Let us see the details: write the previous equation in the following form. Write $M(x,1) = M_1$, and $M(x,u) = M$:

$$(u-1)\,M = (u-1) + x u^2 (u-1) M^2 + x u^2 M - x u M_1$$

$\Downarrow$ Completing the square...

$$\left(M - \frac{1}{2}\,\frac{1-u+u^2x}{u^2x(1-u)}\right)^2 = \overbrace{\frac{x^2u^4 - 2xu^2(u-1)(2u-1) + (1-u^2)}{4u^4x^2(1-u)^2}}^{Q(x,u)} + \frac{M_1}{u(1-u)}$$

So we have an expression of the form $(g_1 M + g_2)^2 = g_3$, where g_3 depends on the unknown function M_1. Now we choose $u = \Theta(x)$ such that $(g_1 M + g_2) = 0$. So, indeed, with this condition we have that g_3 has a double root for $u = \Theta(x)$, and consequently,

$$g_3 = 0, \quad \left.\frac{\partial g_3}{\partial u}\right|_{u=\Theta(x)} = 0 \quad \Rightarrow \text{ two equations, indeterminates } u(x) \text{ and } M_1$$

Doing this we get:

$$\begin{cases} Q(x,u) + \dfrac{M_1}{u(1-u)} = 0 \\ Q_u(x,u) + \dfrac{(2u-1)}{u^2(1-u)^2}\,M_1 = 0 \end{cases}$$

Deleting M_1

$$\Rightarrow \overbrace{(u^2x + (u-1))}^{\text{NOT valid!}}\,\underbrace{(u^2x + (u-1)(2u-3))} = 0$$

$$M_1 = -u\,\frac{3u-4}{(2u-3)^2} \Leftarrow x = \frac{(1-u)(2u-3)}{u^2}$$

Now, by using elimination theory, we can get an equation for M_1:

$$\Rightarrow 27x^2M_1^2 - 18xM_1 + M_1 + 16x - 1 = 0 \Rightarrow M_1 = -\frac{1 - 18x - \sqrt{1-12x}^{\,3}}{54x^2}$$

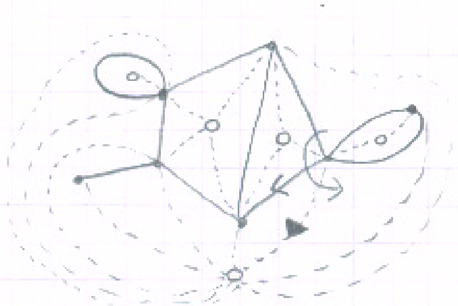
Extracting now the coefficients, we obtain something very interesting:

$$[x^n]\,M_1 = \frac{2}{n+2}\cdot 3^n \cdot C_n \quad !! \quad \text{Why???} \Rightarrow \underline{\text{Bijective explanation}}$$

Bijective proof: Cori-Vauquelin-Schaeffer bijection

The existence of Catalan numbers in the formula for rooted planar maps suggests that there exists some relation between unicellular maps and general ones. This is indeed the situation, and we will prove it now. This proof is due to Gilles Schaeffer.

To start, we will reduce ourselves to rooted maps where faces are of degree 4 (also called quadrangulations); this is what we call the trivial bijection:



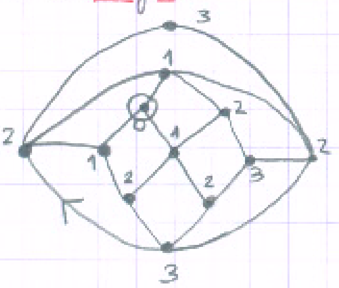
① We draw a (white) vertex for each face, and we link it with each corner of the face.

② We delete the initial edges

③ We root the new object in a canonical way (see the Figure).

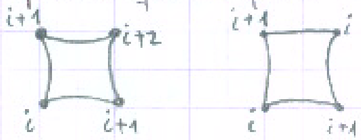
It is clear that now every face has degree 4. Additionally, this operation can be invert the relation. Now, by applying Euler's formula, the number of rooted planar maps with n edges is equal to the number of rooted quadrangulations with n faces. But such a quadrangulation has also $n+2$ vertices (by Euler's relation). We will call this family Q_n, and we will get $|Q_n|$.

The next step is to point a vertex (different possibly from the root vertex), that we call the origin:



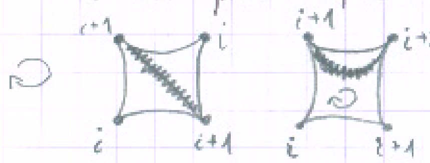
① The origin has label 0, and the rest of the vertices carry the label encoding the distance to the origin.

② Due to the fact that all cycles are of even length, we have two patterns for the faces:

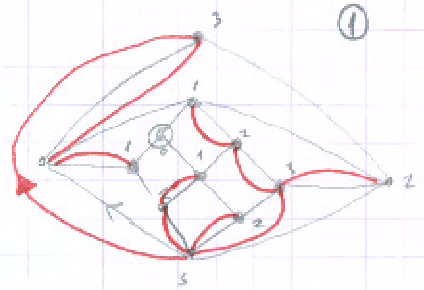


$\Rightarrow$ No vertices with the same label are incident!

