Generating and characteristic functions

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Probability generating function

Let \( X \) be a nonnegative integer-valued random variable.

The probability generating function of \( X \) is defined to be

\[
G_X(s) \equiv E(s^X) = \sum_{k \geq 0} s^k P(X = k)
\]

- If \( X \) takes a finite number of values, the above expression is a finite sum.
- Otherwise, it is a series that converges at least for \( s \in [-1, 1] \) and sometimes in a larger interval.

If \( X \) takes a finite number of values \( x_0, x_1, \ldots, x_n \), \( G_X(s) \) is a polynomial:

\[
G_X(s) = \sum_{k=1}^{n} s^k P(X = k)
= P(X = 0) + P(X = 1) s + \cdots + P(X = n) s^n
\]
Probability generating function

If $X$ takes a countable number of values $x_0, x_1, \ldots, x_k, \ldots$, then

$$G_X(s) = \sum_{k \geq 0} s^k P(X = k)$$

$$= P(X = 0) + P(X = 1) s + \cdots + P(X = k) s^k + \cdots$$

is a series that converges at least for $|s| \leq 1$, because

$$\left| \sum_{k \geq 0} s^k P(X = k) \right| \leq \sum_{k \geq 0} |s|^k P(X = k) \leq \sum_{k \geq 0} P(X = k) = 1$$

Examples

Let $X \sim B(n, p)$.

$$P(X = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, \ldots, n$$

Then

$$G_X(s) = \sum_{k \geq 0} s^k P(X = k) = \sum_{k=0}^{n} s^k \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} \left( \frac{n}{k} \right) (sp)^k q^{n-k} = (q + sp)^n, \quad s \in \mathbb{R}$$

Examples

Let $X$ be a Bernoulli random variable, $X \sim B(p)$.

$$P(X = 0) = q, \quad P(X = 1) = p$$

We have

$$G_X(s) = \sum_{k \geq 0} s^k P(X = k) = q + sp, \quad s \in \mathbb{R}$$

Examples

$X \sim \text{Poiss} (\lambda)$.

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

Then

$$G_X(s) = \sum_{k \geq 0} s^k P(X = k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!}$$

$$= e^{-\lambda} e^{s\lambda} = e^{\lambda(s-1)}, \quad s \in \mathbb{R}$$
Examples

\(X \sim \text{Geom}(p)\)

\[
P(X = k) = q^{k-1}p, \quad k = 1, 2, \ldots, \quad 0 < p < 1
\]

We have

\[
G_X(s) = \sum_{k=0}^{\infty} s^k P(X = k) = \sum_{k=1}^{\infty} s^k q^{k-1}p
\]

\[
= sp \sum_{k=1}^{\infty} (sq)^{k-1} = \frac{sp}{1 - sq}, \quad |s| < \frac{1}{q}
\]

Unicity

If two nonnegative, integer-valued random variables have the same generating function, then they follow the same probability law.

**Theorem**

Let \(X\) and \(Y\) be nonnegative integer-valued random variables such that

\[
G_X(s) = G_Y(s).
\]

Then

\[
P(X = k) = P(Y = k) \quad \text{for all} \quad k \geq 0.
\]

The result is a special case of the uniqueness theorem for power series.

Convolutions

**Theorem (Convolution)**

If \(X\) and \(Y\) are independent random variables and \(Z = X + Y\), then

\[
G_Z(s) = G_X(s) G_Y(s)
\]

**Proof:**

\[
G_Z(s) = E(s^Z) = E(s^{X+Y}) = E(s^X)E(s^Y) = G_X(s)G_Y(s)
\]

Example

Let \(X \sim \text{Bin}(n, p)\), \(Y \sim \text{Bin}(m, p)\) be independent random variables and let

\[
Z = X + Y
\]

We have

\[
G_Z(s) = G_X(s) G_Y(s) = (q + sp)^n (q + sp)^m = (q + sp)^{n+m}
\]

Observe that \(G_Z(s)\) is the probability generating function of a \(\text{Bin}(n + m, p)\) random variable. By the unicity theorem,

\[
X + Y \sim \text{Bin}(n + m, p)
\]
**Convolution theorem**

More generally,

**Theorem**

Let $X_1, X_2, \ldots, X_n$ be independent, nonnegative, integer-valued random variables and set

$$S_n = X_1 + X_2 + \cdots + X_n.$$  

Then

$$G_S(s) = \prod_{k=1}^{n} G_{X_k}(s).$$

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**Example: Negative binomial probability law**

A biased coin such that $P(\text{heads}) = p$ is repeatedly tossed until a total amount of $k$ heads has been obtained.

