villavicencio; it is available at

`http://combinatoricswiki.org/wiki/The_Degree_Diameter_Problem_for_General_Graphs`

3 Directed graphs

As in the case of undirected graphs, there is a natural upper bound on the order, denoted by $n_{d,k}$, of directed graphs (digraphs) of given maximum out-degree d and diameter k. For any given vertex v of a digraph G, we can count the number of vertices at a particular distance from that vertex. Let n_i^*, for $0 \leq i \leq k$, be the number of vertices at distance i from v. Then $n_i^* \leq d^i$, for $0 \leq i \leq k$, and consequently,

$$n_{d,k}^* = \sum_{i=0}^{k} n_i^* \leq 1 + d + d^2 + \cdots + d^k = \begin{cases} \frac{d^{k+1}-1}{d-1} & \text{if } d > 1 \\ k+1 & \text{if } d = 1 \end{cases} \qquad (2)$$

The right-hand side of (2), denoted by $M_{d,k}^*$, is called the *Moore bound* for digraphs. If the equality sign holds in (2) then the digraph is called a *Moore digraph.*

It is well known that Moore digraphs exist only in the trivial cases when $d = 1$ (directed cycles of length $k+1$, C_{k+1}, for any $k \geq 1$) or $k = 1$ (complete digraphs of order $d+1$, K_{d+1}, for any $d \geq 1$). This was first proved by Plesník and Znám in 1974 [23] and later independently by Bridges and Toueg [6]. In the directed version, the general frontier in the study of the upper bound of $n_{d,k}^*$ is defect 1. For diameter $k = 2$, line digraphs of complete digraphs are examples of $(d,2)$-digraphs of defect 1, for any $d \geq 2$, showing that $n_{d,2}^* = M_{d,2}^* - 1$. When $d = 2$ there are two other non-isomorphic $(2,2)$-digraphs of defect 1 but for $d \geq 3$ Gimbert [11, 12] proved that line digraphs of complete digraphs are the only $(d, 2)$-digraphs of defect 1. Moreover, Conde, Gimbert, Gonzalez, Miret and Moreno [8] proved that there are no $(d, 3)$-digraphs with defect 1, for any $d \geq 3$.

On the other hand, focusing on small out-degree instead of diameter, Miller and Fris [16] proved that, for maximum out-degree 2, there are no $(2, k)$-digraphs of defect 1, for any $k \geq 3$. Moreover, Baskoro, Miller, Širáň and Sutton [3] proved, for maximum out-degree 3, that there are no $(3, k)$-digraphs of defect 1, for any diameter greater than or equal to 3. The following conjecture is likely to hold but unlikely to be proved in a simple way.

Conjecture 2 *For maximum out-degree $d \geq 2$ and diameter $k \geq 3$, there are no (d, k)-digraphs with defect 1.*

The study of digraphs of defect two has so far concentrated on digraphs of maximum out-degree $d = 2$. Miller and Širáň [20] proved, for maximum out-degree $d = 2$, that $(2, k)$-digraphs of defect two do not exist, for all $k \geq 3$. For the remaining values of $k \geq 3$ and $d \geq 3$, the question of whether digraphs of defect two exist or not remains completely open.

As in the undirected case, the lower bounds on $n^*_{d,k}$ are obtained from constructions of the corresponding digraphs. The current situation for the best lower bounds in the directed case is much simpler than in the undirected case. In the case of directed graphs, the best known values of $n^*_{d,k}$ are, in almost all cases, given by the corresponding Kautz digraph. One exception is the case of $d = 2$, where the best lower bound for $k \geq 4$ is obtained from Alegre digraph and line digraphs of Alegre digraph.

The difference between lower bound and upper bound on the largest possible order of a digraph of given maximum out-degree and diameter is much smaller than in the undirected case. Correspondingly, it seems much more difficult to find constructions of graphs that would improve the lower bound of $n^*_{d,k}$, and indeed, there has not been any improvement to the lower bound during the last 30 years or so, since the discovery of the Alegre digraph. On the other hand, thanks to the line digraph technique, finding any digraph larger than currently best known would result in much higher "payout" than in the undirected case, giving rise to a whole infinite family of largest known digraphs.

4 Mixed graphs

In many real-world networks, a mixture of both unidirectional and bidirectional connections may exist (e.g., the World Wide Web network, where pages are nodes and hyperlinks describe the connections). For such networks, mixed graphs provide a perfect modeling framework. The idea of "mixed" (or "partially directed") graphs is a generalisation of both undirected and directed graphs.

We start by introducing some definitions which are needed for mixed graphs. Let v be a vertex of a graph G. Denote by $id(v)$ (respectively, $od(v)$) the sum of the number of arcs incident to (respectively, from) v and the number of edges incident with v. Denote by $r(u)$ the number of edges incident with v (i.e., the *undirected degree* of v). A graph G is said to be *regular* of degree d if $od(v) = id(v) = d$, for every vertex v of G. A regular graph G of degree d is said to be *totally regular* with mixed degree d, undirected degree r and *directed degree* $z = d - r$ if for every pair of vertices $\{u, v\}$ of G we have $r(u) = r(v) = r$. Mixed Moore graphs of diameter 2 were first studied by Bosák in [4] and [5] who proved that all mixed Moore graphs are totally regular.