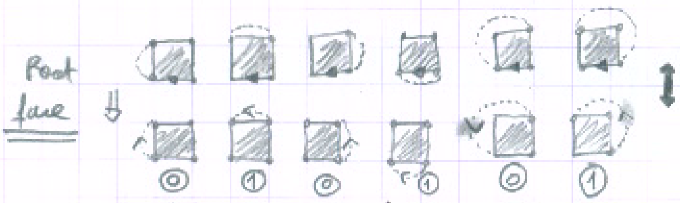
At this point, we are just having Q_n°, such that $|Q_n^\circ| = (n+2)|Q_n|$. But now we do the most important operation:



$\Rightarrow$



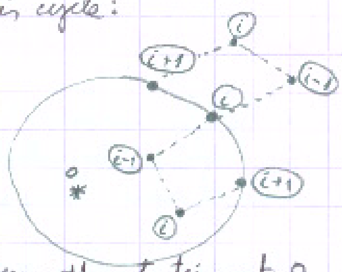
It is a tree!!



Let us argue that with this construction we obtain a graph which is connected and without cycles:

a) The resulting graph does NOT have cycles. Assume the contrary: consider a cycle C in the graph, and let i the smallest label in this cycle:

Any path starting at 0 and finishing at v have lenght $> i$!

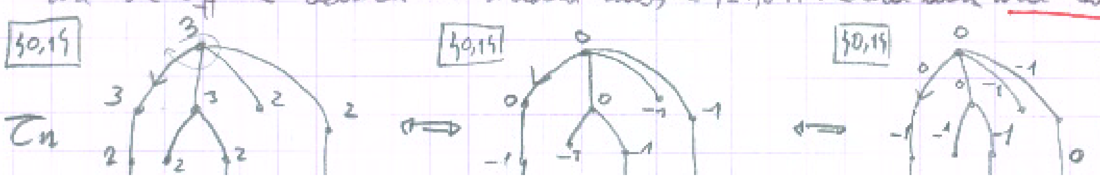


Any path starting at 0 and finishing at i have lenght $> i$!

$\Rightarrow$ Hence, it is NOT possible to have cycles!!

b) Is connected: the resulting graph has no cycles, $n+1 = n+2-1$ vertices and n edges (1 edge for each face). Hence, this means that this graph is a tree.

So, we are obtaining a tree whose vertices are labelled, and the labels belong to $\{1,2,3,\dots\}$, and the difference between two incident labels $\in \{\pm 1, 0\}$. We call them well-labelled trees:

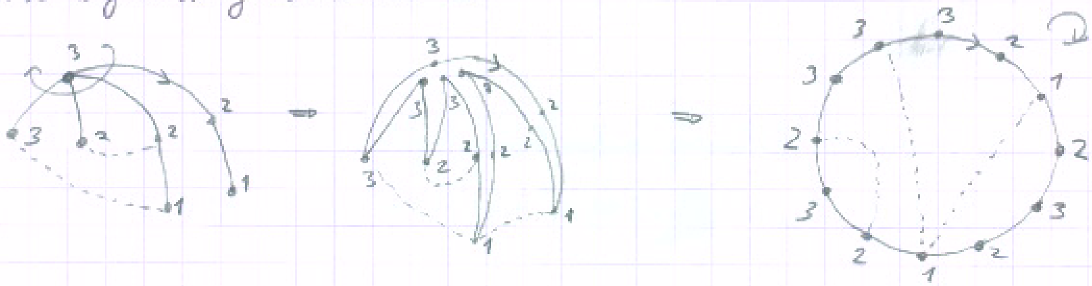


$\Rightarrow$ Well labelled trees are in bijection with

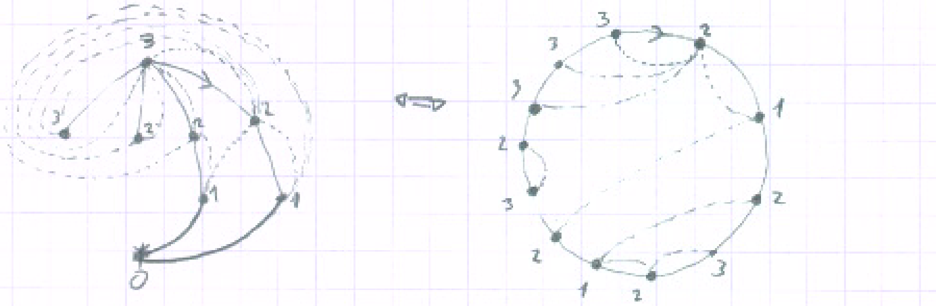
So, we have proved that:

$Q_n^\bullet \subseteq \{0,1\} \times Z_n = \{0,1\} \times \mathcal{C}_n \times \{0,\pm 1\}^n$ (which tells that $(n+2)|Q_n| \leq 2 \cdot 3^n \cdot C_n$)