Let $X$ be the number of tosses.

Notice that

$$X = X_1 + X_2 + \cdots + X_k,$$

where

$$X_i \sim \text{Geom}(p)$$

is the number of tosses between the $(i-1)$-th and the $i$-th head.

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**Convolution theorem**

A case of particular importance is:

**Corollary**

If, in addition, $X_1, X_2, \ldots, X_n$ are equidistributed, with common probability generating function $G_X(s)$, then

$$G_S(s) = (G_X(s))^n.$$
Example: Negative binomial probability law

Recall that if $\alpha \in \mathbb{R}$, then the Taylor series expansion about 0 of the function $(1 + x)^\alpha$ is, for $x \in (-1, 1)$,

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{2} x^2 + \ldots + \frac{\alpha(\alpha - 1)\ldots(\alpha - r + 1)}{r!} x^r + \ldots$$

We can write

$$(1 + x)^\alpha = \sum_{r \geq 0} \binom{\alpha}{r} x^r$$

where

$$\binom{\alpha}{r} = \frac{\alpha(\alpha - 1)\ldots(\alpha - r + 1)}{r!}$$

Example: Negative binomial probability law

Consider the series expansion of $G_X(s)$:

$$G_X(s) = (sp)^k(1 - sq)^{-k} = (sp)^k \sum_{r=0}^{\infty} \binom{-k}{r}(-sq)^r$$

where

$$\binom{-k}{r} = (-k)(-k-1)\ldots(-k-r+1)$$

$$\frac{1}{r!} = (-1)^r \binom{k+r-1}{k-1}$$

Therefore,

$$G_X(s) = \sum_{r=0}^{\infty} \binom{k+r-1}{k-1} p^k q^r s^{k+r} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} p^k q^{n-k} s^n$$

Hence,

$$P(X = n) = \begin{cases} 0, & n < k \\ \binom{n-1}{k-1} p^k q^{n-k}, & n = k, k+1, \ldots \end{cases}$$

This is the negative binomial probability law.

Properties

- $G_X(0) = P(X = 0)$
- $G_X(1) = 1$

We have

$$G_X(1) = \left[ \sum_{k=0}^{\infty} s^k P(X = k) \right]_{s=1} = \sum_{k=0}^{\infty} P(X = k) = 1$$
Properties

**Proposition**

Let $R$ be the radius of convergence of $G_X(s)$. If $R > 1$, then

$$E(X) = G_X'(1)$$

Indeed,

$$G_X'(s) = \frac{d}{ds} \sum_{k \geq 0} s^k P(X = k) = \sum_{k \geq 1} k s^{k-1} P(X = k)$$

Hence,

$$G_X'(1) = \sum_{k \geq 1} k P(X = k) = E(X)$$

Examples

Let $X \sim \text{Bin}(n, p)$.

$$E(X) = G_X'(1) = \frac{d}{ds} (q + sp)^n \bigg|_{s=1} = np(q + p)^{n-1} = np$$

More generally,

**Proposition**

- $E(X) = G_X'(1) \equiv \lim_{s \to 1^-} G_X'(s)$
- $E(X(X - 1) \cdots (X - k + 1)) = G_X^{(k)}(1) \equiv \lim_{s \to 1^-} G_X^{(k)}(s)$

Examples

Let $X \sim \text{Poiss}(\lambda)$.

$$E(X) = G_X'(1) = \frac{d}{ds} e^{\lambda(s-1)} \bigg|_{s=1} = \lambda e^{\lambda(s-1)} \bigg|_{s=1} = \lambda$$