Let G be a mixed graph of diameter k, maximum degree d and maximum out-degree z. Let $r = d - z$. Then the order $n(z, r, k)$ of G is bounded by

$$\begin{aligned} n(z,r,k) &\leq M_{z,r,k} \qquad (3) \\ &= 1 + (z+r) + z(z+r) + r(z+r-1) \\ &\qquad + \cdots + z(z+r)^{k-1} + r(z+r-1)^{k-1} \end{aligned}$$

We shall call $M_{z,r,k}$ *the mixed Moore bound* for mixed graphs of maximum degree d, maximum out-degree z and diameter k. A mixed graph of maximum degree d, maximum out-degree z, diameter k and order $M_{z,r,k}$ is called a *mixed Moore graph*. Note that $M_{z,r,k} = M_{d,k}$ when $z = 0$ and $M_{z,r,k} = M^*_{d,k}$ when $r = 0$ ($d = r + z$).

A mixed graph G is said to be a *proper* mixed graph if G contains at least one arc and at least one edge.. Most of the known proper mixed Moore graphs of diameter 2, constructed by Bosák, can be considered isomorphic to Kautz digraphs of the same degree and order (with the exception of order $n = 18$). Indeed, they are the Kautz digraphs $Ka(d, 2)$ with all digons (a *digon* is a pair of arcs with the same end points and opposite direction) considered as undirected edges.

Mixed Moore graphs for $k \geq 3$ have been categorised in [22]. Suppose $d \geq 1$, $k \geq 3$. A finite graph G is a mixed Moore graph of degree d and diameter k if and only if either $d = 1$ and G is Z_{k+1} (the directed cycle on $k+1$ vertices), or $d = 2$ and G is C_{2k+1} (the undirected cycle on $2k+1$ vertices).

It remains to consider Moore graphs of diameter 2. Mixed Moore graphs of diameter 2 were studied by Bosák in [5] using matrix and eigenvalue techniques. Bosák proved that any mixed Moore graph of diameter 2 is totally regular with undirected degree r and directed degree z, where these two parameters r and z must satisfy a tight arithmetic condition obtained by eigenvalue analysis. Thus, apart from the trivial cases $z = 1$ and $r = 0$ (graph Z_3), $z = 0$ and $r = 2$ (graph C_5), there must exist a positive odd integer c such that

$$c \mid (4z-3)(4z+5) \text{ and } r = \frac{1}{4}(c^2+3). \tag{4}$$

Mixed Moore graphs of diameter $k = 2$ and order $n \leq 100$ are summarized in Table 1, where $d = z + r$ and the values of r and z are derived from (4) (see [5]).

5 Conclusion

In this paper we have given an overview of the degree/diameter problem and we pointed out some research directions concerning the three parameters order, diameter and maximum degree for undirected graphs, resp., maximum out-degree for directed graphs, resp., maximum mixed degree for mixed graphs. More specifically, we have been interested in the questions of optimising one of these three parameters (the order) given the values of the other two parameters. We finish by presenting a list of some related open problems in this area.

1. *Does there exist a Moore graph of diameter 2 and degree 57?* This is the best known open problem in this area; it has been open for half a century.

2. *Find graphs (resp. digraphs) which have larger number of vertices than the currently largest known graphs (resp., digraphs).*

n	d	z	r	*existence*	*uniqueness*
3	1	1	0	Z_3	$\surd$
5	2	0	2	C_5	$\surd$
6	2	1	1	$Ka(2,2)$	$\surd$
10	3	0	3	Petersen graph	$\surd$
12	3	2	1	$Ka(3,2)$ (Figure 1)	$\surd$
18	4	1	3	Bosák graph	$\surd$
20	4	3	1	$Ka(4,2)$	$\surd$
30	5	4	1	$Ka(5,2)$	$\surd$
40	6	3	3	unknown	unknown
42	6	5	1	$Ka(6,2)$	$\surd$
50	7	0	7	Hoffman-Singleton graph	$\surd$
54	7	4	3	unknown	unknown
56	7	6	1	$Ka(7,2)$	$\surd$
72	8	7	1	$Ka(8,2)$	$\surd$
84	9	2	7	unknown	unknown
88	9	6	3	unknown	unknown
90	9	8	1	$Ka(9,2)$	$\surd$

Table 1: Mixed Moore graphs of diameter 2 and order ≤ 100.

3. *Prove the diregularity or otherwise of digraphs close to Moore bound for defect greater than one.* Clearly, undirected graphs close to the Moore bound must be regular. It is also easy to see that digraphs close to the directed Moore bound must be out-regular. However, even for quite small defect (as little as 2), there exist digraphs which are in-regular but not out-regular (that is, all vertices have the same in-degree but not the same out-degrees).

4. *Investigate the degree/diameter problem for regular graphs, digraphs and mixed graphs.*

5. *Investigate the existence (and uniqueness) of mixed Moore graphs of diameter $k = 2$ and orders $40, 54, 88, 90$ and when $n > 100$.*

6. *Find large proper mixed graphs and construct a Table of the largest known proper mixed graphs.*

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