So now the next step is to see that we can reconstruct a map which is quadrangular starting from a well-labelled tree. For each well-labelled tree we construct a polygon whose edges arise by walking around the tree:

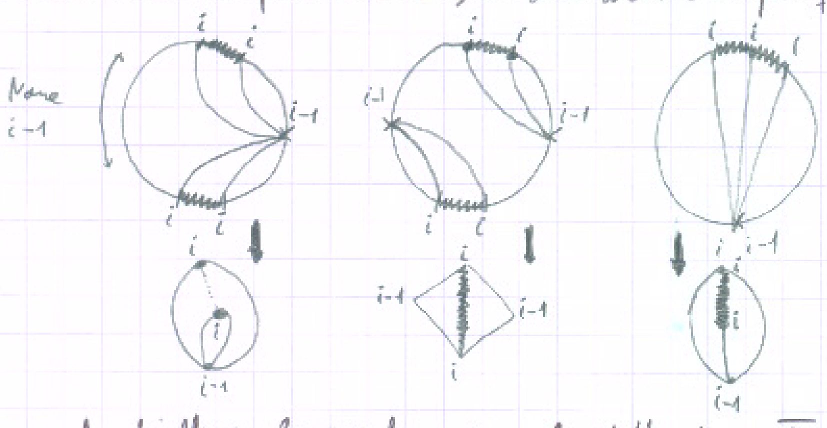


We say that a pair of replicates of the same edge are a couple. Now we will say which is the rule we use to construct our quadrangulation. We do the following: we link a vertex with label (i) with the next vertex in the order with label (i-1):



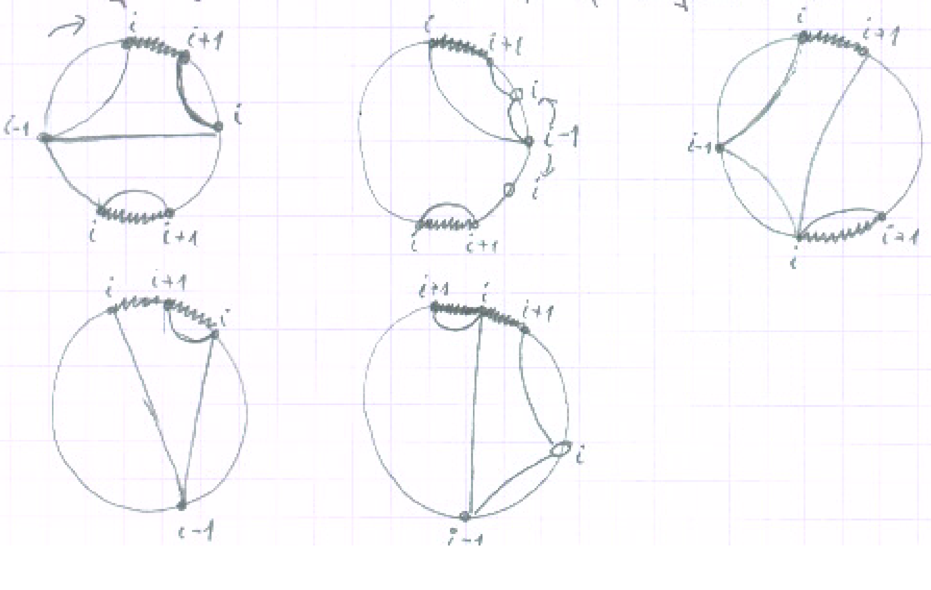
Finally we put the vertex (0), we connect with all the (1)s; we finally apply the rule with the 0, 1 to reconstruct the root.

So it remains to show that with this method all the faces we obtain are of degree 4. Let us see all possible cases; we start with a couple of the form $\overline{i\,i}$, $i \geq 2$



$\Rightarrow$ each couple of the form $\overline{ii}$ defines a quadrangle

And finally, when we have a couple of the form $\overline{i\,i{+}1}$, $i \geq 2$:

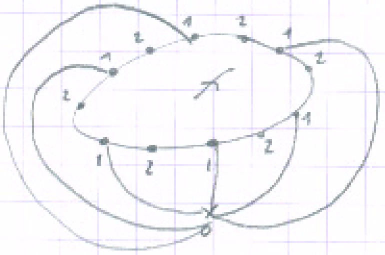


$\Rightarrow$ In all the cases we get faces of degree 4

$\Downarrow$

Something similar happens when considering couples of the form $\overline{i{+}1\,i}$

It remains to show that when adding the vertex with 0 (the origin), we are also just creating faces of degree 4:



$\Rightarrow$ Now all faces are of degree 4!

So, we have proved the following combinatorial bijection.

Theorem / (Schaeffer '98) $Q_n^\circ \simeq \mathcal{C}_n \times \{\pm 1, 0\}^n \times \{0, 1\}$ $\Rightarrow (n+2)\,|Q_n| = C_n \cdot 3^n \cdot 2 \Rightarrow |Q_n| = \frac{2 \cdot 3^n}{n+2} C_n$.