Analogously,

$$E(X(X - 1)) = G_X''(1) = \lambda^2 e^{\lambda(s-1)} \bigg|_{s=1} = \lambda^2$$

Hence,

$$E(X^2) = \lambda^2 + \lambda, \quad \text{Var}(X) = E(X^2) - (E(X))^2 = \lambda$$
Examples

\( X \sim \text{Geom}(p). \)

\[
E(X) = G'_0(1) = \frac{d}{ds} \frac{sp}{(1 - sq)} \bigg|_{s=1} = \frac{p}{(1 - sq)^2} \bigg|_{s=1} = \frac{1}{p}
\]

Analogously,

\[
E(X(X - 1)) = G''_0(1) = \frac{2pq}{(1 - sq)^3} \bigg|_{s=1} = \frac{2q}{p^2}
\]

Therefore

\[
E(X^2) = \frac{2q}{p^2} + \frac{1}{p}, \quad \text{Var}(X) = E(X^2) - (E(X))^2 = \frac{q}{p^2}
\]

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Examples

Let \( X \) be a \textbf{negative binomial} random variable.

\[
E(X) = G'_0(1) = \frac{d}{ds} \left( \frac{sp}{1 - sq} \right)^k \bigg|_{s=1}
\]

\[
= k \left( \frac{sp}{1 - sq} \right)^{k-1} \frac{p}{(1 - sq)^2} \bigg|_{s=1} = \frac{k}{p}
\]

This result can also be obtained from \( X = X_1 + \cdots + X_k \), with each \( X_i \sim \text{Geom}(p) \).

\[
E(X) = \sum_{i=1}^k E(X_i) = \frac{k}{p}
\]

Moment generating function

The \textbf{moment generating function} of a random variable \( X \) is defined as

\[
\phi_X(t) \equiv E \left( e^{tx} \right) = \begin{cases} 
\sum_{i} e^{tx_i} P(X = x_i), & \text{if } X \text{ is discrete} \\
\int_{-\infty}^{\infty} e^{tx} f_X(x) \, dx, & \text{if } X \text{ is continuous}
\end{cases}
\]

provided that the sum or the integral converges.
The moment generating function specifies **uniquely** the probability distribution.

**Theorem**

Let $X$ and $Y$ be random variables. If there exists $h > 0$, such that $\Phi_X(t) = \Phi_Y(t)$ for $|t| < h$, then $X$ and $Y$ are identically distributed.

**Examples**

Let $X \sim \text{Bin}(n, p)$.

$$\Phi_X(t) = \sum_{k=0}^{n} e^{tk} P(X = k) = \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k q^{n-k}$$

$$= (q + pe^t)^n, \quad t \in \mathbb{R}$$

Let $X \sim \text{Exp}(\mu)$.

$$\Phi_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) \, dx$$

$$= \int_{0}^{\infty} \mu e^{-(\mu-t)x} \, dx = \frac{\mu}{\mu - t}, \quad t < \mu$$

For continuous random variables, $\Phi_X(t)$ is related to the Laplace transform of $f_X(x)$.

Let $X \sim \text{Poiss}(\lambda)$.

$$\Phi_X(t) = \sum_{k=0}^{\infty} e^{tk} P(X = k) = \sum_{k=0}^{\infty} \frac{(e^t \lambda)^k}{k!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^t \lambda)^k}{k!} = e^{\lambda(e^t-1)}, \quad t \in \mathbb{R}$$
Examples

Let $Z \sim N(0, 1)$. 

$$
\Phi_Z(t) = \int_{-\infty}^{\infty} e^{tz} f_Z(z) \, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \, dz = e^{t^2/2}, \quad t \in \mathbb{R}
$$

because 

$$
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \, dz = P(-\infty < N(t, 1) < \infty) = 1
$$

Power series expansion

Notice that 

$$
\Phi_X(t) = \frac{d}{dt} E\left(e^{tX}\right) = E\left(\frac{d}{dt} e^{tX}\right) = E\left(X e^{tX}\right)
$$

Therefore, 

$$
\Phi_X'(0) = E(X)
$$

Analogously, 

$$
\Phi_X''(t) = \frac{d}{dt} \Phi_X'(t) = \frac{d}{dt} E\left(X e^{tX}\right) = E\left(X^2 e^{tX}\right)
$$

Thus, 

$$
\Phi_X''(0) = E(X^2)
$$

Examples

More generally, if 

$$
X = \sigma Z + m,
$$

then $X \sim N(m, \sigma^2)$. 

We have 

$$
\Phi_X(t) = E\left(e^{tX}\right) = E\left(e^{(\sigma Z + m)}\right) = e^{tm} E\left(e^{t\sigma Z}\right) = e^{tm} \Phi_Z(\sigma t) = e^{\sigma^2 t^2 + tm}
$$

For instance, if $X \sim \text{Exp} (\mu)$, 

$$
\Phi_X(t) = \frac{\mu}{\mu - t}, \quad t < \mu
$$

and 

$$
E(X) = \Phi_X'(0) = \frac{d}{dt} \left(\frac{\mu}{\mu - t}\right)_{t=0} = \frac{\mu}{(\mu - t)^2}_{t=0} = 1/\mu
$$

Analogously, 

$$
E(X^2) = \Phi_X''(0) = \frac{d}{dt} \left(\frac{\mu}{(\mu - t)^2}\right)_{t=0} = \frac{2\mu}{(\mu - t)^3}_{t=0} = 2/\mu^2
$$
Power series expansion

More generally,

\[ \Phi_X(t) = E \left( e^{tX} \right) = E \left( 1 + tX + \frac{(tX)^2}{2!} + \cdots + \frac{(tX)^k}{k!} + \cdots \right) \]

\[ = 1 + E(X) t + \frac{E(X^2)}{2!} t^2 + \cdots + \frac{E(X^k)}{k!} t^k + \cdots \]

This is the power series expansion of \( \Phi_X(t) \),

\[ \Phi_X(t) = \sum_{k=0}^{\infty} \frac{\phi_X^{(k)}(0)}{k!} t^k \]

Power series expansion

For instance, let \( X \sim \text{Exp}(\mu) \). If \(|t| < \mu\) we have

\[ \Phi_X(t) = \frac{\mu}{\mu - t} = \frac{1}{1 - (t/\mu)} = 1 + \frac{t}{\mu} + \left( \frac{t}{\mu} \right)^2 + \cdots \]

Hence,

\[ \frac{E(X^n)}{n!} = \frac{1}{\mu^n} \]

Therefore,

\[ E(X^n) = \frac{n!}{\mu^n} \]

Power series expansion

For instance, let \( X \sim \text{Exp}(\mu) \). If \(|t| < \mu\) we have

\[ \Phi_X(t) = \frac{\mu}{\mu - t} = \frac{1}{1 - (t/\mu)} = 1 + \frac{t}{\mu} + \left( \frac{t}{\mu} \right)^2 + \cdots \]

Hence,

\[ \frac{E(X^n)}{n!} = \frac{1}{\mu^n} \]

Therefore,

\[ E(X^n) = \frac{n!}{\mu^n} \]

Convolution theorem

The convolution theorem applies also to moment generating functions.

Theorem

Let \( X_1, X_2, \ldots, X_n \) be independent random variables and let \( S = X_1 + X_2 + \cdots + X_n \). Then,

\[ \Phi_S(t) = \prod_{k=1}^{n} \Phi_{X_k}(t) \]
Examples

$X \sim \text{Poiss}(\lambda_X), \ Y \sim \text{Poiss}(\lambda_Y)$, independent.

Let $Z = X + Y$.

We have

$$\Phi_Z(t) = \Phi_X(t)\Phi_Y(t) = e^{\lambda_X(e^{t-1})}e^{\lambda_Y(e^{t-1})} = e^{(\lambda_X + \lambda_Y)(e^{t-1})}$$

Hence,

$Z \sim \text{Poiss}(\lambda_X + \lambda_Y)$

---

Examples

$X \sim \mathcal{N}(m_X, \sigma_X^2), \ Y \sim \mathcal{N}(m_Y, \sigma_Y^2)$, independent.

Let $Z = X + Y$.

We have

$$\Phi_Z(t) = \Phi_X(t)\Phi_Y(t)$$
$$= e^{\frac{\sigma_X^2}{2} + tm_X}e^{\frac{\sigma_Y^2}{2} + tm_Y} = e^{\frac{(\sigma_X^2 + \sigma_Y^2)^2}{2} + t(m_X + m_Y)}$$

Therefore

$Z \sim \mathcal{N}(m_X + m_Y, \sigma_X^2 + \sigma_Y^2)$

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Characteristic function

The characteristic function of a random variable $X$ is the complex-valued function of the real argument $\omega$

$$M_X : \mathbb{R} \rightarrow \mathbb{C}$$

$$\omega \mapsto M_X(\omega)$$

defined as

$$M_X(\omega) \equiv E \left( e^{i\omega X} \right) = E \left( \cos(\omega X) \right) + i \left( E \left( \sin(\omega X) \right) \right)$$
**Characteristic function**

Therefore,

\[ M_X(\omega) = \begin{cases} \sum_k e^{i\omega x_k} P(X = x_k), & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{i\omega x} f_X(x) \, dx, & \text{if } X \text{ is continuous} \end{cases} \]

- The characteristic function exists for all \( \omega \) and for all random variables.
- If \( X \) is continuous, \( M_X(\omega) \) is the Fourier transform of \( f_X(x) \).
  (Notice the change of sign from the usual definition.)
- If \( X \) is discrete, \( M_X(\omega) \) is related to Fourier series.

**Properties**

- \( |M_X(\omega)| \leq M_X(0) = 1 \) for all \( \omega \in \mathbb{R} \).

\[ |M_X(\omega)| = \left| E\left( e^{i\omega X}\right) \right| \leq E\left( \left| e^{i\omega X}\right| \right) = E(1) = 1 \]

On the other hand,

\[ M_X(0) = E\left( e^{i0X}\right) = E(1) = 1 \]

- \( \overline{M_X(\omega)} = M_X(-\omega) \).

\[
\overline{M_X(\omega)} = E\left( e^{i\omega X}\right) = E\left( e^{-i\omega X}\right) = E(\cos(\omega X)) - i E(\sin(\omega X)) = E(\cos(-\omega X)) + i E(\sin(-\omega X)) = M_X(-\omega)
\]

- \( M_X(\omega) \) is uniformly continuous in \( \mathbb{R} \).

**Examples**

Let \( X \sim \text{Binom}(n, p) \). Then,

\[ M_X(\omega) = (pe^{i\omega} + q)^n \]

If \( X \sim \text{Poiss}(\lambda) \), then

\[ M_X(\omega) = e^{\lambda(e^{i\omega} - 1)} \]

If \( X \sim \text{N}(m, \sigma^2) \), then

\[ M_X(\omega) = e^{i\omega m - \frac{1}{2} \sigma^2 \omega^2} \]
Characteristic function and moments

**Theorem**

If \( E(X^n) < \infty \) for some \( n = 1, 2, \ldots \), then

\[
M_X(\omega) = \sum_{k=0}^{\infty} \frac{E(X^k)}{k!} (i\omega)^k + o(|\omega|^n) \quad \text{as} \quad \omega \to 0.
\]

So,

\[
E(X^k) = \frac{M_X^{(k)}(0)}{i^k} \quad \text{for} \quad k = 1, 2, \ldots, n.
\]

In particular, if \( E(X) = 0 \) and \( \text{Var}(X) = \sigma^2 \), then

\[
M_X(\omega) = 1 - \frac{1}{2} \sigma^2 \omega^2 + o(\omega^2) \quad \text{as} \quad \omega \to 0.
\]

Indeed,

\[
M_X(\omega) = E\left(e^{i\omega X}\right) = \frac{\sum_{k=0}^{\infty} (i\omega X)^k}{k!} = \sum_{k=0}^{\infty} \frac{i^k E(X^k)}{k!} \omega^k
\]

But this is the Taylor’s series expansion of \( M_X(\omega) \):

\[
M_X(\omega) = \sum_{k=0}^{\infty} \frac{M_X^{(k)}(0)}{k!} \omega^k
\]

Therefore

\[
i^k E(X^k) = M_X^{(k)}(0)
\]

Convolution theorem

**Theorem**

Let \( X_1, X_2, \ldots, X_n \) be independent random variables and let

\[
S = X_1 + X_2 + \cdots + X_n.
\]

Then

\[
M_S(\omega) = \prod_{k=1}^{n} M_{X_k}(\omega)
\]
**Convolution theorem**

In the case $n = 2$ we have essentially the convolution theorem for Fourier transforms.

If $X$ and $Y$ are continuous and independent random variables and $Z = X + Y$, then

$$f_Z = f_X * f_Y$$

This implies

$$F(f_Z) = F(f_X) \cdot F(f_Y),$$

that is,

$$M_Z(\omega) = M_X(\omega)M_Y(\omega)$$

---

**Unicity**

**Theorem**

Let $X$ have probability distribution function $F_X$ and characteristic function $M_X$. Let $\bar{F}_X(x) = (F_X(x) + F_X(x^-))/2$.

Then

$$\bar{F}_X(b) - \bar{F}_X(a) = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-i\omega b} - e^{-i\omega a}}{i\omega} M_X(\omega) \, d\omega.$$ 

$M_X$ specifies uniquely the probability law of $X$. Two random variables have the same characteristic function if and only if they have the same distribution function.

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**Inversion**

**Theorem (Inversion of the Fourier transform)**

Let $X$ be a continuous r.v. with density $f_X$ and characteristic function $M_X$. Then

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} M_X(\omega) \, d\omega$$

at every point $x$ at which $f_X$ is differentiable.

**Inversion**

In the discrete case, $M_X(\omega)$ is related to *Fourier series*.

**Theorem**

If $X$ is an integer-valued random variable, then

$$P(X = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-ik\omega}}{e^{-i\omega k}} M_X(\omega) \, d\omega$$

---
Joint characteristic functions

The joint characteristic function of the random variables $X_1, X_2, \ldots, X_n$ is defined to be

$$M_X(\omega_1, \omega_2, \ldots, \omega_n) = \mathbb{E} \left( e^{i(\omega_1 X_1 + \omega_2 X_2 + \cdots + \omega_n X_n)} \right)$$

Using vectorial notation one can write

$$\omega = (\omega_1, \omega_2, \ldots, \omega_n)^t, \quad X = (X_1, X_2, \ldots, X_n)^t$$

and

$$M_X(\omega) = \mathbb{E} \left( e^{i\omega^t X} \right)$$

Joint moments

The joint characteristic function allows us to calculate joint moments. For instance, given $X, Y$:

$$m_{kl} = \mathbb{E} (X^k Y^l) = \frac{1}{i^{k+l}} \frac{\partial^{k+l} M_{XY}(\omega_1, \omega_2)}{\partial \omega_1^k \partial \omega_2^l} \bigg|_{(\omega_1, \omega_2) = (0, 0)}$$

Marginal characteristic functions

Marginal characteristic functions are easily derived from the joint characteristic function.

For instance, given $X, Y$:

$$M_X(\omega) = \mathbb{E} \left( e^{i\omega X} \right)$$

$$= \mathbb{E} \left( e^{i(\omega_1 X_1 + \omega_2 Y)} \right) \bigg|_{\omega_2 = 0} = M_{XY}(\omega, 0)$$

Analogously,

$$M_Y(\omega) = M_{XY}(0, \omega)$$

Independent random variables

**Theorem**

The random variables $X_1, X_2, \ldots, X_n$ are independent if and only if

$$M_X(\omega_1, \omega_2, \ldots, \omega_n) = M_{X_1}(\omega_1) M_{X_2}(\omega_2) \cdots M_{X_n}(\omega_n)$$

If the random variables are independent, then

$$M_X(\omega_1, \omega_2, \ldots, \omega_n)$$

$$= \mathbb{E} \left( e^{i(\omega_1 X_1 + \omega_2 X_2 + \cdots + \omega_n X_n)} \right)$$

$$= \mathbb{E} \left( e^{i\omega_1 X_1} \right) \mathbb{E} \left( e^{i\omega_2 X_2} \right) \cdots \mathbb{E} \left( e^{i\omega_n X_n} \right)$$

$$= M_{X_1}(\omega_1) M_{X_2}(\omega_2) \cdots M_{X_n}(\omega_n)$$